TESTING MONO-VS. MULTIFRACTAL WITH BOOTSTRAPPED WAVELET LEADERS

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Summary

In many applications where data possess **scaling** properties, it is of importance to decide whether the data are better modelled with **mono- or multifractal** processes. However, so far no appropriate test is available. For this purpose, we propose here to use a **bootstrap test procedure** to decide whether the second cumulant of the log of the wavelet coefficients or wavelet Leaders of the data is zero. We study the p-value and the power of the tests through numerical simulation, using synthetic multifractal processes, and end up with a **powerful procedure for practically discriminating mono- vs. multifractal processes**.

MULTIFRACTAL, LEADERS & CUMULANTS

Wavelet Leaders

$X(t),\ t \in [0, n)$ - Process under analysis

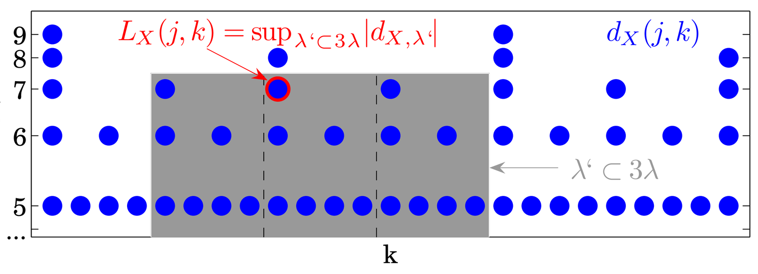
$d_X(j,k) = \langle \psi_{j,k} | X \rangle$ - Discrete wavelet transform (DWT) of X

$$L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{X,\lambda'}|$$

Index $\lambda_{j,k} = [k2^j, (k+1)2^j)$

$3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$

The supremum is taken on the $d_X(\cdot,\cdot)$ in the time neighborhood $3\lambda_{j,k}$ over all finer scales $2^{j'} < 2^j$ (cf. [1]).



Scaling and Multifractal

$$\frac{1}{n_j} \sum_{k=1}^{n_j} |L_X(j,k)|^q = F_q |2^j|^{\zeta(q)}$$

Statistical orders $q \in [q_*^-, q_*^+]$

$a = 2^j \in [a_m, a_M]$, $\frac{a_M}{a_m} >> 1$

$\zeta(q)$ - Scaling exponents of X

$\zeta(q) = qH$ - monofractal $\qquad \zeta(q) \neq qH$ - multifractal

$\zeta(q)$ closely related to multifractal spectrum of X [1].

Log-Cumulants

$$\mathbb{E}|L_X(j,\cdot)|^q = \qquad = F_q |2^j|^{\zeta(q)}$$

$$\ln \mathbb{E} e^{q \ln |L_X(j,\cdot)|} = \sum_{p=1}^{\infty} C_p^j \frac{q^p}{p!} = \ln F_q + \zeta(q) \ln 2^j$$

C_p^j - *cumulant of* $\ln |L_X(j,\cdot)|$ *of order* $p \geq 1$

Combining the two equations yields:

$$\forall p \geq 1 : \qquad C_p^j = c_p^0 + c_p \ln 2^j$$

$$\ln \mathbb{E} e^{q \ln |d_X(j,\cdot)|} = \sum_{p=1}^{\infty} C_p^j \frac{q^p}{p!} = \underbrace{\sum_{p=1}^{\infty} c_p^0 \frac{q^p}{p!}}_{\ln F_q} + \underbrace{\sum_{p=1}^{\infty} c_p \frac{q^p}{p!}}_{\zeta(q)} \ln 2^j$$

- Therefore: $\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$
- Measurements of $\zeta(q)$ replaced by those of the log-cumulants c_p.
- Estimates $\hat{c}_p$ of log-cumulants c_p: linear regression of $\hat{C}_p^j$ vs. j.
- Emphasizes difference monofractal - multifractal processes:

  monofractal: $\forall p \geq 2 :\ c_p \equiv 0$; $\qquad$ multifractal: $\exists p \geq 2 :\ c_p \neq 0$
- Empirically: Mostly $c_2 \neq 0 \Longrightarrow$ multifractal

