EXTREME VALUES, HEAVY TAILS AND LINEARIZATION EFFECT: A CONTRIBUTION TO EMPIRICAL MULTIFRACTAL ANALYSIS

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Multifractal processes (e.g. Compound Poisson Motion):
Ensemble and **sample moments** of increments do not always coincide.
This can be explained through **extreme values**, **heavy tail** marginal distributions and **dependence** structure of multifractal processes.

$$\mathbb{E}\,|A(t+a)-A(t)|^q \sim a^{\lambda(q)}$$
$$\frac{1}{n_a}\sum_{k=1}^{n_a}|A(a(k+1))-A(ak)|^q \sim a^{\zeta(q)}$$
$$\lambda(q)\neq\zeta(q)$$

Compound Poisson Motion

- Compound Poisson Cascade (CPC): $Q_r(t)=C\prod_{(t_i,r_i)\in\mathcal{C}_r(t)}W_i,\quad r>0$
  - (t_i,r_i) random points of Poisson measure
  - W_i positive iid multipliers associated with (t_i,r_i)
- Compound Poisson Motion (CPM): $A(t)=\lim_{r\to 0}\int_0^t Q_r(s)ds$
- Increments $T_A(a,t)=A(t+a)-A(t)$:

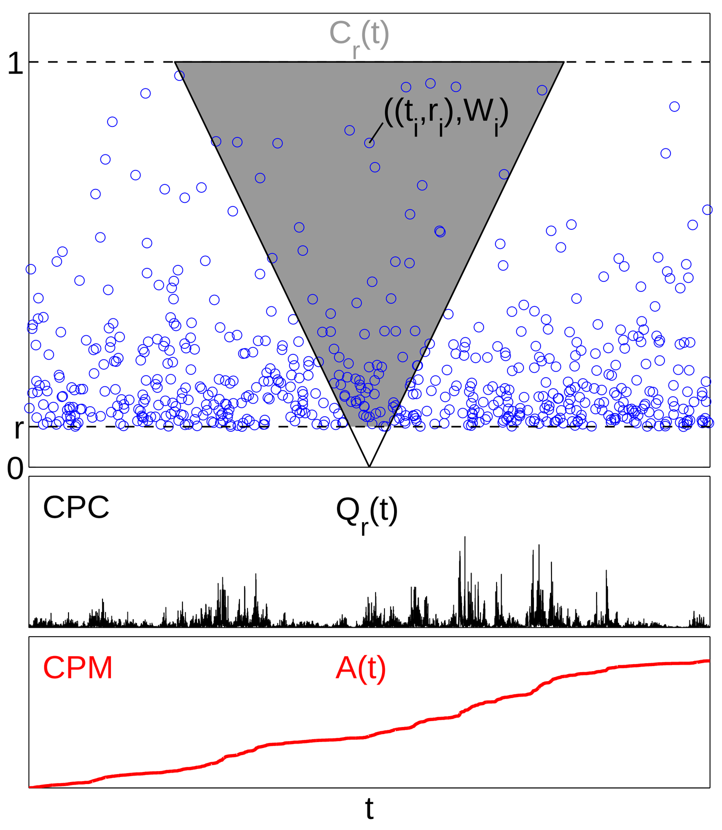
  $\to$ stationary

  $\to$ finite moments for $0<q<q_c^+$:

  $\mathbb{E}|T_A(a,t)|^q\sim a^{\lambda(q)}$

  $q_c^+=\ \sup\{q\geq 1,\lambda(q)-1\geq 0\}$

  $\lambda(q)=q+c((1-\mathbb{E}W^q)-q(1-EW))$

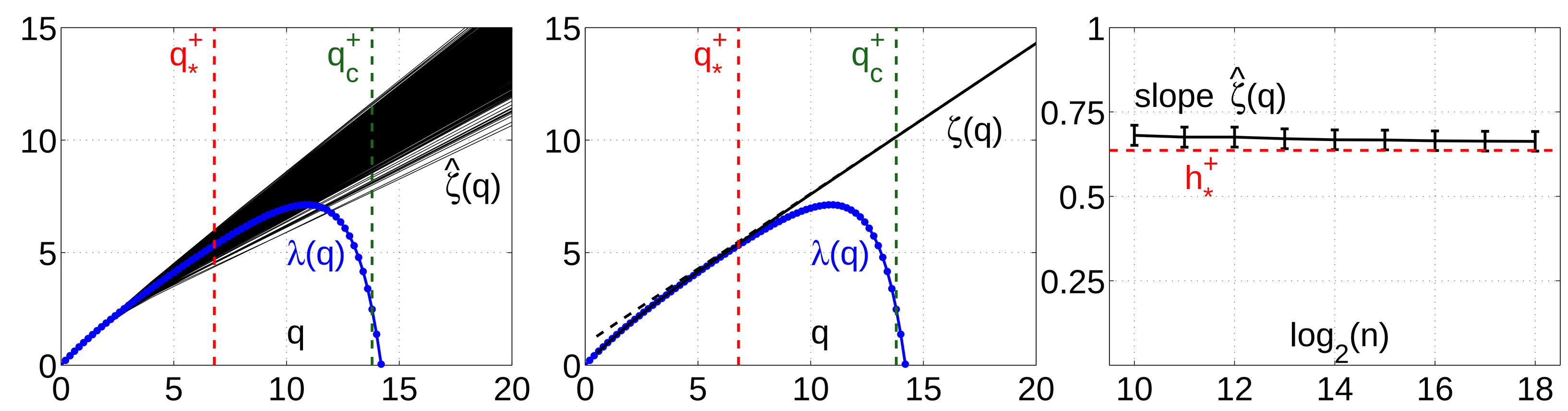


- Multifractal Properties: $|T_A(a,t)|\simeq c|a|^h,\ a\to 0$

$$D_A(h)=\begin{cases}D_\lambda(h), & \text{if } D_\lambda(h)\geq 0,\\ -\infty, & \text{otherwise.}\end{cases}\qquad D_\lambda(h)=\min_{q\neq 0}(1+qh-\lambda(q))$$

Linearization Effect

- Estimation:
  - $S_n(q,a)=\frac{1}{n_a}\sum_{k=1}^{n_a}|T_X(a,ak)|^q$
  - $\widehat{\zeta}(q)=\sum w_j\log_2 S_n(q,2^j)$



- Observation:

$$\zeta(q)=\langle\widehat{\zeta}(q)\rangle_R=\begin{cases}\lambda(q), & \text{if } q\leq q_*^+,\\ 1+qh_*^+, & \text{if } q>q_*^+\end{cases}$$

  - $h_*^+=\ \min{}_h\{D_A(h)=0\}$
  - $q_*^+=(dD_A/dh)_{h=h_*^+}$
  - $q_*^+\leq q_c^+$

All results shown here:

