MULTIFRACTAL ANALYSIS OF SELF-SIMILAR PROCESSES

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Summary Two classes of processes are classically in competition for modeling *scale invariance in applications*: *self-similar processes* and *multiplicative cascades*. They imply fundamentally different underlying (additive or multiplicative) mechanisms. Their identification relies on properties often associated with the former class: self-similarity, monofractality, linear scaling function, null c_2 parameter. By performing a wavelet leader based analysis of the multifractal properties of self-similar processes, the present work contributes to a better disentangling of these different properties.

Scale Invariance

- Wavelet leaders $\longrightarrow$ *local supremum of wavelet coefficients*

$$L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_X(\lambda')| \qquad (1)$$

$d_X(j,k)$ – DWT coefficients of locally bounded function
$3\lambda_{j,k}$ – dyadic cube $[k2^j,(k+1)2^j)$ and its 2 neighbors

- Scaling function: $S(2^j,q) = \frac{1}{n_j}\sum_k L_X(j,k)^q \overset{2^j\to 0}{=} F_q 2^{j\zeta(q)}$

$$\zeta(q) = \liminf_{2^j\to 0} \log_2 S(2^j,q)/\log_2 2^j \qquad (2)$$

Cumulant expansion

- $C_p(2^j) = c_p^0 + c_p \ln 2^j$ $\quad C_p(2^j)$ – p-th cumulant of $\ln L_X(j,k)$
- $\zeta(q) = \sum_{p\geq 1} c_p \frac{q^p}{p!}$ $\quad$ polynomial expansion around $q=0$

Multifractal Spectrum

- Local regularity:

locally bounded function $X(x)$ $\longrightarrow$ *local power law behavior*

$|X(x)-X(x_0)| \leq C|x-x_0|^\alpha$ $\quad C>0,\ \alpha>0$

largest such α: Hölder exponent $h(\mathbf{x}_0)$

- Multifractal spectrum:

$\longrightarrow$ *geometric structure of subsets* $E_h:\ h(x_i)=h$

$$\mathcal{D}(h) = \dim_{Hausdorff}\{x:\ h(x)=h\} \qquad (3)$$

- Multifractal formalism:

Legendre transform $\longrightarrow$ *upper bound for multifractal spectrum*

$$\mathcal{L}(h) = \min_q(1+qh-\zeta(q)) \geq \mathcal{D}(h) \qquad (4)$$

Minimum regularity

$\mathcal{D}(h),\ L_X$: locally bounded functions only – equiv. $h_m>0$

$$h_m = \liminf_{2^j\to 0} \frac{\ln \sup_k |d_X(j,k)|}{\ln 2^j} \qquad (5)$$

- Fractional Integration
  - if $h_m<0$: $\longrightarrow$ $FI_\gamma(X)$ locally bounded for $\gamma \geq \max(0,-h_m)$
  - apply multifractal formalism to: $d_X^{(\gamma)}(j,k) = 2^{\gamma j} d_X(j,k)$

H-sssi Processes

H-self-similar (H-ss) with stationary increments (si):

$$\{X_H(at)\}_{t\in\mathbb{R}} \overset{d}{=} \{a^H X_H(t)\}_{t\in\mathbb{R}}, \quad a>0$$

Multiplicative Cascades
$\longrightarrow$ *strictly concave scaling function*
$\longrightarrow$ $c_2 < 0$
$\longrightarrow$ *multifractal*

• Fractional Brownian motion (fBm)

- $B_H(t) = \int_{\mathbb{R}} f(t,s)dB(s)$

$dB(s)$: white Gaussian noise $\mathcal{N}(0,\sigma)$
$f(t,s) = (t-s)_+^{H-1/2} - (-s)_+^{H-1/2}$
$H\in(0,1)$ $\quad (t)_+ = t$ if $t\geq 0$, $(t)_+=0$ elsewhere

- only Gaussian H-sssi process
- canonical reference for scale invariance modeling:

1. $\forall q\in\mathbb{R}$: $\zeta(q)=qH$ $\longrightarrow$ *strictly linear scaling function:* $\longrightarrow$ $c_2\equiv 0$
2. $\mathcal{D}(h) = \begin{cases} 1 & \text{if } h=H, \\ -\infty & \text{elsewhere.}\end{cases}$ $\longrightarrow$ *monofractal*

