A Bayesian estimator for the multifractal analysis of multivariate data

H. Wendt[1]

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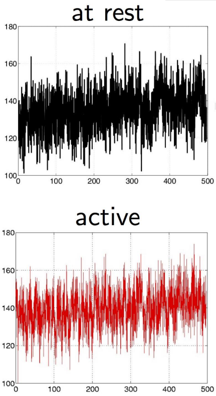
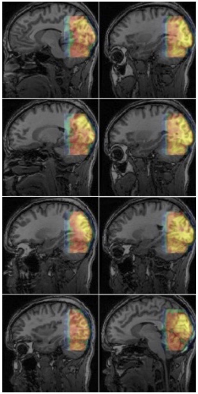
GdR ISIS, 8 Feb. 2018

CNRS, IRIT, Heriot Watt University, ENS de Lyon

Empirical data: signals / images

Macroscopic Brain activity

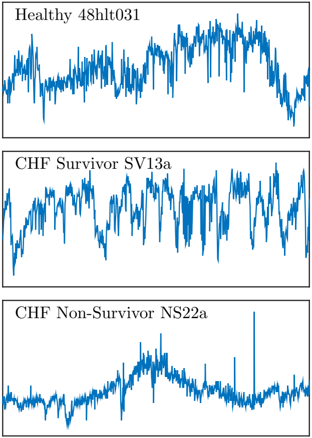




Collab. P. Ciuciu (CEA)

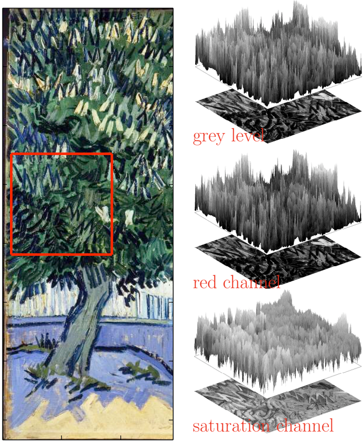
Empirical data: signals / images

Human Heart Rate



Collab. Y. Yamamoto (U Tokyo) and K. Kiyono (U. Osaka)

Van Gogh's Painting



Collab. Van Gogh Museum, Amsterdam

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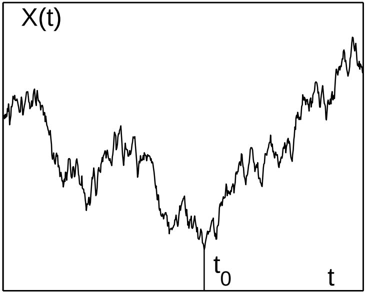
Multifractal spectrum

- Local regularity of $X(t)$ at t_0

  Hölder exponent

  $h(t_0) \triangleq \sup_\alpha \{\alpha : \ |X(t) - X(t_0)| < C|t - t_0|^\alpha\} \qquad 0 < \alpha$

  $h(t_0) \to 1 \Rightarrow$ smooth, very regular,
  $h(t_0) \to 0 \Rightarrow$ rough, very irregular



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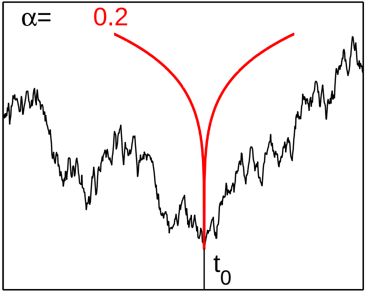

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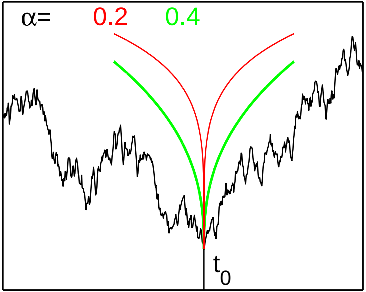

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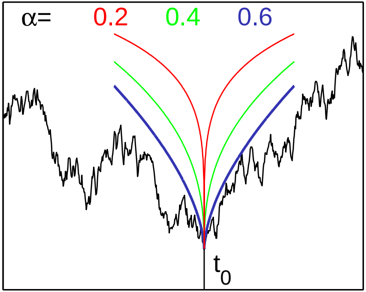

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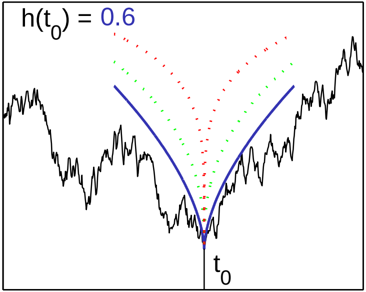

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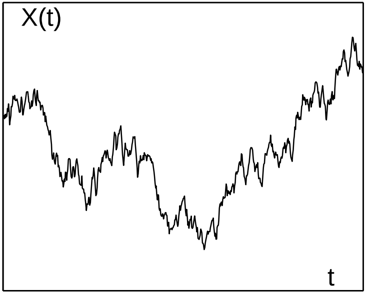
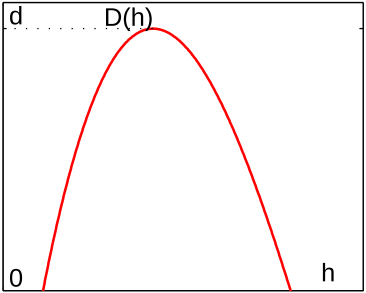
- Multifractal Spectrum $\mathcal{D}(h)$: Fluctuations of regularity $h(t)$
  - Set of points that share same regularity $\{t_i | h(t_i) = h\}$
  - Fractal (or Haussdorf) Dimension of each set:

$$\mathcal{D}(h) \triangleq \dim_H \{t : \ h(t) = h\}$$





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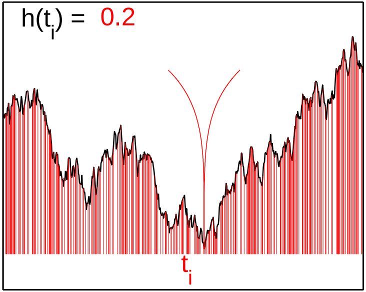

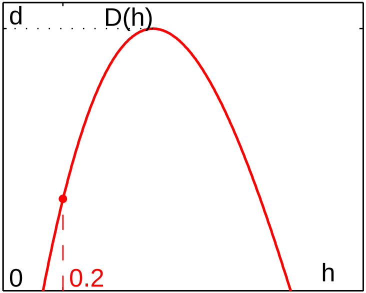

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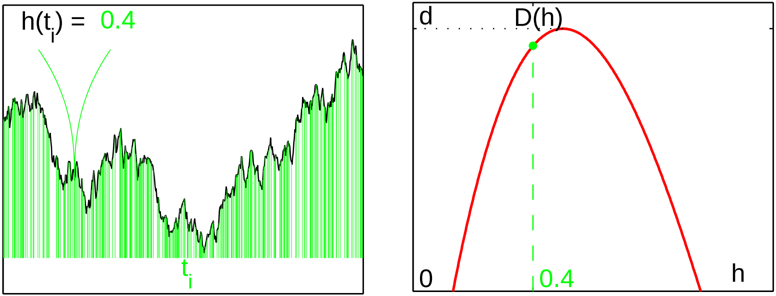

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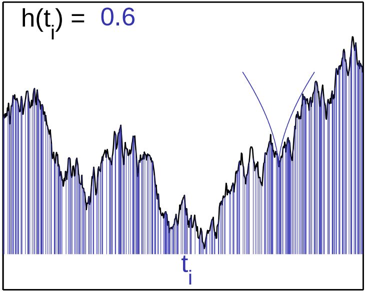

