BAYESIAN JOINT ESTIMATION OF THE MULTIFRACTALITY PARAMETER OF IMAGE PATCHES USING GAMMA MARKOV RANDOM FIELD PRIORS

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Summary

Multifractal (MF) analysis enables homogeneous image texture to be modeled and studied via the regularity fluctuations of image amplitudes. In this work, we propose a **Bayesian approach for the local (patch-wise) MF analysis** of images with **heterogeneous MF properties**. To this end, a joint Bayesian model for image patches is formulated using spatially **smoothing gamma Markov Random Field priors**, yielding a fast algorithm and significantly improving the state-of-the-art performance. Numerical simulations based on synthetic MF images illustrate the benefits of the proposed MF analysis procedure for images.

Multifractal analysis

- Hölder exponent:

Roughness of image X around $\boldsymbol{t}_0$

$\longrightarrow$ comparison to a local power law behavior

$||X(\boldsymbol{t})-X(\boldsymbol{t}_0)|| \le C||\boldsymbol{t}-\boldsymbol{t}_0||^{\alpha}$ $\quad C>0, \quad \alpha>0$

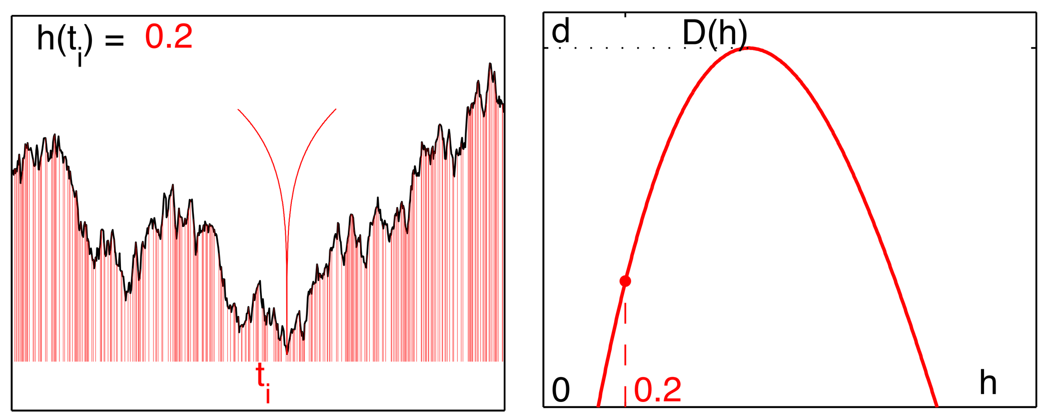
largest such α: Hölder exponent $h(\boldsymbol{t}_0)$

$$X(\boldsymbol{t}_0)\begin{cases}\text{rough} & h(\boldsymbol{t}_0)\sim 0\\ \text{smooth} & h(\boldsymbol{t}_0)\sim 1\end{cases}$$

- Multifractal spectrum:

$$\boxed{\mathcal{D}(h)=\dim_{\text{Hausdorff}}\{\boldsymbol{t}: h(\boldsymbol{t})=h\}}$$

$\longrightarrow$ geometric structure of iso-Hölder sets of points $\boldsymbol{t}_i$



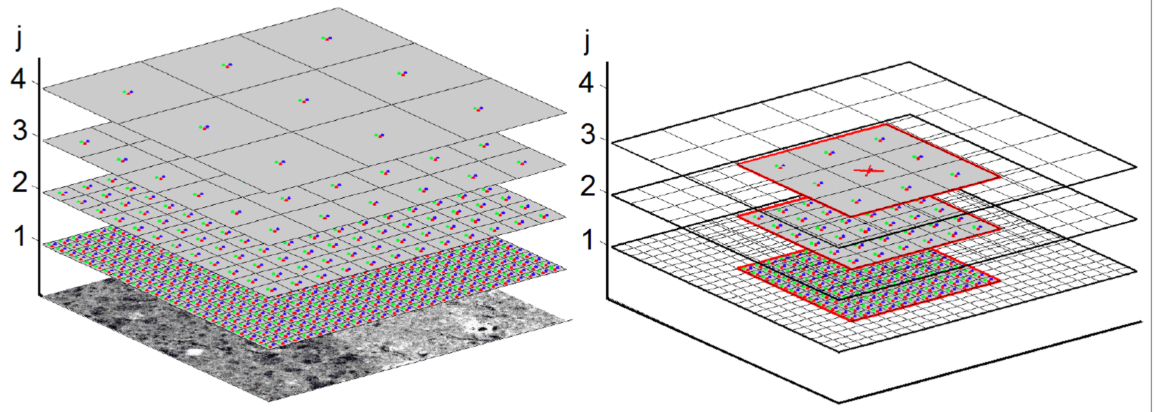
- Wavelet-leaders:

Local supremum of wavelet coefficients

$d_X^{(m)}(j,\boldsymbol{k})$ – 2D DWT coefficients $(m=1,2,3)$

$9\lambda_{j,\boldsymbol{k}}$ – dyadic cube centred at $\boldsymbol{k}2^j$ and its 8 neighbors

$$\boxed{l(j,\boldsymbol{k})=\sup_{m\in(1,2,3),\lambda'\subset 9\lambda_{j,\boldsymbol{k}}}|d_X^{(m)}(\lambda')|}$$



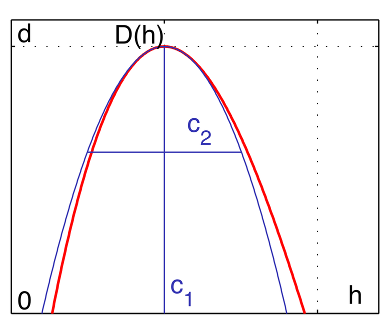
- Multifractal formalism:

- $\mathcal{D}(h)\le 1-(h-c_1)^2/(2|c_2|)+\cdots$

$C_p(j)=c_p^0+c_p\ln 2^j$

p-th cumulant of $\ln l(j,\boldsymbol{k})$

- c_2 – multifractality parameter



$$\boxed{C_2(j)=\text{Var}\left[\ln l(j,\boldsymbol{k})\right]=c_2^0+c_2\ln 2^j}$$

$\longrightarrow$ classical estimation by linear regression (LF)

Statistical model for log-leaders

- Time-domain statistical model: [Combrexelle15]

Centered log-leaders $\boldsymbol{\ell}_j=[\log l(j,\cdot)]\sim$ Gaussian random field

– radial symmetric covariance model $\varrho_j(\boldsymbol{\Delta k};\boldsymbol{\theta})$

– parametrized by $\boldsymbol{\theta}=[\theta_1,\theta_2]^T=[c_2,c_2^0]^T$

– assuming independence between scales:

$$p(\boldsymbol{\ell}|\boldsymbol{\theta})=\prod_{j=j_1}^{j_2}p(\boldsymbol{\ell}_j|\boldsymbol{\theta})\propto\prod_{j=j_1}^{j_2}|\boldsymbol{\Sigma}(\boldsymbol{\theta})|^{-\frac{1}{2}}\,e^{\left(-\frac{1}{2}\boldsymbol{\ell}_j^T\boldsymbol{\Sigma}(\boldsymbol{\theta})^{-1}\boldsymbol{\ell}_j\right)},\quad \boldsymbol{\ell}=[\boldsymbol{\ell}_{j_1}^T,\dots,\boldsymbol{\ell}_{j_2}^T]^T$$

$\boldsymbol{\Sigma}(\boldsymbol{\theta})$ - covariance matrix induced by $\varrho_j(\boldsymbol{\Delta k};\boldsymbol{\theta})$

– Whittle approximation of $p(\boldsymbol{\ell}_j|\boldsymbol{\theta})$

$$p(\boldsymbol{\ell}_j|\boldsymbol{\theta})\propto\exp\left(-\sum_{\mathbf{m}}\log\phi_j(\mathbf{m};\boldsymbol{\theta})+\frac{\boldsymbol{y}_j^*(\mathbf{m})\boldsymbol{y}_j(\mathbf{m})}{\phi_j(\mathbf{m};\boldsymbol{\theta})}\right)$$

$\boldsymbol{y}_j=DFT(\boldsymbol{\ell}_j)$ - Fourier transform of centered log-leaders

$\phi_j(\mathbf{m};\boldsymbol{\theta})$ - spectral density associated with $\varrho_j(\boldsymbol{\Delta k};\boldsymbol{\theta})$

- Data augmented model in the Fourier domain: [Combrexelle16]

– statistical interpretation of Whittle approximation

$\longrightarrow\boldsymbol{y}=[\boldsymbol{y}_{j_1}^T,\dots,\boldsymbol{y}_{j_2}^T]^T$ complex Gaussian random variable

– reparametrization $\boldsymbol{v}=\psi(\boldsymbol{\theta})\in\mathbb{R}_\star^{+2}$

$$\boldsymbol{y}\sim\mathcal{CN}(v_1\tilde{\boldsymbol{F}}+v_2\tilde{\boldsymbol{G}})$$

independent positivity contraints on v_i

$\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{G}}$ - diagonal, positive-definite, known and fixed

