HYPERBOLIC WAVELET LEADERS FOR ANISOTROPIC MULTIFRACTAL TEXTURE ANALYSIS

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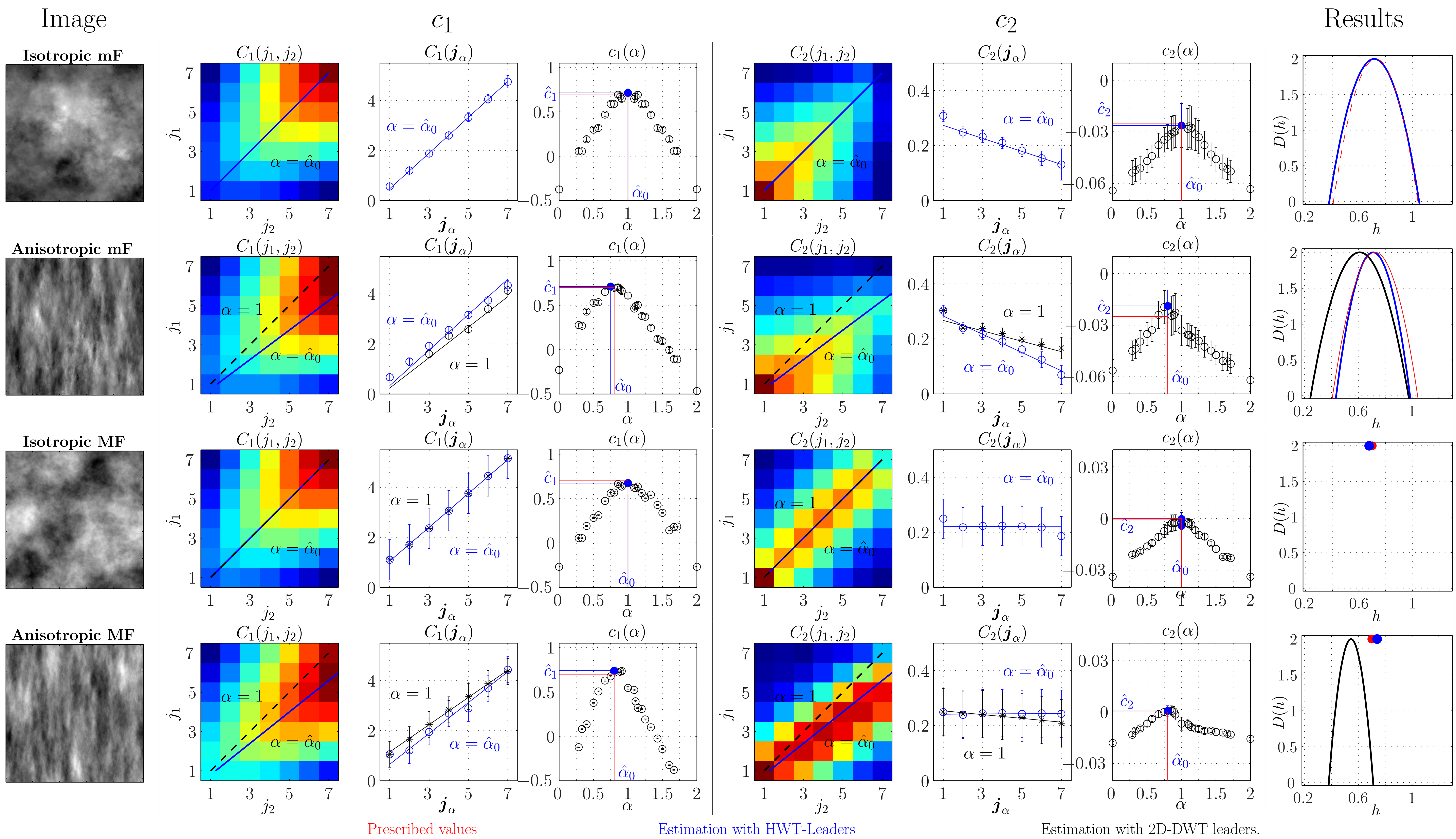


Goals

Define 2D process that incorporates jointly anisotropy and multifractality
Define the corresponding analysis tool, the hyperbolic wavelet leaders

Anisotropic multifractal textures

$$X_{\alpha_0,H_0,\lambda_0}(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{(e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle}-1)}{||\boldsymbol{\xi}||_{2,\alpha_0}^{(H_0+1)}} G_{\lambda_0}(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

$\mathbf{x}=(x_1,x_2)$ and $\boldsymbol{\xi}=(\xi_1,\xi_2)$;

- $0 < H_0 < 1$: global scale free parameter of the process.
- $\lambda_0 > 0$: *intermittency* parameter, controls the degree of variation of the local regularity along space,

  $G_{\lambda_0}(\boldsymbol{\xi}) = TF\left\{e^{\omega(\mathbf{x})} dW(\mathbf{x})\right\}$, $dW(\mathbf{x})$ 2D-Wiener measure;

  $\omega(\mathbf{x})$ Gaussian stationary process with $\mathrm{cov}(\omega(\mathbf{x}),\omega(\mathbf{y})) = -\lambda_0^2 \log(||\mathbf{x}-\mathbf{y}||_2)$,
- $\lambda_0 = 0$, we set $\omega(\mathbf{x}) \equiv 1$.
- $0 < \alpha_0 < 2$: global anisotropy parameter

  pseudo norme : $||\boldsymbol{\xi}||_{2,\alpha_0} = \sqrt{|\xi_1|^{2/\alpha_0} + |\xi_2|^{2/(2-\alpha_0)}}$.