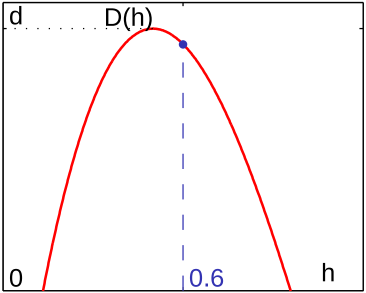

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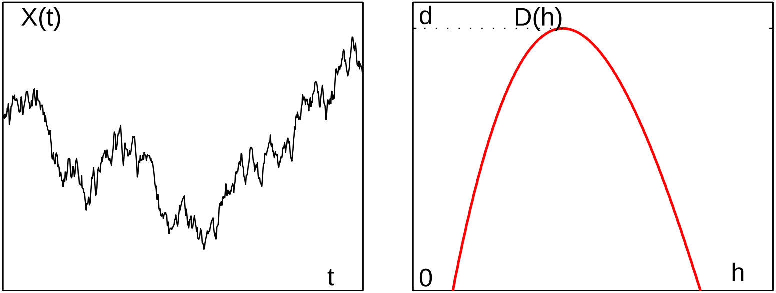
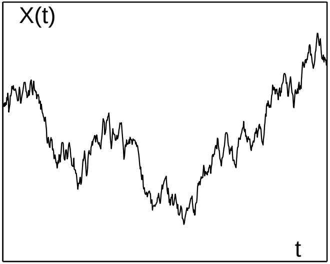
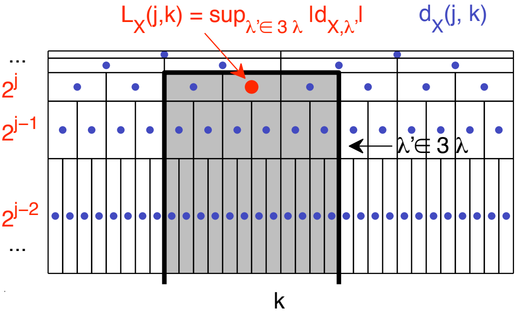


Multifractal formalism

- $D(h)$ in practice $\rightarrow$ multifractal formalism [Parisi85]
- Multiresolution quantities: wavelet leaders $\{\ell(j\cdot,\cdot)\}$ [Jaffard04]

$$\ell(j,k) \triangleq \sup_{\lambda' \subset 3\lambda_{j,k}} |d(\lambda')|, \qquad d(j,k): \text{ DWT coefficient}$$





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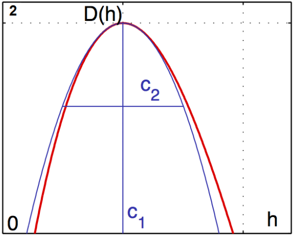
- Polynomial expansion [Castaing93]

$$D(h) \approx 1 + \frac{c_2}{2!}\left(\frac{h-c_1}{c_2}\right)^2 - \frac{c_3}{3!}\left(\frac{h-c_1}{c_2}\right)^3 + \dots$$

$\rightarrow$ c_p tied to cumulants of $l(j,k) \triangleq \ln \ell(j,k)\}$



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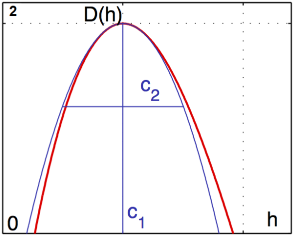
- Multifractality parameter c_2
  - $\sim$ fluctuations of regularity
  - tied to the variance of log-leaders

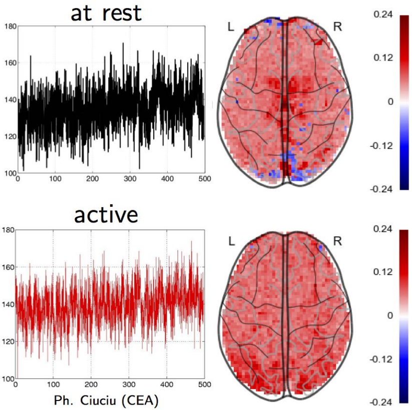
$$\boxed{\mathrm{Var}\,[\,\ln \ell(j,\cdot)\,] = c_2^0 + c_2 \ln 2^j}$$

self-similar processes $\rightarrow$ $c_2 = 0$
multifractal multiplicative cascades $\rightarrow$ $c_2 < 0$



Estimation of the multifractality parameter

- Estimation of c_2 is challenging
  - linear regression-based estimation [Castaing93]
    - X *poor estimation performance* $\longrightarrow$ *need (very) long time series*



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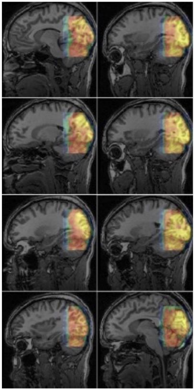

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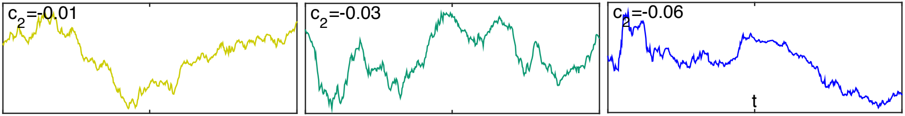
1. Bayesian estimation for c_2 for single time series [TIP15,ICASSP16]
    - robust semiparametric model for log-leaders

Synthetic multiplicative cascades with different c_2



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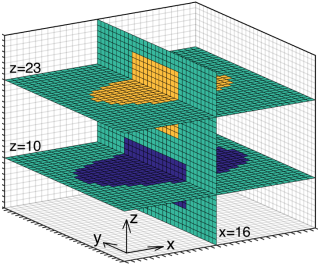
1. Bayesian estimation for c_2 for single time series [TIP15,ICASSP16]
    - robust semiparametric model for log-leaders
2. Bayesian estimation for c_2 for multivariate data [IWSSIP16,EUSIPCO16,ICIP16,HW18]
    - regularization using Markov field joint prior



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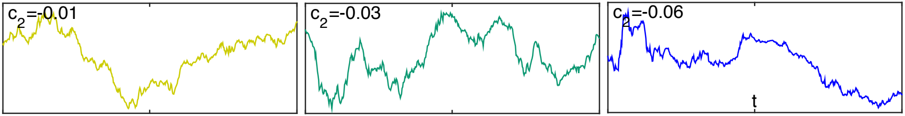
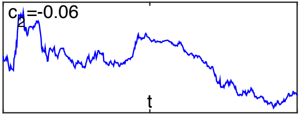


Bayesian model for single time series



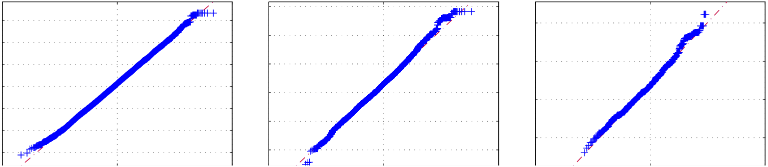
Gaussian random field model for log-leaders

- Marginal distributions: log-Normal always good fit for multiscale histograms of multifractal cascades [Mandelbrot90]

$\longrightarrow$ log-leaders well approximated by Gaussian [ICASSP13,TIP15]

$$l(j,k) = \ln \ell(j,k) \sim \mathcal{N}(\mathbb{E}[l(j,k)], \mathrm{Var}\,[l(j,k)])$$

log-leaders $l(2,\cdot)$ log-leaders $l(3,\cdot)$ log-leaders $l(4,\cdot)$



empirical marginals (qq-plots)

Multifractal random walk (MRW) [Bacry01,Robert10]