– data augmentation of the model

$$\begin{cases}\boldsymbol{y}|\boldsymbol{\mu},v_2\sim\mathcal{CN}(\boldsymbol{\mu},v_2\tilde{\boldsymbol{G}}) & \text{observed data}\\ \boldsymbol{\mu}|v_1\quad\sim\mathcal{CN}(0,v_1\tilde{\boldsymbol{F}}) & \text{hidden mean}\end{cases}$$

$\longrightarrow$ associated with augmented likelihood

$$p(\boldsymbol{y},\boldsymbol{\mu}|\boldsymbol{\theta})\propto v_2{}^{-N_Y}\exp\Big(-\frac{1}{v_2}(\boldsymbol{y}-\boldsymbol{\mu})^H\tilde{\boldsymbol{G}}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\Big)$$
$$\times\, v_1{}^{-N_Y}\exp\Big(-\frac{1}{v_1}\boldsymbol{\mu}^H\tilde{\boldsymbol{F}}^{-1}\boldsymbol{\mu}\Big)$$

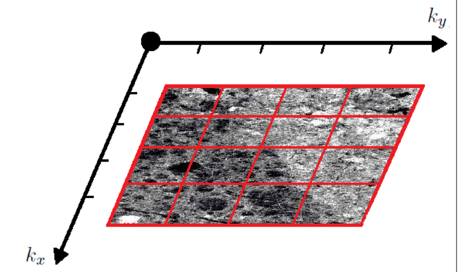
$\checkmark$ inverse-gamma $\mathcal{IG}$ priors on v_i are conjugate

[Robert05] Monte Carlo Statistical Methods, Springer, New York, USA, 2005
[Dikmen10] *Gamma Markov Random Fields for Audio Source Modeling*, IEEE T. Audio, Speech, and Language Process., vol. 18, no. 3, pp. 589-601, March 2010
[Combrexelle15] *Bayesian estimation of the multifractality parameter for image texture using a Whittle approximation*, IEEE T. Image Process., vol. 24, no. 8, pp. 2540-2551, Aug. 2015
[Combrexelle16] *A Bayesian framework for the multifractal analysis of images using data augmentation and a Whittle approximation*, Proc. ICASSP, Shanghai, China, March 2016

Bayesian model for image patches

- Likelihood:

- $\{\boldsymbol{X}_{\boldsymbol{k}=(k_x,k_y)}\}$ - partition of image $\boldsymbol{X}$ into non-overlapping patches
- $\boldsymbol{Y}=\{\boldsymbol{y}_{\boldsymbol{k}}\}$ - collection of Fourier coefficien
- $\boldsymbol{M}=\{\boldsymbol{\mu}_{\boldsymbol{k}}\}$ - collection of latent variables
- $\boldsymbol{V}_i=\{v_{i,\boldsymbol{k}}\}$ - collection of MF parameter



+ assuming indepence between patches

$$\longrightarrow\ p(\boldsymbol{Y},\boldsymbol{M}|\boldsymbol{V})\propto\prod_{\boldsymbol{k}}p(\boldsymbol{y}_{\boldsymbol{k}},\boldsymbol{\mu}_{\boldsymbol{k}}|\boldsymbol{v}_{\boldsymbol{k}}),\quad \boldsymbol{V}=\{\boldsymbol{V}_1,\boldsymbol{V}_2\}$$

- Gamma Markov random field (GMRF) prior:

- assuming smooth spatial evolution of MF parameters
- positive auxiliary variables $\mathbf{Z}=\{\mathbf{Z}_1,\mathbf{Z}_2\}$, $\boldsymbol{Z}_i=\{z_{i,\boldsymbol{k}}\}$, $i=1,2$

$\longrightarrow$ induce positive correlation between neighbouring $v_{i,\boldsymbol{k}}$

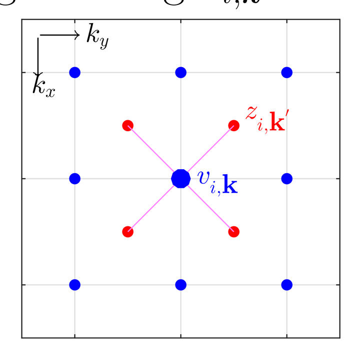
- each $v_{i,\boldsymbol{k}}$ is connected to 4 variables $z_{i,\boldsymbol{k}'}$

$\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})=\{(k_x,k_y)+(i_x,i_y)\}_{i_x,i_y=0,1}$

via edges with weights a_i

- each $z_{i,\boldsymbol{k}}$ is connected to 4 variables $v_{i,\boldsymbol{k}'}$