$\alpha_0 \equiv 1$: we recover the isotropic case

$\lambda_0 \equiv 0$: we recover the classical exactly self-similar Gaussian OSGRF.

$\alpha_0 \equiv 1$ and $\lambda_0 \equiv 0$ gives the Gaussian isotropic exactly self-similar 2D-fBf.

Anistropic multifractal Description.

- Anisotropic singularity spectrum : $D_\alpha(h) = d_H(\{\mathbf{x} \mid h_\alpha(\mathbf{x}) = h\})$
- Anisotropic local regularity exponent : $h_\alpha(\mathbf{x}) = \sup_s\{s;\ |X_{\alpha_0,H_0,\lambda_0}(\mathbf{y}) - P_{\mathbf{x}}(\mathbf{y})| \le C||\mathbf{y}-\mathbf{x}||_{2,\alpha}^s\}$

  C a constant and $P_{\mathbf{x}}(\mathbf{y})$ a polynomial of the form $P_{\mathbf{x}}(y_1,y_2) = \sum_{(\beta_1,\beta_2)\in\mathbb{N}^2} a_{\beta_1,\beta_2} y_1^{\beta_1} y_2^{\beta_2}$.

Scale-free properties.
$X_{\alpha_0,H_0,\lambda_0}(\mathbf{x})$ -characterized by

$$D_{\alpha_0}(h) = 2 - (h-c_1)^2/c_2.$$

with $c_1 = c_1(\alpha_0) = H_0 - \lambda^2$ and $c_2 = c_2(\alpha_0) = \lambda_0^2$.

Conclusions

Anisotropy $\Rightarrow$ Hyperbolic Wavelet Transform
Multifractality $\Rightarrow$ Hyperbolic Wavelet Leaders
$\Big\}$ $\Rightarrow$ Robustness of estimation.

Hyperbolic wavelet leaders

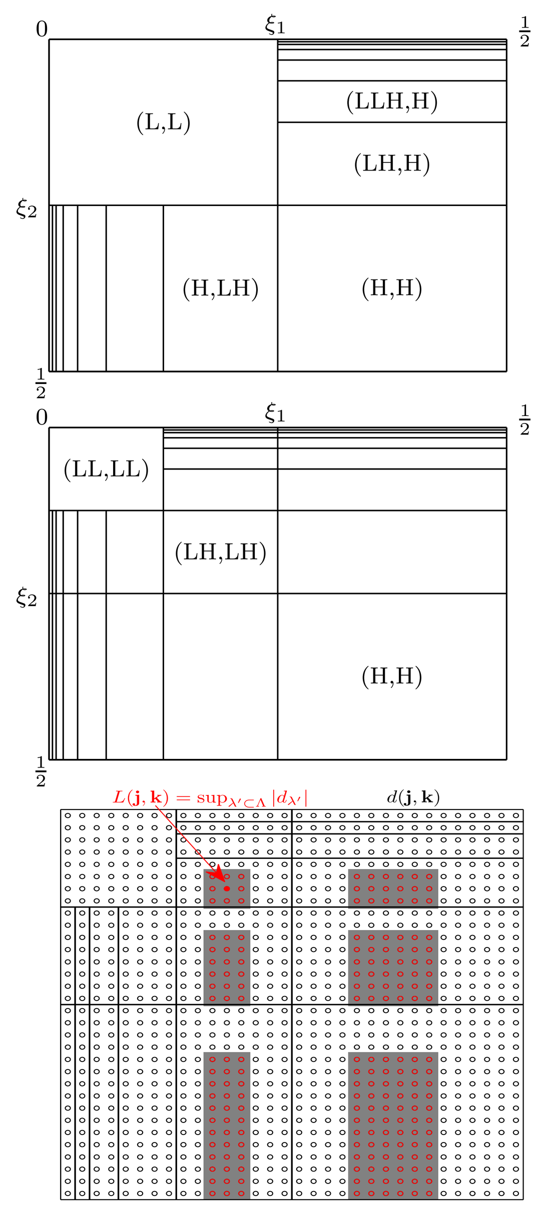
Hyperbolic Wavelet Transform: two independent dilation factors $\boldsymbol{j} = (j_1, j_2)$.