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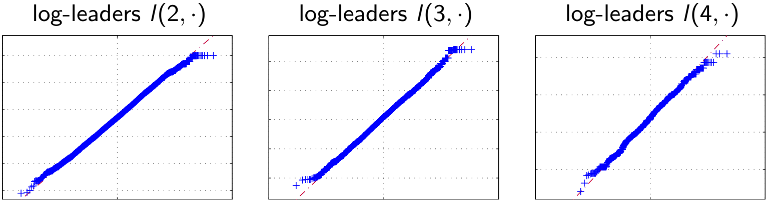


empirical marginals (qq-plots)

Log-Poisson Cascade [Mandelbrot]

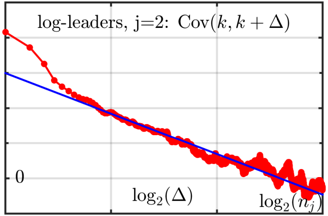
Gaussian random field model for log-leaders

- Mean

$$\boxed{\mathbb{E}[l(j,k)] = c_1^0 + jc_1 \ln 2}$$

(discarded below)

- Variance-covariance
  - asymptotic covariance decay: [Arneodo98]
    → linear in $\log(\Delta k)$
    → controlled by c_2



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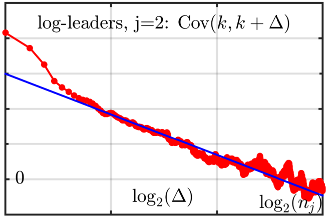
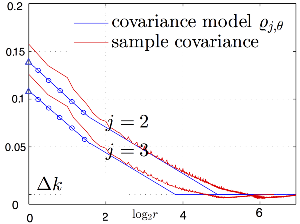




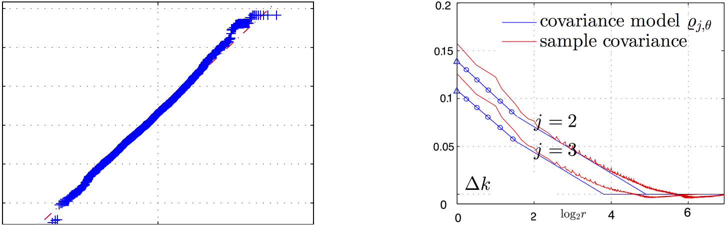
$\rightarrow$ piecewise logarithmic model $\varrho_{j,(c_2,c_2^0)}(\Delta k)$ [ICASSP15,TIP15]

with parameters (c_2, c_2^0) $\longrightarrow$ Covariance matrix $\Sigma_{j,(c_2,c_2^0)}$

From a standard likelihood...

- Likelihood w.r.t. (c_2, c_2^0)
  - log-leaders at scale j: $\boldsymbol{l}_j \triangleq (l(j,1), l(j,2), \dots)$

$$p(\boldsymbol{l}_j | (c_2, c_2^0)) \propto (\det \Sigma_{j,(c_2,c_2^0)})^{-\frac{1}{2}} \exp\left(-(\boldsymbol{l}_j^T \Sigma_{j,(c_2,c_2^0)}^{-1} \boldsymbol{l}_j)/2\right)$$



empirical marginals (qq-plot) and covariance

- ✗ inversion of $\Sigma_{j,(c_2,c_2^0)}$ prohibitive $\rightarrow$ Whittle approximation
- ✗ constraints: $\Sigma_{j,(c_2,c_2^0)}$ p.d. $\rightarrow$ reparametrization
- ✗ conjugacy of priors for θ $\rightarrow$ data augmentation

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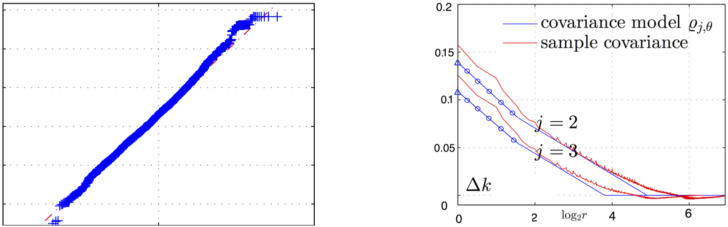

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...to a Data Augmented Likelihood

[TIP15,ICASSP16]

1. Whittle approximation $\Longrightarrow$ Fourier transform (DFT) of centered log-leaders $\boldsymbol{l}_j$

$$\boldsymbol{y}_j = DFT(\boldsymbol{l}_j) \quad \longrightarrow \quad p(\boldsymbol{l}_j | (c_2, c_2^0)) \propto \boldsymbol{f}_{j,(c_2,c_2^0)}^{-1} \exp\left(\frac{\boldsymbol{y}_j^* \boldsymbol{y}_j}{\boldsymbol{f}_{j,(c_2,c_2^0)}}\right)$$

$$\longrightarrow \quad \approx \textit{diagonal covariance } \boldsymbol{F}_{(c_2,c_2^0)}$$

$p(\boldsymbol{l}_j | (c_2, c_2^0))$

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2. Reparametrization $\Longrightarrow$ independent positivity constraints on parameters

$$\boldsymbol{v} = \psi((c_2, c_2^0)) \in \mathbb{R}_\star^{+2} \quad \longrightarrow \quad \textit{separable } \boldsymbol{F}_{(c_2,c_2^0)} = v_1 \boldsymbol{F}_1 + v_2 \boldsymbol{F}_2$$

$\boldsymbol{F}_1$, $\boldsymbol{F}_2$ diagonal, positive definite, known and fixed

---

$$p(\boldsymbol{l}_j|(c_2, c_2^0)) \quad \longrightarrow \quad p(\boldsymbol{l}_j|\boldsymbol{v})$$

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$$\boldsymbol{v} = \psi((c_2, c_2^0)) \in \mathbb{R}_\star^{+2} \quad \longrightarrow \quad \textit{separable } \boldsymbol{F}_{(c_2,c_2^0)} = v_1 \boldsymbol{F}_1 + v_2 \boldsymbol{F}_2$$

$\boldsymbol{F}_1$, $\boldsymbol{F}_2$ diagonal, positive definite, known and fixed

3. Data augmentation $\Longrightarrow$ hidden mean $\boldsymbol{\mu}_j$ for $\boldsymbol{y}_j$

$\Longrightarrow$ complex Gaussian model for $\boldsymbol{y} = [\boldsymbol{y}_{j_1}^T, ..., \boldsymbol{y}_{j_2}^T]^T$

$$\begin{cases} \boldsymbol{y}|\boldsymbol{\mu}, v_2 & \sim \mathcal{CN}(\boldsymbol{\mu}, v_2 \boldsymbol{F}_2) \quad \text{observed data} \\ \boldsymbol{\mu}|v_1 & \sim \mathcal{CN}(\boldsymbol{0}, v_1 \boldsymbol{F}_1) \quad \text{hidden mean} \end{cases}$$

$$p(\boldsymbol{l}_j|(c_2, c_2^0)) \quad \longrightarrow \quad p(\boldsymbol{l}_j|\boldsymbol{v}) \quad \longrightarrow \quad p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v}) \propto p(\boldsymbol{y}|\boldsymbol{\mu}, v_2)\, p(\boldsymbol{\mu}|v_1)$$

Augmented likelihood based Bayesian model [ICASSP16]

- Augmented likelihood w.r.t. $\boldsymbol{v} = \psi((c_2, c_2^0))$

$$p(\boldsymbol{y}, \boldsymbol{\mu} | \boldsymbol{v}) \propto {v_2}^{-N_Y} \exp\left(-\frac{1}{v_2}(\boldsymbol{y}-\boldsymbol{\mu})^H \boldsymbol{F}_2^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \times {v_1}^{-N_Y} \exp\left(-\frac{1}{v_1}\boldsymbol{\mu}^H \boldsymbol{F}_1^{-1}\boldsymbol{\mu}\right)$$

- Prior distribution for parameters
  v_i as variance of Gaussian $\rightarrow$ conjugate inverse-gamma prior $\mathcal{IG}(\alpha_i, \beta_i)$