$\boldsymbol{k}'\in\mathcal{V}_z(\boldsymbol{k})=\{(k_x,k_y)+(i_x,i_y)\}_{i_x,i_y=-1,0}$



- associated distribution [Dikmen10]

$$p(\boldsymbol{V}_i,\boldsymbol{Z}_i|a_i)=\frac{1}{C(a_i)}\prod_{\boldsymbol{k}}e^{-(4a_i+1)\log v_{i,\boldsymbol{k}}}\,e^{(4a_i-1)\log z_{i,\boldsymbol{k}}}\times e^{-\frac{a_i}{v_{i,\boldsymbol{k}}}\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})}z_{i,\boldsymbol{k}'}}$$

$\checkmark$ conditionals for $v_{i,\boldsymbol{k}}$ ($z_{i,\boldsymbol{k}}$) are inverse-gamma $\mathcal{IG}$ (gamma $\mathcal{G}$)

- Posterior distribution & Bayesian estimator:

- Posterior distribution via Bayes' theorem

$$p(\boldsymbol{V},\boldsymbol{Z},\boldsymbol{M}|\boldsymbol{Y},a_i)\propto\underbrace{p(\boldsymbol{Y}|\boldsymbol{V}_2,\boldsymbol{M})\,p(\boldsymbol{M}|\boldsymbol{V}_1)}_{\text{augmented likelihood}}\times\underbrace{\prod_i p(\boldsymbol{V}_i,\boldsymbol{Z}_i|a_i)}_{\text{independent GMRF priors}}$$

- Marginal posterior mean (MMSE) estimator

$$\boldsymbol{V}_i^{\text{MMSE}}=\underbrace{\mathbb{E}[\boldsymbol{V}_i|\boldsymbol{Y},a_i]}_{\text{untractable}}\approx\frac{1}{N_{mc}-N_{bi}}\sum_{q=N_{bi}}^{N_{mc}}\boldsymbol{V}_i^{(q)}$$

with $\{\boldsymbol{V}_i^{(q)}\}_{q=1}^{N_{mc}}$ distributed according to $p(\boldsymbol{V},\boldsymbol{Z},\boldsymbol{M}|\boldsymbol{Y},a_i)$

$\longrightarrow$ computation via Markov chain Monte Carlo algorithm

- Gibbs sampler: [Robert05]

$\longrightarrow$ iterative sampling according to conditional distributions

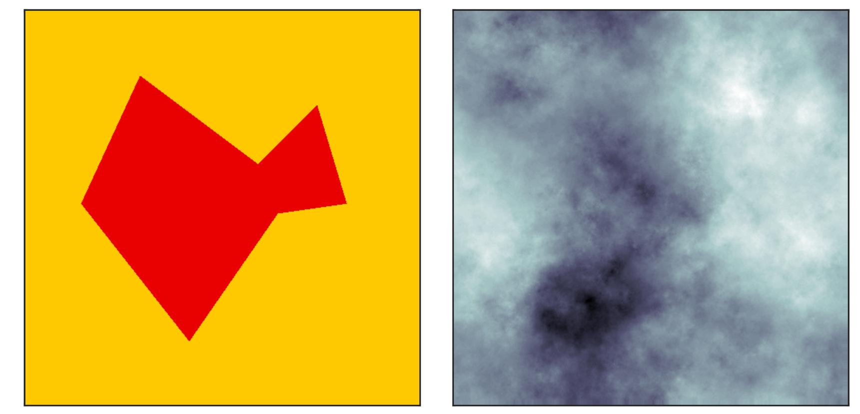
$$\boldsymbol{\mu}_{\boldsymbol{k}}|\boldsymbol{Y},\boldsymbol{V}\sim\mathcal{CN}\Big(v_{1,\boldsymbol{k}}\tilde{\boldsymbol{F}}\boldsymbol{\Gamma}_{\boldsymbol{v_k}}^{-1}\boldsymbol{y}_{\boldsymbol{k}},\Big((v_{1,\boldsymbol{k}}\tilde{\boldsymbol{F}})^{-1}+(v_{2,\boldsymbol{k}}\tilde{\boldsymbol{G}})^{-1}\Big)^{-1}\Big)$$
$$v_{1,\boldsymbol{k}}|\boldsymbol{M},\boldsymbol{Z}_1\sim\mathcal{IG}\Big(N_Y+4a_i,\boldsymbol{\mu}_{\boldsymbol{k}}^H\tilde{\boldsymbol{F}}^{-1}\boldsymbol{\mu}_{\boldsymbol{k}}+\beta_{1,\boldsymbol{k}}\Big)$$
$$v_{2,\boldsymbol{k}}|\boldsymbol{Y},\boldsymbol{M},\boldsymbol{Z}_2\sim\mathcal{IG}\Big(N_Y+4a_i,(\boldsymbol{y}_{\boldsymbol{k}}-\boldsymbol{\mu}_{\boldsymbol{k}})^H\tilde{\boldsymbol{G}}^{-1}(\boldsymbol{y}_{\boldsymbol{k}}-\boldsymbol{\mu}_{\boldsymbol{k}})+\beta_{2,\boldsymbol{k}}\Big)$$
$$z_{i,\boldsymbol{k}}|\boldsymbol{V}_i\sim\mathcal{G}(4a_i,\tilde{\beta}_{i,\boldsymbol{k}})$$