BOOTSTRAP HYPOTHESIS TEST

Null Hypothesis: $H_{\text{null}} : c_{2,\text{null}} \equiv 0$

Test Statistic: $T = c_2 - c_{2,\text{null}}$

Null Distribution: $F_{\text{null}} = \Pr(T \leq t | H_{\text{null}})$ (quantiles: t_α)

$(1-2\alpha)$ Test: $\quad d = \begin{cases} 1 & \text{if } t \notin [t_\alpha, t_{1-\alpha}] \\ 0 & \text{otherwise} \end{cases}$

Nonparametric Bootstrap: Estimate F_{null} from **single sample** X.

1. At each scale j, draw B bootstrap resamples blockwise, with replacement, from sample of original Leaders (Block length: Λ):

$$\{L_X(j,1), \cdots, L_X(j,n_j)\} \xrightarrow[\text{with replacement}]{\text{Repeat } B \text{ times: draw blockwise}} \{L_X^{*(1)}(j,\cdot), \cdots, L_X^{*(n_j)}(j,\cdot)\}$$

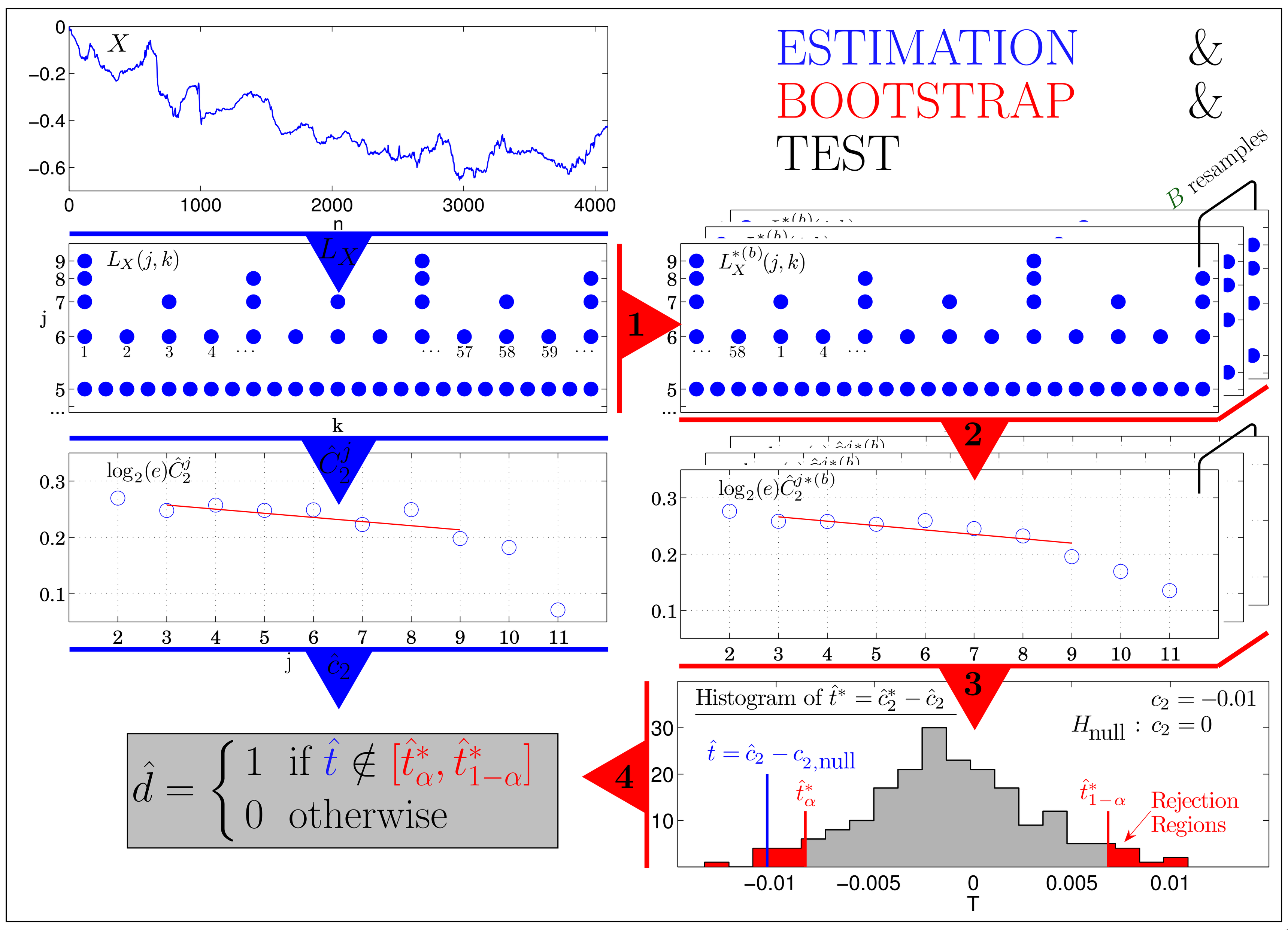
2. Calculate, for each of the B resamples, and for each scale j:

$$\hat{C}_2^{j*(b)} \xrightarrow{\text{linear fit}} \hat{c}_2^{*(b)}, \quad \hat{t}^{*(b)} = \hat{c}_2^{*(b)} - c_{2,\text{null}}, \quad b = 1, \cdots, B$$

3. Calculate approximate null distribution:

$$\hat{F}_{\text{null}}(\tau) = \frac{1}{B} \sum_{b=1}^{B} I(t^{*(b)} \leq \tau)$$

4. Perform test using the approximate α-quantiles $\hat{t}_{1-\alpha}^*$, $\hat{t}_\alpha^*$ of $\hat{F}_{\text{null}}(\tau)$:



MONTE CARLO SIMULATION AND RESULTS

Fractional Brownian Motion (FBM): Gaussian mono-fractal

$\zeta(q) = qH$ for $q \in (-\infty, \infty)$

Multifractal Random Walk (MRW): non Gaussian multi-fractal

$\zeta(q) = (H - c_2)q + c_2 q^2/2$ for $q \in \left[-\sqrt{2/|c_2|}, \sqrt{2/|c_2|}\right]$

$H_{\text{FBM}} = 0.8$	$N_{\text{MC}} = 1000$	Daubechies Wavelet $L = 6$
$H_{\text{MRW}} = \{0.79, \cdots, 0.72\}$	$n_1 = 2^{12}$	Bootstrap $B = 200$
$-c_2 = \{0.01, \cdots, 0.08\}$	$n_2 = 2^{15}$	$\Lambda = 6$