- Posterior distribution

$$p(\boldsymbol{v}, \boldsymbol{\mu} | \boldsymbol{y}) \propto p(\boldsymbol{y}, \boldsymbol{\mu} | \boldsymbol{v}) p(v_1) p(v_2)$$

- Bayesian estimators
  $\rightarrow$ marginal posterior mean estimator (MMSE) $\boldsymbol{v}^{\text{MMSE}} = \mathbb{E}[\boldsymbol{v}|\boldsymbol{y}]$

- Gibbs sampler
  $p(\boldsymbol{\mu}|\boldsymbol{v}, \boldsymbol{y})$ closed-form Gaussian distribution
  $p(\boldsymbol{v}_i|\boldsymbol{v}_{j\neq i}, \boldsymbol{\mu}, \boldsymbol{y})$ closed-form inverse-gamma distributions

  all standard distributions $\rightarrow$ no Metropolis-Hasting moves

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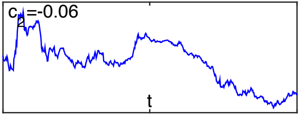
  all standard distributions → no Metropolis-Hasting moves

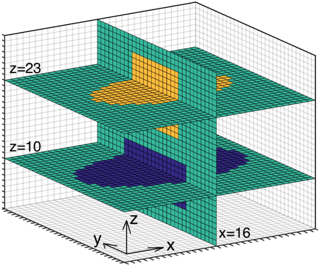
Estimation: Markov Chain Monte Carlo Algorithm

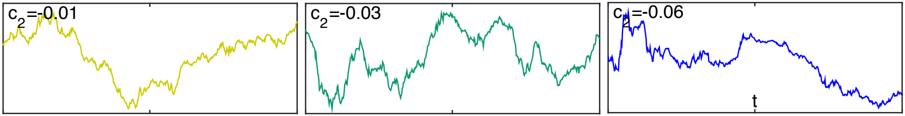
- Performance for synthetic data *(further details later)*
  - $N = 512$, $c_2 = -0.01, \ldots, -0.08$
  - estimation performance improved by factor up to ~ 4
  - about 5 to 2 times slower than linear regression

	LF	IG
$\lvert b \rvert$	0.0158	0.0051
std	0.0800	0.0255
rmse	0.0819	0.0262



Bayesian model for multivariate time series

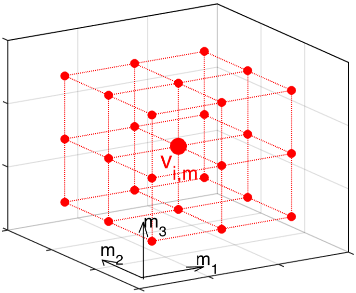


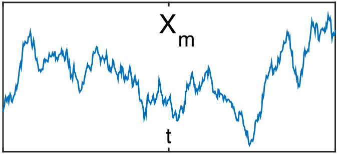


Hierarchical Bayesian model

for volumetric time series (voxels), $X_{\boldsymbol{m}}$, $\boldsymbol{m} \triangleq (m_1, m_2, m_3)$, of length N
(other data structures possible)

1. Augmented likelihood $p(\boldsymbol{y}_{\boldsymbol{m}}, \boldsymbol{\mu}_{\boldsymbol{m}} | \boldsymbol{v}_{\boldsymbol{m}})$
   - $\boldsymbol{y}_{\boldsymbol{m}}$: Fourier coefficients of log-leaders of $X_{\boldsymbol{m}}$
   - $\boldsymbol{\mu}_{\boldsymbol{m}}$: latent variables
   - $\boldsymbol{v}_{\boldsymbol{m}}$: parameter vector





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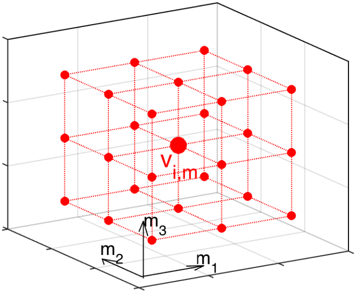
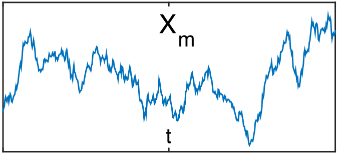


2. Prior independence between voxels

$$p(\boldsymbol{Y}, \boldsymbol{M} | \boldsymbol{V}) \propto \prod_{\boldsymbol{m}} p(\boldsymbol{y_m}, \boldsymbol{\mu_m} | \boldsymbol{v_m})$$

- $\boldsymbol{Y} \triangleq \{\boldsymbol{y_m}\}$
- $\boldsymbol{M} \triangleq \{\boldsymbol{\mu_m}\}$
- $\boldsymbol{V} \triangleq \{\boldsymbol{V}_1, \boldsymbol{V}_2\}$ ($\boldsymbol{V}_i \triangleq \{\theta_{i,\boldsymbol{m}}\}$, $i = 1, 2$)



Hierarchical Bayesian model

for volumetric time series (voxels), $X_{\boldsymbol{m}}$, $\boldsymbol{m} \triangleq (m_1, m_2, m_3)$, of length N
(other data structures possible)

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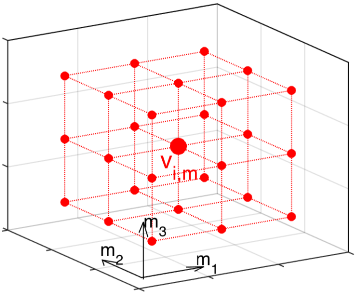

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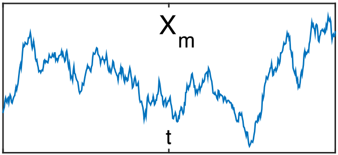

3. Design of regularizing priors on $\boldsymbol{V}$

Gamma Markov random field (GaMRF)

$\longrightarrow$ smooth evolution of multifractal parameters $\mathbf{v}$
($\sim$ variances of Gaussians)

- Positive auxiliary variables $\boldsymbol{Z} = \{\boldsymbol{Z}_1, \boldsymbol{Z}_2\}$, $\boldsymbol{Z}_i = \{z_{i,\boldsymbol{m}}\}$
  $\longrightarrow$ **induce dependence** between neighboring elements of $\boldsymbol{V}_i$

- $v_{i,\boldsymbol{m}}$: connected to 8 variables $z_{i,\boldsymbol{m}'} \in \mathcal{V}_v(\boldsymbol{m})$
  $$\mathcal{V}_v(\boldsymbol{m}) \triangleq \{\boldsymbol{m} + (i_1, i_2, i_3)\}_{i_1,i_2,i_3=0,1}$$
  via edges with weights ρ_i, $i = 1, 2$

- and vice-versa $z_{i,\boldsymbol{m}}$ to $v_{i,\boldsymbol{m}'} \in \mathcal{V}_z(\boldsymbol{m})$
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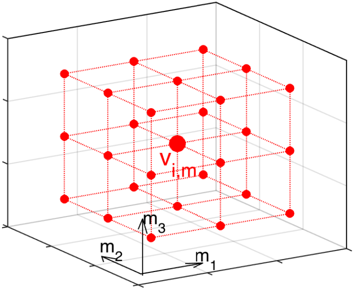

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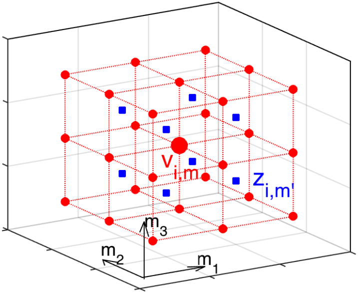