with $\beta_{i,\boldsymbol{k}}=a_i\sum_{\boldsymbol{k}'\in\mathcal{V}_v(\boldsymbol{k})}z_{i,\boldsymbol{k}'}$ and $\tilde{\beta}_{i,\boldsymbol{k}}=(a_i\sum_{\boldsymbol{k}'\in\mathcal{V}_z(\boldsymbol{k})}v_{i,\boldsymbol{k}'}^{-1})^{-1}$

$\checkmark$ standard distributions $\longrightarrow$ no acceptance/reject moves

Numerical experiments

- Scenario



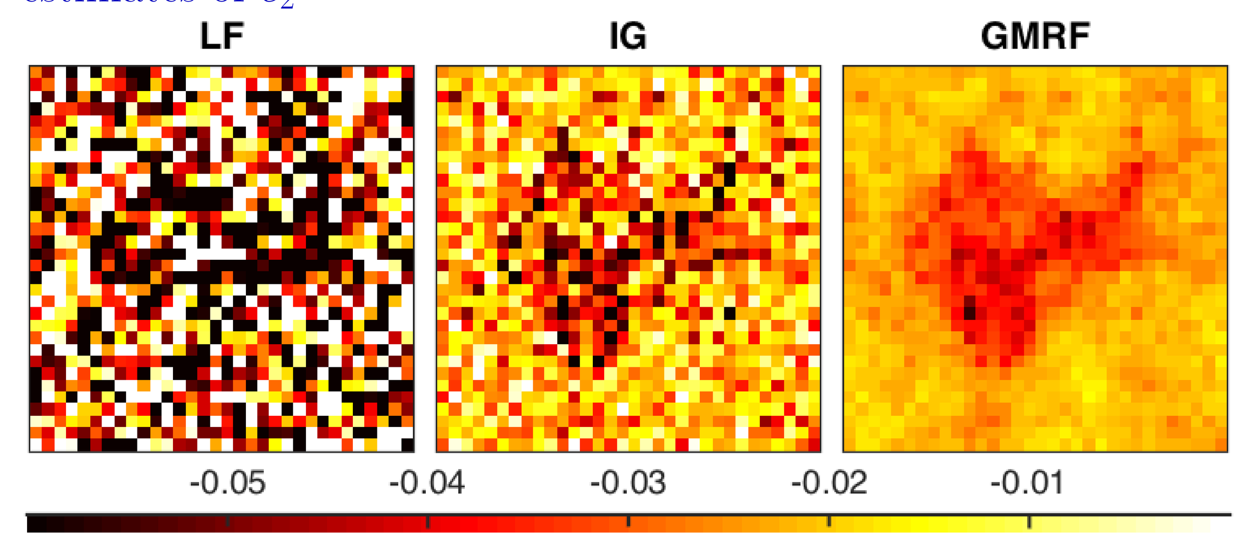
- synthetic multifractal 2048×2048 image

$\longrightarrow$ 2D multifractal random walk (MRW)

- piece-wise constant values of $c_2\in\{-0.04,-0.02\}$
- decomposition into non-overlapping 64×64 patches

- Illustration on a single realization

- estimates of c_2



- k-means classification

- Estimation performance

evaluated on 100 realizations of heterogeneous 2D MRW

	|bias|	STD	RMSE	classification %	time (s)
LF	0.0055	0.0406	0.0413	54.2	∼ **10**
IG	**0.0018**	0.0123	0.0125	76.5	∼ 50
GMRF	0.0027	**0.0032**	**0.0044**	**94.6**	∼ 50

Conclusion & future work

Take-away message

- smooth joint estimation of c_2 for image patches
- GMRF prior inducing positive correlation
- STD/RMSE reduced by $\sim 4-10$ vs. state-of-the-art estimators
- efficient MCMC algorithm (only ~ 5 times LF)

Future work

- automatic estimation of a_i
- additional MF parameters (c_1, c_3, ...)