• Stable processes

- $L_{\{\cdot\}}(t) = \int_{\mathbb{R}} f(t,s)M(ds)$

$M(ds)$: symmetric α-stable measure $S\alpha S(\sigma)$

Levy-stable process $L_\alpha(t)$:
- $f(t,s) = 1(t-s>0) - 1(-s>0)$, $\quad \alpha\in(0,2]$
- H-sssi with $H=1/\alpha$

Linear fractional stable motion (LFSM) $L_{H,\alpha}(t)$:
- $f(t,s) = (t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha}$, $\quad \alpha\in(1/2,2)$
- H-sssi with $H\in(0,1)$

1. $-1<q<\alpha$: $\mathbb{E}|L_{\{\cdot\}}(t+\tau) - L_{\{\cdot\}}(t)|^q = \mathbb{E}|L_{\{\cdot\}}(1)|^q|\tau|^{qH}$
2. Levy: $\mathcal{D}(h) = \begin{cases} h\alpha & \text{if } 0\leq h\leq 1/\alpha, \\ -\infty & \text{elsewhere.}\end{cases}$ $\longrightarrow$ *multifractal*

LFSM: *multifractal properties not yet studied theoretically*

• Hermite processes

- $\mathcal{H}_{p,H}(t) = \int_{-\infty}^{+\infty} dB(s_1)\int_{-\infty}^{s_1} dB(s_2)\ldots\int_{-\infty}^{s_{p-1}} dB(s_p)\int_0^t \Pi_{k=1}^p (u-s_k)_+^{(H-1)/p-1/2}du$,

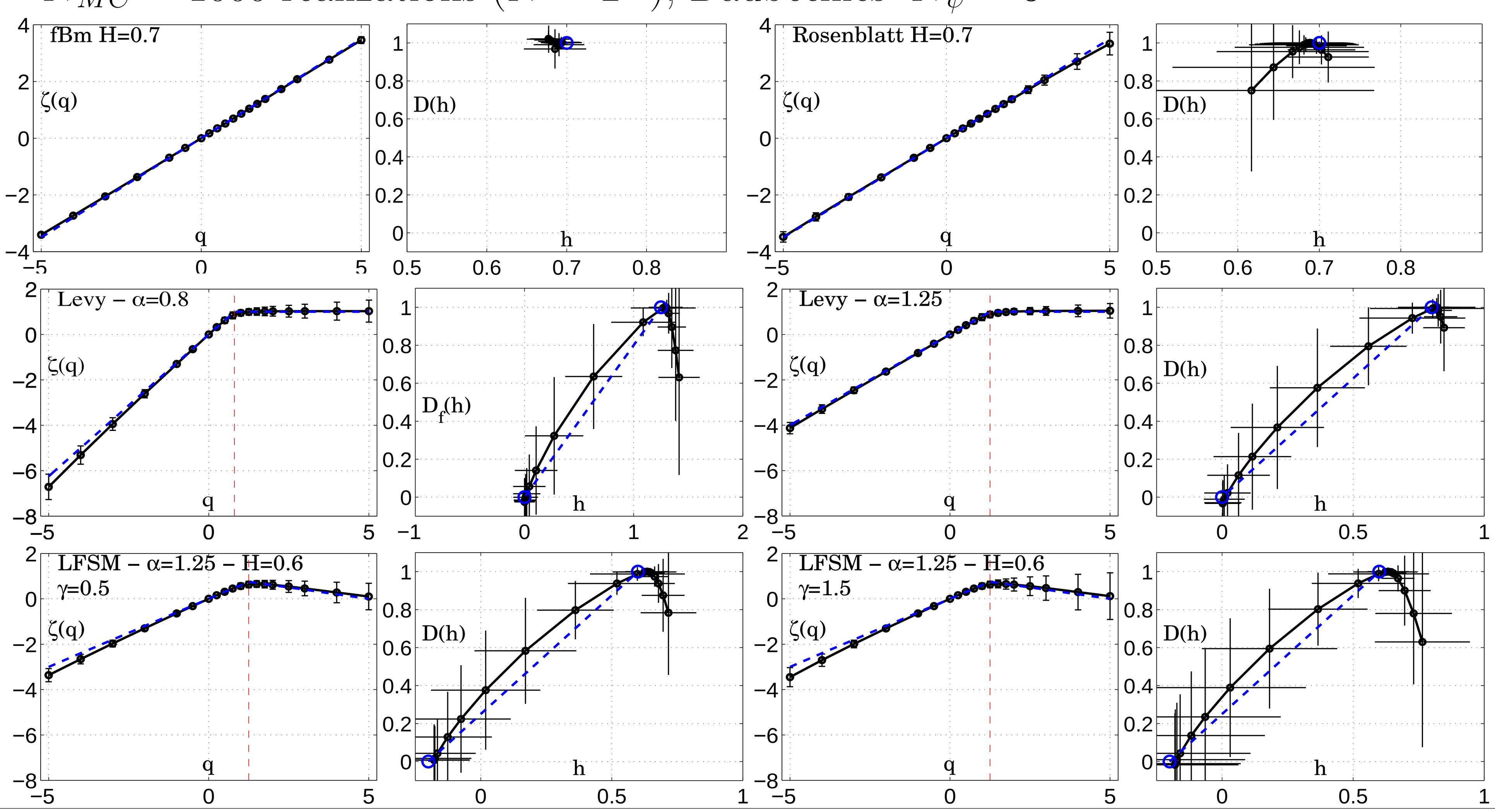
$dB(s_k)$: independent realizations of white Gaussian noise $\mathcal{N}(0,\sigma)$, $\quad p\in\mathbb{N}^*$
- H-sssi with $H\in(1/2,1)$,
- $p=1$: fBm, $p\geq 2$: non Gaussian