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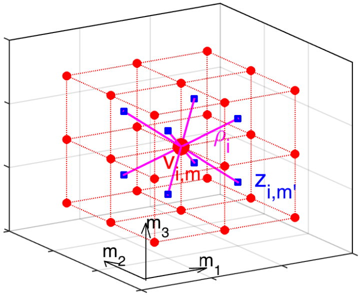

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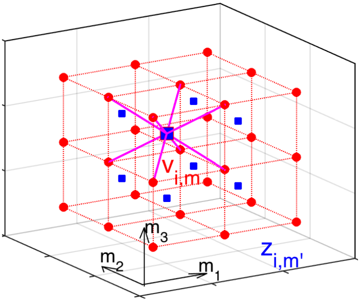

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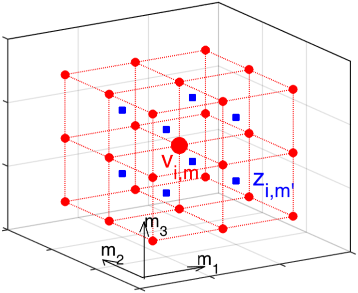

Bayesian model with GaMRF prior

- GaMRF prior: associated density [Dikmen10]

$$p(\boldsymbol{V}_i, \boldsymbol{Z}_i|\rho_i) \propto \prod_{\boldsymbol{m},n} \mathrm{e}^{(8\rho_i - 1)\log z_{i,\boldsymbol{m}}}\, \mathrm{e}^{-(8\rho_i+1)\log v_{i,\boldsymbol{m}}} \times \mathrm{e}^{-\frac{\rho_i}{v_{i,\boldsymbol{m}}}\sum_{\boldsymbol{m}'\in\mathcal{V}_v \boldsymbol{m}} z_{i,\boldsymbol{m}'}}$$

$$z_{i,\boldsymbol{m}}|\boldsymbol{V}_i \sim \mathcal{G}\Big(8\rho_i, \big(\rho_i \sum_{\boldsymbol{m}'\in\mathcal{V}_z(\boldsymbol{m})} v_{i,\boldsymbol{k}'}^{-1}\big)^{-1}\Big) \quad \rightarrow \text{gamma conditionals}$$

$$\boldsymbol{v}_{i,\boldsymbol{m}}|\boldsymbol{Z}_i \sim \mathcal{IG}\Big(8\rho_i, \rho_i \sum_{\boldsymbol{m}'\in\mathcal{V}_v(\boldsymbol{m})} z_{i,\boldsymbol{m}'}\Big) \quad \rightarrow \text{inverse-gamma conditionals}$$

Bayesian model with GaMRF prior

- Augmented likelihood

$$p(\boldsymbol{Y}, \boldsymbol{M}|\boldsymbol{V}) = p(\boldsymbol{Y}|\boldsymbol{V}_2, \boldsymbol{M})\ p(\boldsymbol{M}|\boldsymbol{V}_1)$$

- GaMRF prior: associated density [Dikmen10]

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- Posterior distribution

$$p(\boldsymbol{V}, \boldsymbol{Z}, \boldsymbol{M}|\boldsymbol{Y}, \rho_1, \rho_2) \ \propto \underbrace{p(\boldsymbol{Y}|\boldsymbol{V}_2, \boldsymbol{M})\ p(\boldsymbol{M}|\boldsymbol{V}_1)}_{\textit{augmented likelihood}} \times \underbrace{p(\boldsymbol{V}_1, \boldsymbol{Z}_1|\rho_1)\ p(\boldsymbol{V}_2, \boldsymbol{Z}_2|\rho_2)}_{\textit{independent GaMRF priors}}$$

Gibbs sampler

- Marginal posterior mean estimator

$$\boldsymbol{V}_i^{\mathrm{MMSE}} = \mathbb{E}[\boldsymbol{V}_i | \boldsymbol{Y}, \rho_i] \approx (N_{mc} - N_{bi})^{-1} \sum_{q=N_{bi}}^{N_{mc}} \boldsymbol{V}_i^{(q)}$$

- Sampling of $\boldsymbol{M}$ and parameters $\boldsymbol{V}$

$p(\boldsymbol{\mu_m} | \boldsymbol{V}, \boldsymbol{Y} \quad)$ closed-form Gaussian distribution

$p(\boldsymbol{v}_{i,\boldsymbol{m}} | \boldsymbol{V}_{j \neq i}, \boldsymbol{M}, \boldsymbol{Y} \quad)$ closed-form inverse-gamma distributions

all standard distributions $\rightarrow$ no Metropolis-Hasting moves

$\rightarrow$ efficient sampling scheme, tailored for large datasets

Gibbs sampler with GaMRF prior

- Marginal posterior mean estimator

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- Sampling of auxiliary variables $\boldsymbol{Z}$

$p(z_{i,\boldsymbol{m}} | \boldsymbol{V}, \boldsymbol{M}, \boldsymbol{Y}, \boldsymbol{\rho})$ closed-form gamma distributions

all standard distributions → no Metropolis-Hasting moves

→ efficient sampling scheme, tailored for large datasets

(Hyperparameters ρ_i fixed manually)

Gibbs sampler with independent $\mathcal{IG}$ priors

- Marginal posterior mean estimator

$$\boldsymbol{V}_i^{\mathrm{MMSE}} = \mathbb{E}[\boldsymbol{V}_i | \boldsymbol{Y}, \rho_i] \approx (N_{mc} - N_{bi})^{-1} \sum_{q=N_{bi}}^{N_{mc}} \boldsymbol{V}_i^{(q)}$$

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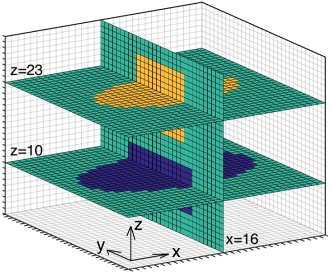
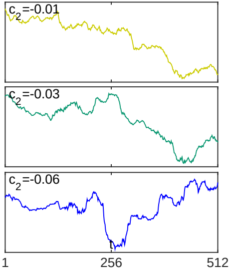
(Hyperparameters ρ_i fixed manually)

Synthetic multifractal time series

- Multifractal Random Walk
  $\sim$ Mandelbrot's celebrated multiplicative cascades
- cube of 32^3 voxels of length $N = 512$
  - 3 zones with constant $c_2 \in \{-0.01, -0.03, -0.06\}$





- Comparison of estimators for c_2 ($N_\psi = 2$, $j \in [2, 4]$)
  - LF – univariate linear regression based estimation
  - IG – univariate Bayesian estimation
  - GaMRF – joint Bayesian estimator