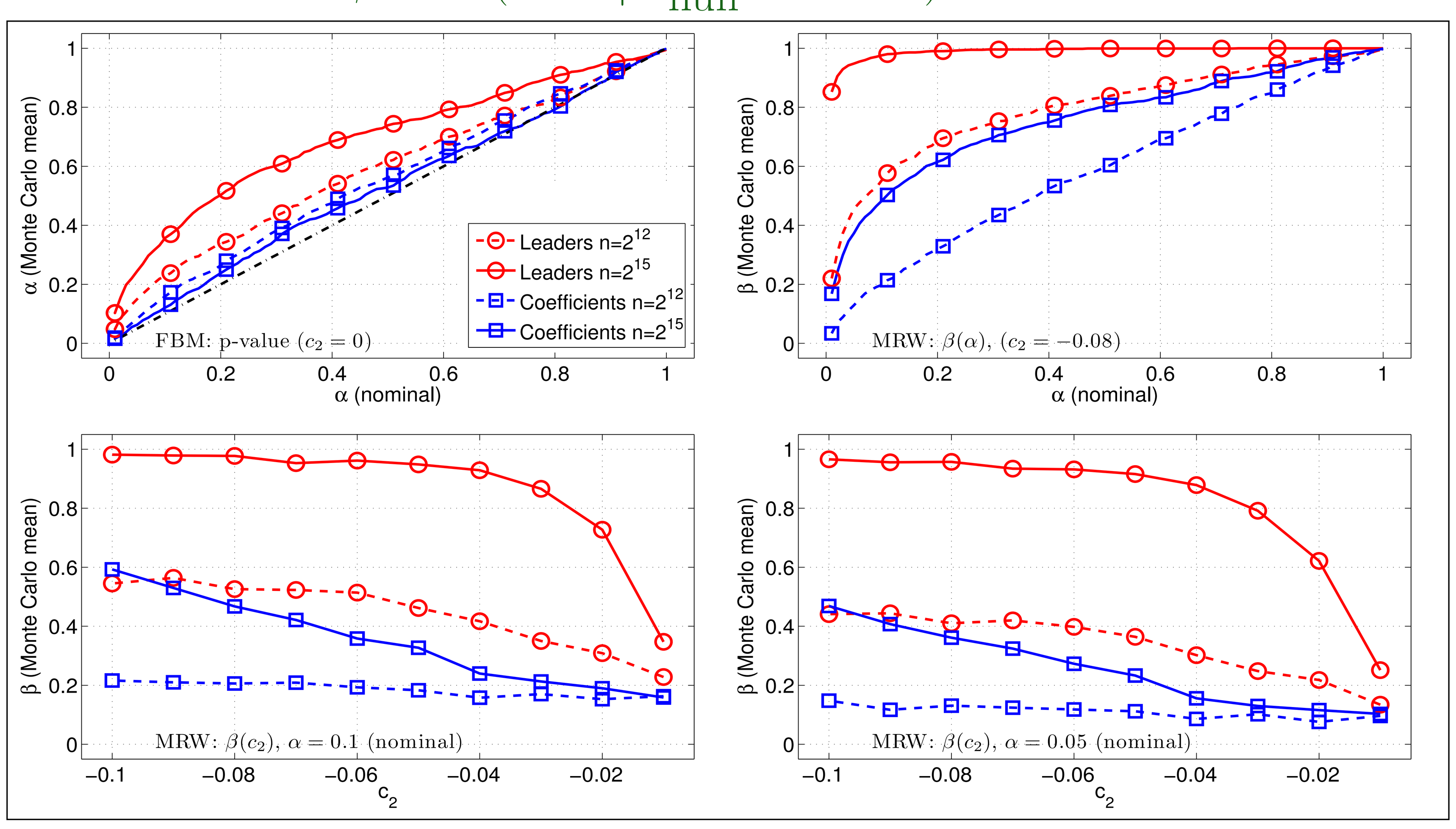
- Tests based on Coefficients d_X:
  - Significance α closer to nominal value
- Tests based on Leaders L_X:
  - Significantly larger power β
  - Maintain large power for c_2 close to zero

$\Rightarrow$ Powerful and reliable test of monofractal vs. multifractal

- Perspectives: Improved test statistics, advanced bootstrap tests (e.g. pivoting), parametric bootstrap, and tests on c_3.

Significance: $\alpha = \Pr(d = 1 | H_{\text{null}} \text{ true })$

Power: $\qquad \beta = \Pr(d = 1 | H_{\text{null}} \text{ not true })$



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[1] S. Jaffard, B. Lashermes and P. Abry, *Wavelet leaders in multifractal analysis*, in *Wavelet Analysis and Applications*, 2005, University of Macau, China.

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