Log-Normal W_i

$(\mu=-0.4,\sigma^2=0.08)$

$h_*^+\approx 0.64$

$q_*^+\approx 6.8$

$q_c^+\approx 13.8$

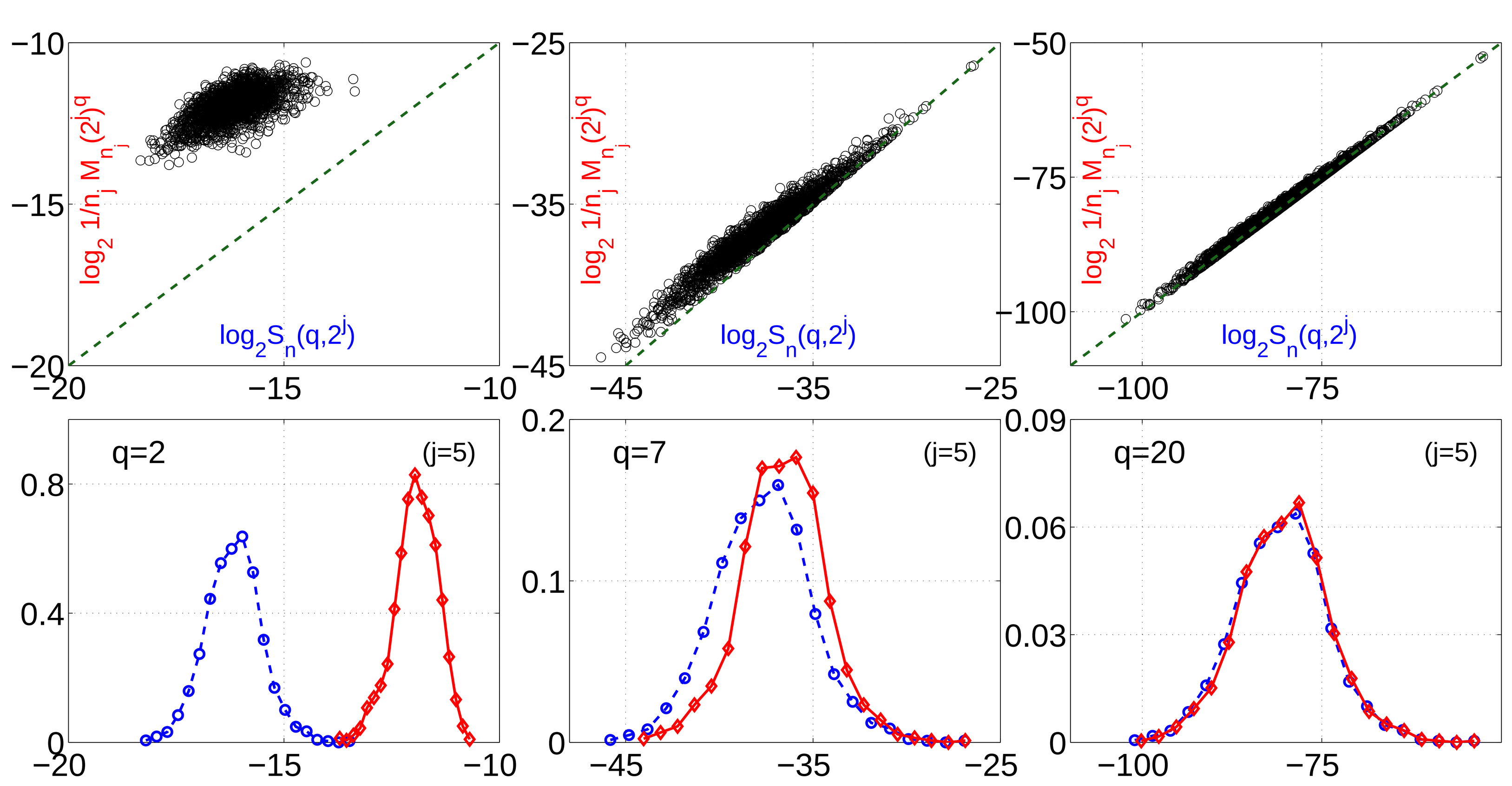
Extreme Values and Heavy Tails

Extreme Values

- Maxima of increments:
  - $M_{n_j}(2^j)=\max\{|T_A(2^j,2^jk)|,\ k=1,\dots,n_j\}$
- Theory:
  - $q\to+\infty:\ S_n(q,2^j)\simeq\frac{1}{n_j}M_{n_j}(2^j)^q$
  - Independence: $\frac{1}{n_j}M_{n_j}(2^j)^q\to S_n(q,2^j)$ for $q>q_c^+$
- Observation:

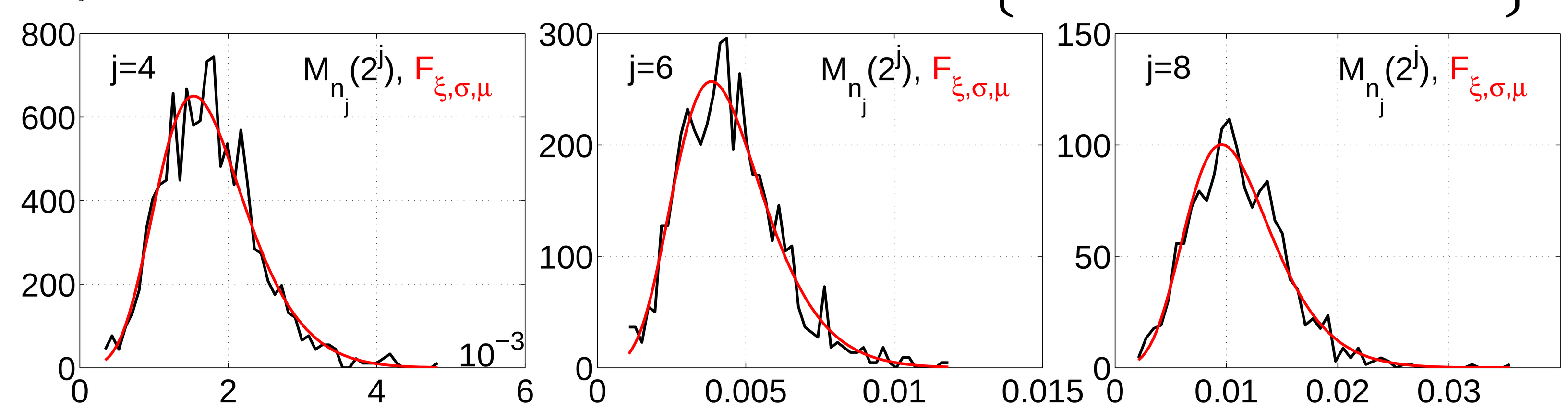
  - for $q>q_*^+$: $\frac{1}{n_j}M_{n_j}(2^j)^q\not\to S_n(q,2^j)$