Illustration for single realization: estimates

estimates for c_2

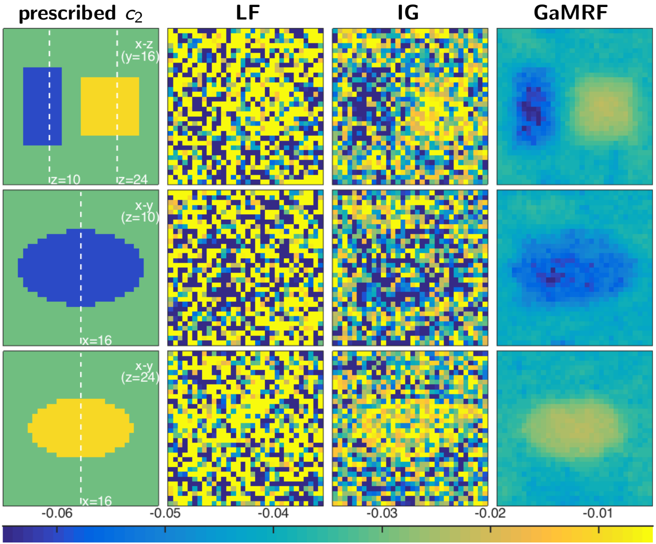


Illustration for single realization: estimates

estimates for c_2

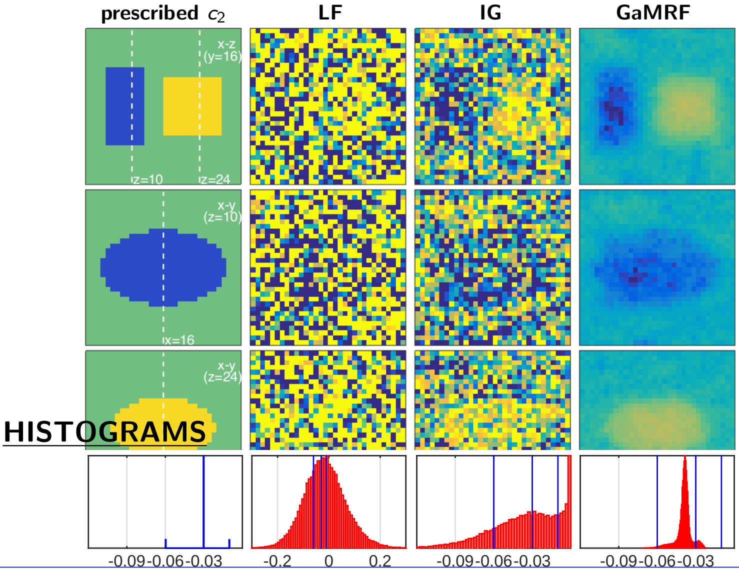
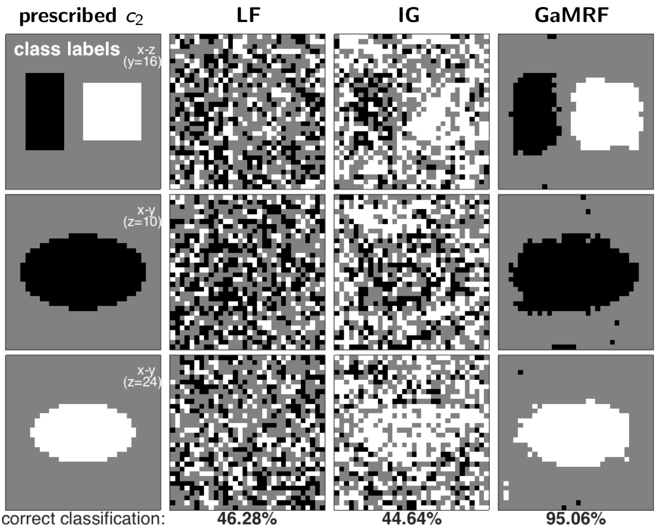


Illustration for single realization: histogram thresholding

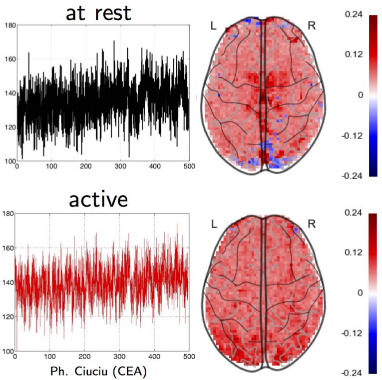
k-means classification



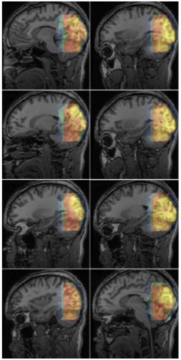
Scale-free dynamics and infraslow macroscopic brain activity

model	$1/f^{\beta}$	$\subset$	self-similar	$\subset$	multifractal
analysis	Fourier	$\longrightarrow$	wavelets	$\longrightarrow$	wavelet leaders
parameters	β		Hurst H		multifractal spectrum $D(h)$

[Shimizu'04,He'11,Ciuciu'12]



Ph. Ciuciu (CEA)



Ph. Ciuciu (CEA)

[Ciuciu-EMBC'17]

[Wendt-ISBI'18]

Collab. P. Ciuciu (CEA, NeuroSpin, France), P. Abry (ENS Lyon, France)

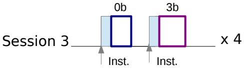
fMRI data: Experimental design and acquisition

Verbal n-back working memory task ($n = 3$).

– serially presented upper-case letters (displayed 1s, separation 2s)

$\rightarrow$ Is letter same as that presented 3 stimuli before?

– each run: alternating sequence of 8 blocks



Data acquisition.

– resting-state fMRI images first: participant at rest, with eyes closed

– 543 scans (9min10s) / 512 scans (8min39s) for rest / task

– fMRI data acquisition at 3 Tesla (Siemens Trio, Germany)

– multi-band GE-EPI (TE=30ms, TR=1s, FA=61, MB=2) sequence (CMRR, USA), 3mm isotropic resolution, FOV of $192 \times 192 \times 144 \text{mm}^3$

Shown results: $(-c_2)$ maps.

– for single subject (arbitrarily chosen from 40 participants).

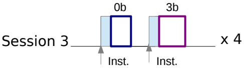
fMRI data: Experimental design and acquisition

Verbal n-back working memory task ($n = 3$).

– serially presented upper-case letters (displayed 1s, separation 2s)

$\rightarrow$ Is letter same as that presented 3 stimuli before?

– each run: alternating sequence of 8 blocks



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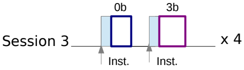

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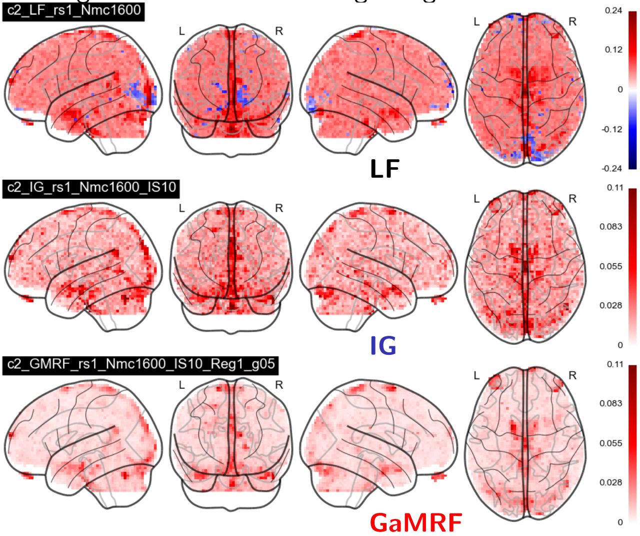
Resting-state analysis

$((-c_2)$ maps)

Left sagittal Coronal Right sagittal Axial



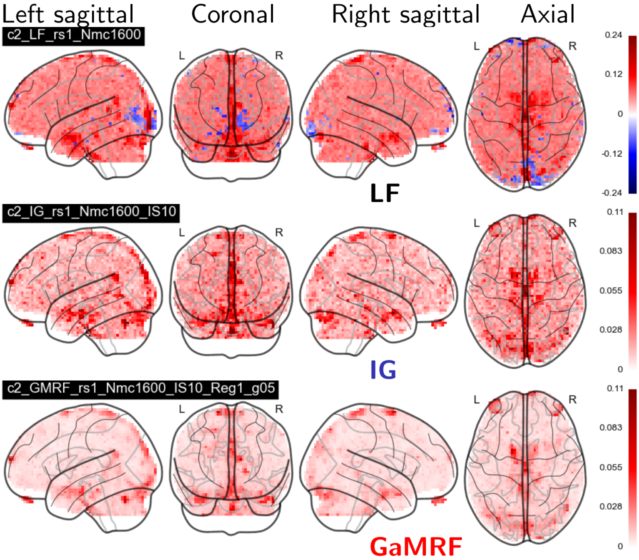
LF:
- poor estimation (var!)