1. $\forall q>-1$: $\mathbb{E}\,|\mathcal{H}_{p,H}(t+\tau) - \mathcal{H}_{p,H}(t)|^q = \mathbb{E}\,|\mathcal{H}_{p,H}(1)|^q\,|\tau|^{qH}$
2. *multifractal properties not yet studied theoretically for* $p\geq 2$

Results

Process	γ	c_1	$\hat{c}_1^\gamma - \gamma$	c_2	$\hat{c}_2^\gamma$
fBm$_{H=0.7}$	0	0.7	0.690 ± 0.017	0	0.004 ± 0.005
$\mathcal{H}_{p,H=(2,0.7)}$	0	0.7	0.689 ± 0.042	0	-0.001 ± 0.015
$\mathcal{H}_{p,H=(3,0.7)}$	0	0.7	0.695 ± 0.055	0	-0.011 ± 0.032
$\mathcal{H}_{p,H=(4,0.7)}$	0	0.7	0.687 ± 0.074	0	-0.005 ± 0.043
$L_{\alpha=1.25}$	0	0.8	0.820 ± 0.090	0	0.002 ± 0.192
$L_{\alpha=0.8}$	0	1.25	1.293 ± 0.156	0	-0.026 ± 0.462
$L_{\alpha,H=(1.75,0.85)}$	0	0.85	0.801 ± 0.044	0	-0.003 ± 0.060
$L_{\alpha,H=(1.75,0.85)}$	1.0	0.85	0.824 ± 0.039	0	-0.005 ± 0.050
$L_{\alpha,H=(1.50,0.70)}$	0	0.7	0.712 ± 0.067	0	-0.013 ± 0.109
$L_{\alpha,H=(1.50,0.70)}$	1.0	0.7	0.708 ± 0.061	0	-0.014 ± 0.116
$L_{\alpha,H=(1.25,0.60)}$	0.5	0.6	0.635 ± 0.084	0	-0.028 ± 0.151
$L_{\alpha,H=(1.25,0.60)}$	1.5	0.6	0.634 ± 0.079	0	-0.032 ± 0.158

- $N_{MC}=1000$ realizations ($N=2^{15}$), Daubechies' $N_\psi=3$



Conclusions

- Conjecture for H-sssi processes:
  - $\zeta(q)$ piecewise linear
  - $\zeta(q)$ necessarily controlled by H in a neighborhood of $q=0$, hence $c_2\equiv 0$

– H-sssi does neither imply linear $\zeta(q)$, nor monofractality
– $c_2\equiv 0$ does neither imply linear $\zeta(q)$, nor monofractality

- Conjecture for LFSM: $\mathcal{D}(h) = \begin{cases} 1+\alpha(h-H) & \text{if } H-1/\alpha\leq h\leq H, \\ -\infty & \text{elsewhere.}\end{cases}$