Image $X(x_1,x_2)$: $d_X(\boldsymbol{j},\mathbf{k}) = \langle X(x_1,x_2), \psi_{j_1,j_2,k_1,k_2}(x_1,x_2)\rangle$.

where $\psi_{j_1,j_2,k_1,k_2}(x_1,x_2) = 2^{-(j_1+j_2)/2}\psi_0(x_1 - 2^{j_1}k_1, x_2 - 2^{j_2}k_2)$

with $\psi_0(x_1,x_2) = \psi_0(x_1)\psi_0(x_2)$ the mother-wavelet (here D3).

Efficiently computed using alternate iterations of the classical 1D and 2D pyramidal algorithms of the DWT.



Hyperbolic wavelet leaders.

Let $\beta > 0$and let $9\lambda_{\boldsymbol{j}}(\mathbf{k})$ the hyperbolic dyadic rectangles

$$9\lambda_{\boldsymbol{j}}(\mathbf{k}) =]\frac{2^{j_1}k_1-1}{2^{j_1}}, \frac{2^{j_1}k_1+2}{2^{j_1}}]\times]\frac{2^{j_2}k_2-1}{2^{j_2}}, \frac{2^{j_2}k_2+2}{2^{j_2}}].$$

Hyperbolic wavelet leaders at scales $\boldsymbol{j}$ and location $\mathbf{k}$: local suprema of the HWT coefficients accros finer scales

$$L_X^{(\beta)}(\boldsymbol{j},\mathbf{k}) = \sup_{\boldsymbol{j}',\mathbf{k}'\subset 9\lambda_{\boldsymbol{j}}(\mathbf{k})} 2^{\frac{(j_1+j_2)\beta}{2}} |d_X(\boldsymbol{j}',\mathbf{k}')|.$$

When X has an anisotropic local regularity exponent $h_{\alpha_0}(\mathbf{x}_0) > 0$ at location $\mathbf{x}_0 = (x_{0,1}, x_{0,2})$, then for $\mathbf{k} = (2^{-j_1}x_{0,1}, 2^{-j_2}x_{0,2})$:

$$L_X^{(\beta)}(\boldsymbol{j},\mathbf{k}) \le 2^{(h_{\alpha_0}+\beta)(\mathbf{x}_0)\max(\frac{j_1}{\alpha_0},\frac{j_2}{(2-\alpha_0)})}, \quad (1)$$

Parameter β permits to ensure that the process X has enough regularity for the corner stone Eq. (1) to hold.

Anistropic multifractal spectrum estimation.

α analysis anisotropy angle $(0 < \alpha < 2)$: $\boldsymbol{j}_\alpha(j) = (\alpha j, (2-\alpha)j)$;

Conjecture : $\mathbb{E}(|L_X(\alpha j, (2-\alpha)j, \mathbf{k})|^q) \approx 2^{\frac{j}{\alpha}\left(q - (\tau_{\alpha_0}(q) + (1+\beta)q)\max(\frac{\alpha}{\alpha_0}, \frac{2-\alpha}{2-\alpha_0})\right)}$,

where $\tau_{\alpha_0}(q)$ Legendre transform of D_{α_0}: $\tau_{\alpha_0}(q) = \min_{h_{\alpha_0}}(qh_{\alpha_0} - D_{\alpha_0}(h_{\alpha_0}))$.

$$C_1(\boldsymbol{j}_\alpha) = \frac{1}{n_{\boldsymbol{j}_\alpha}} \sum_{\mathbf{k}\in\mathbb{Z}^2} \left[\log L_X^{(\beta)}(\boldsymbol{j}_\alpha,\mathbf{k})\right], \qquad C_1(\alpha j, (2-\alpha)j) \sim c_1(\alpha)\ln 2^j$$

$$C_2(\boldsymbol{j}_\alpha) = \frac{1}{n_{\boldsymbol{j}_\alpha}} \sum_{\mathbf{k}\in\mathbb{Z}^2} \left[\log L_X^{(\beta)}(\boldsymbol{j}_\alpha,\mathbf{k})\right]^2 - [C_1(\boldsymbol{j}_\alpha)]^2, \quad \Longrightarrow \quad C_2(\alpha j, (2-\alpha)j) \sim -c_2(\alpha)\ln 2^j$$

where $n_{\boldsymbol{j}_\alpha}$ denotes the number of hyperbolic wavelet leaders available at scales $\boldsymbol{j}_\alpha$.

$\hat{c}_p(\alpha)$ obtained by linear regressions of $C_p(\alpha j, (2-\alpha)j)$ versus j,
then $(\alpha_0, \hat{c}_1, \hat{c}_2)$ obtained as $\hat{\alpha}_{c_p} = \mathrm{argmax}_\alpha\, c_p(\alpha)$, $\hat{c}_p = c_p(\hat{\alpha}_{c_p})$

The use of the analysis angle $\alpha = 1$ amounts to recovering the classical 2D-DWT leaders.

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