IG & GaMRF:
- estimation var decrease

GaMRF:
- enhanced MF contrast

$\longrightarrow$ **significant MF in default mode network (DMN)**

Resting-state analysis ($(-c_2)$ maps)



LF:
- poor estimation (var!)

IG & GaMRF:
- estimation var decrease
- increase of MF in DMN

GaMRF:
- enhanced MF contrast

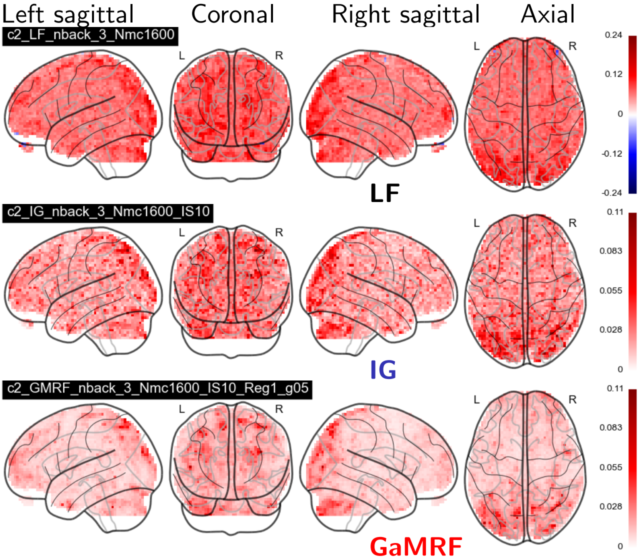
scale-free dynamics in DMN for resting-state fMRI reported before, but for H only [He JNS'11].

$\longrightarrow$ *evidence for richer, MF resting state brain dynamics*

$\longrightarrow$ **significant MF in default mode network (DMN)**

Task analysis: 3-back run

$((-c_2)$ maps$)$



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$\longrightarrow$ **overall MF increase; working memory network (WMN), visual, sensory.**

Task analysis: 3-back run

$((-c_2)$ maps)

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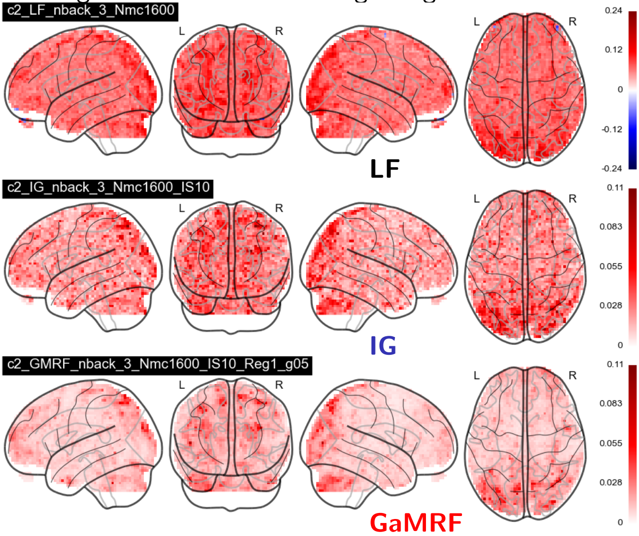

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overall increase in MF during task [Ciuciu FPhys'12]

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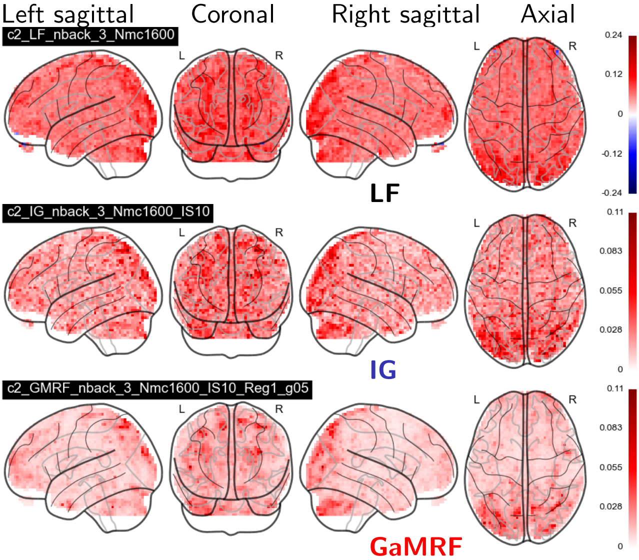
Task analysis: 3-back run

(($-c_2$) maps)



LF:
- poor estimation (var!)

IG & GaMRF:
- estimation var decrease

overall increase in MF during task [Ciuciu FPhys'12]

GaMRF:
significant MF in
- bilateral parietal regions belonging to WMN
- occipital cortex (visual)
- cerebellum (sensory)
involved in task

$\longrightarrow$ **overall MF increase; working memory network (WMN), visual, sensory.**

Conclusions and perspectives

Multifractal analysis:

- Bayesian estimation for c_2 of multivariate time series
  - hierarchical Bayesian model with smoothing priors:

$$\begin{cases} \text{data augmented Fourier domain likelihood} & (\sim \mathcal{CN}) \\ \text{GaMRF joint prior for } c_2 \text{ of different data components} \end{cases}$$

  $\rightarrow$ efficient inference via a Gibbs sampler (large data sets)

  $\rightarrow$ significantly improved estimation performance (gain: factor ~ 10)

- Current model:
  GaMRF hyperparameter, integral scale; EM algorithm
- Multivariate priors
  joint estimation-segmentation in time / space
- Estimation of parameters of *multivariate multifractal models*

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Thank you for your attention

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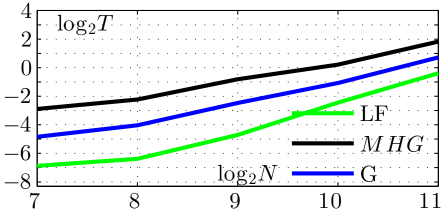
Estimation performance for c_2

	LF	IG	GaMRF
$\lvert b \rvert$	0.0158	0.0051	0.0092
std	0.0800	0.0255	0.0020
rmse	**0.0819**	**0.0262**	**0.0094**

$$\mathrm{b} = \widehat{\mathbb{E}}[\hat{c}_2] - c_2, \quad \mathrm{std} = \sqrt{\widehat{\mathrm{Var}}[\hat{c}_2]}, \quad \mathrm{rmse} = \sqrt{\mathrm{b}^2 + \mathrm{std}^2}$$

(100 independent realizations)

Computation time:



Model: time-domain statistical model of log-leaders

1. Marginal distribution of log-leaders approximated by Gaussian

$$l(j,\cdot,\cdot) = \ln L(j,\cdot,\cdot) \;\sim\; \mathcal{N}\left(\cdot, c_2^0 + c_2 \ln 2^j\right)$$

2. Intra-scale parametric covariance model

$$\mathrm{Cov}[l(j,k), l(j,k+\Delta r)] \approx \varrho_j(\Delta r; \boldsymbol{v}), \quad \boldsymbol{v} = (c_2, c_2^0)$$

- Likelihood of centered log-leaders $\boldsymbol{l}_j$ stacked in $\boldsymbol{l} = [\boldsymbol{l}_{j_1}^T, ..., \boldsymbol{l}_{j_2}^T]^T$

  $\rightarrow$ scale-wise product of Gaussian likelihoods

$$p(\boldsymbol{l}|\boldsymbol{v}) \propto \prod_{j=j_1}^{j_2} |\boldsymbol{\Sigma}_{j,\boldsymbol{v}}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\boldsymbol{l}_j^T \boldsymbol{\Sigma}_{j,\boldsymbol{v}}^{-1} \boldsymbol{l}_j\right), \text{ with } \boldsymbol{\Sigma}_{j,\boldsymbol{v}} \text{ induced by } \varrho_j(\Delta r; \boldsymbol{v})$$