Suggests: moments as if infinite for $q>q_*^+$ and NOT for $q>q_c^+$

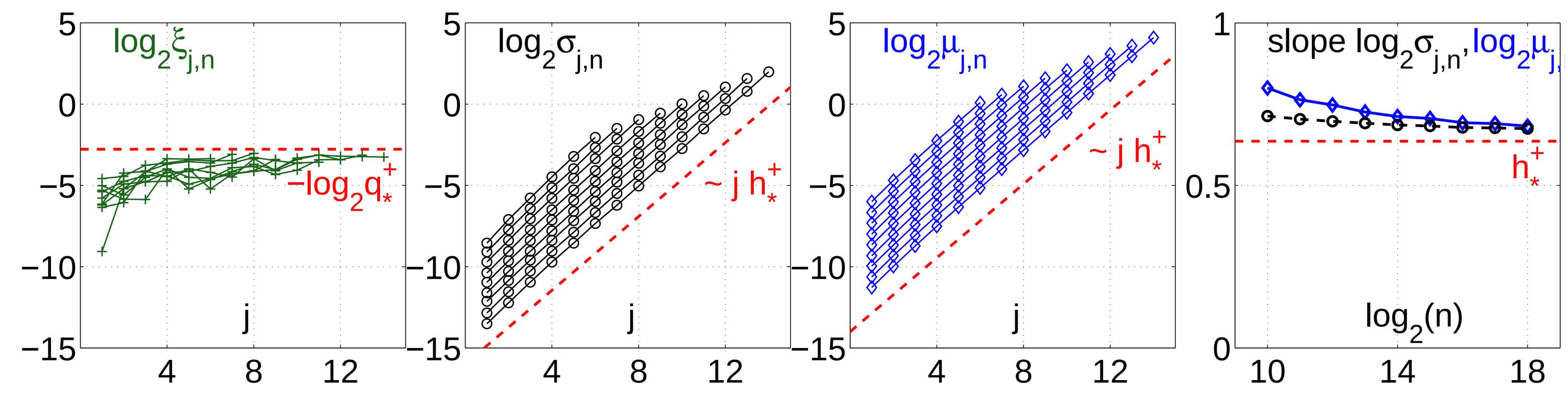


Generalized Extreme Value Fits

- $M_{n_j}\to$ GEV distribution: $F_{\xi,\sigma,\mu}(x)=\exp-\left\{(1+\xi((x-\mu)/\sigma)^{-1/\xi}\right\}$



- Extreme value fits:
  - $\xi_{j,n}\simeq\xi_0\simeq 1/q_*^+>1/q_c^+$
  - $\sigma_{j,n}\simeq\sigma_{0,n}2^{jh_*^+},\ \mu_{j,n}\simeq\mu_{0,n}2^{jh_*^+}$



$$\{M_{n_j}(2^j)\}_{j=j_1,\dots,j_2}\overset{d}{\simeq}\{2^{jh_*^+}(\sigma_{0,n}\Lambda_{\xi_0}^j+\mu_{0,n})\}_{j=j_1,\dots,j_2},\quad\Lambda_{\xi_0}^j\sim F_{\xi_0,1,0}$$

- Linearization Effect:
  - $\widehat{\zeta}(q)\simeq 1+q\left(h_*^++\sum w_j\log_2(\sigma_{0,n}\Lambda_{\xi_0}^j+\mu_{0,n})\right)$
  - $\langle\zeta(q)\rangle_R\simeq 1+qh_*^+,\qquad 1+q_*^+(d\lambda/dq)_{q=q_*^+}-\lambda(q_*^+)=0$

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Conclusions and Perspectives

- Moments of $|T_X(a,t)|$ as if infinite for $q>q_*^+$ and not for $q>q_c^+$
- $M_{n_j}\sim 2^{jh_*^+}\to$ coherent with multifractal analysis
- Heavy tails and dependence structure $\to$ Linearization effect
- Extension to wavelet Leaders
- Extension to other multifractal processes