  X evaluation of $p(\boldsymbol{l}|\boldsymbol{v})$ numerically instable

[TIP15]

Model: Whittle approximation

- Evaluation of the Gaussian likelihood in the spectral domain

$$p_W(\boldsymbol{l}|\boldsymbol{v}) \propto \prod_{j=j_1}^{j_2} |\boldsymbol{\Gamma}_{j,\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}_j^H \boldsymbol{\Gamma}_{j,\boldsymbol{v}}^{-1} \boldsymbol{y}_j\right)$$

  - $\boldsymbol{y}_j$ Fourier coefficients of $\boldsymbol{l}_j$
  - $\boldsymbol{\Gamma}_{j,\boldsymbol{v}}$ parametric spectral density associated with $\varrho_j(\Delta r; \boldsymbol{v})$

    $\rightarrow$ closed-form expression via Hankel transform

$$\boldsymbol{\Gamma}_{j,\boldsymbol{v}} = c_2\, \mathbf{F}_{1,j} + c_2^0\, \mathbf{F}_{2,j}, \quad \mathbf{F}_{i,j} = \mathrm{diag}(\mathbf{f}_{i,j})$$

- Estimation of $\boldsymbol{v}$ embedded in a Bayesian framework
  - space-domain likelihood (approximated) + common priors

  X non-standard posterior distribution $\rightarrow$ acceptance/reject moves

[TIP15]

Model: Fourier-domain statistical model

- Whittle approximation

$$p_W(\boldsymbol{l}|\boldsymbol{v}) \propto \prod_{j=j_1}^{j_2} |\boldsymbol{\Gamma}_{j,\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}_j^H \boldsymbol{\Gamma}_{j,\boldsymbol{v}}^{-1} \boldsymbol{y}_j\right)$$

  - $\boldsymbol{y}_j$ Fourier coefficients of $\boldsymbol{l}_j$
  - $\boldsymbol{\Gamma}_{j,\boldsymbol{v}} = c_2\, \mathbf{F}_{1,j} + c_2^0\, \mathbf{F}_{2,j}$ parametric spectral density

$$\Updownarrow$$

- Generative model for $\boldsymbol{y} = [\boldsymbol{y}_{j_1}^T, ..., \boldsymbol{y}_{j_2}^T]^T$

$$p(\boldsymbol{y}|\boldsymbol{v}) \propto |\boldsymbol{\Gamma}_{\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}^H \boldsymbol{\Gamma}_{\boldsymbol{v}}^{-1} \boldsymbol{y}\right)$$

  - complex Gaussian model $\boldsymbol{y} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Gamma}_{\boldsymbol{v}})$
  - $\boldsymbol{\Gamma}_{\boldsymbol{v}} = c_2 \boldsymbol{F}_1 + c_2^0 \boldsymbol{F}_2$ and $\boldsymbol{F}_i = \text{block}(\boldsymbol{F}_{i,j_1}, \ldots, \boldsymbol{F}_{i,j_2})$

X model non-separable in (c_2, c_2^0)

Model: Reparametrization

- Non-separable constraints on (c_2, c_2^0)

$$\boldsymbol{v} \in \mathcal{A} = \{(c_2, c_2^0) \in \mathbb{R}_\star^- \times \mathbb{R}_\star^+ | \boldsymbol{\Gamma}_{\boldsymbol{v}} = c_2 \mathbf{F}_1 + c_2^0 \mathbf{F}_2 \text{ positive-definite}\}$$

- Design of a linear diffeomorphism ψ

  1. mapping joint constraints into independent positivity constraints

$$\psi \colon \mathcal{A} \to \mathbb{R}_\star^{+2}$$
$$\colon \boldsymbol{v} \mapsto \psi(\boldsymbol{v}) \triangleq \boldsymbol{v}$$

  2. yielding more convenient likelihood

$$p(\boldsymbol{y}|\boldsymbol{v}) \propto |\boldsymbol{\Gamma}_{\boldsymbol{v}}|^{-1} \exp\left(-\boldsymbol{y}^H \boldsymbol{\Gamma}_{\boldsymbol{v}}^{-1} \boldsymbol{y}\right) \quad \text{with}$$

$$\text{for } \boldsymbol{v} \in \mathbb{R}_\star^{+2} \begin{cases} \boldsymbol{\Gamma}_{\boldsymbol{v}} = \tilde{\theta}_1 \tilde{\boldsymbol{F}}_1 + \tilde{\theta}_2 \tilde{\boldsymbol{F}}_2 & \text{positive-definite} \\ \tilde{\theta}_i \tilde{\boldsymbol{F}}_i & \text{positive-definite} \end{cases}$$

$\rightarrow$ separability of the likelihood via data augmentation

Model: Data augmentation

- Definition of an augmented model

$$\begin{cases} \boldsymbol{y}|\boldsymbol{\mu}, \tilde{\theta}_2 \sim \mathcal{CN}(\boldsymbol{\mu}, \tilde{\theta}_2 \tilde{\boldsymbol{F}}_2) & \text{observed data} \\ \boldsymbol{\mu}|\tilde{\theta}_1 \sim \mathcal{CN}(\boldsymbol{0}, \tilde{\theta}_1 \tilde{\boldsymbol{F}}_1) & \text{hidden mean} \end{cases}$$

with

$$p(\boldsymbol{y}|\boldsymbol{v}) = \int p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v}) d\boldsymbol{\mu}$$

- Virtues of the augmented likelihood $p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v})$

$$p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v}) \propto \tilde{\theta}_2{}^{-N_Y} \exp\left(-\frac{1}{\tilde{\theta}_2}(\boldsymbol{y}-\boldsymbol{\mu})^H \tilde{\boldsymbol{F}}_2^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \times \tilde{\theta}_1{}^{-N_Y} \exp\left(-\frac{1}{\tilde{\theta}_1}\boldsymbol{\mu}^H \tilde{\boldsymbol{F}}_1^{-1}\boldsymbol{\mu}\right)$$

√ separable in $(\tilde{\theta}_1, \tilde{\theta}_2)$

√ conjugate to inverse-gamma priors

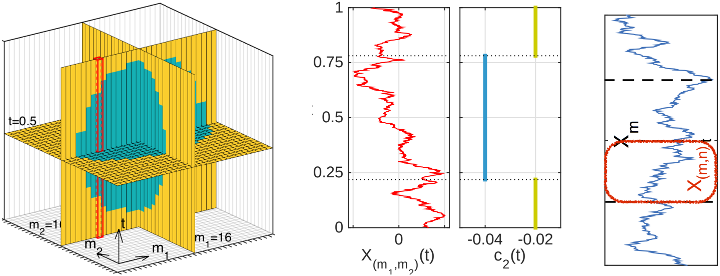
MCMC algorithm

- Strategy of Gibbs sampler
  - iterative sampling according to conditional laws
  - non-standard conditional laws $\rightarrow$ Metropolis-within-Gibbs
  - computation of acceptance ratio at each iteration

$$r_{c_2} = \sqrt{\frac{\det \boldsymbol{\Sigma}(\boldsymbol{v}^{(t)})}{\det \boldsymbol{\Sigma}(\boldsymbol{v}^{(\star)})}} \times \prod_{j=j_1}^{j_2} \exp\left(-\frac{1}{2}\boldsymbol{l}_j^T \left(\boldsymbol{\Sigma}_{j,\boldsymbol{v}}(\boldsymbol{v}^{(\star)})^{-1} - \boldsymbol{\Sigma}_{j,\boldsymbol{v}}(\boldsymbol{v}^{(t)})^{-1}\right)\boldsymbol{l}_j\right)$$

Time block wise estimation (2D+time)

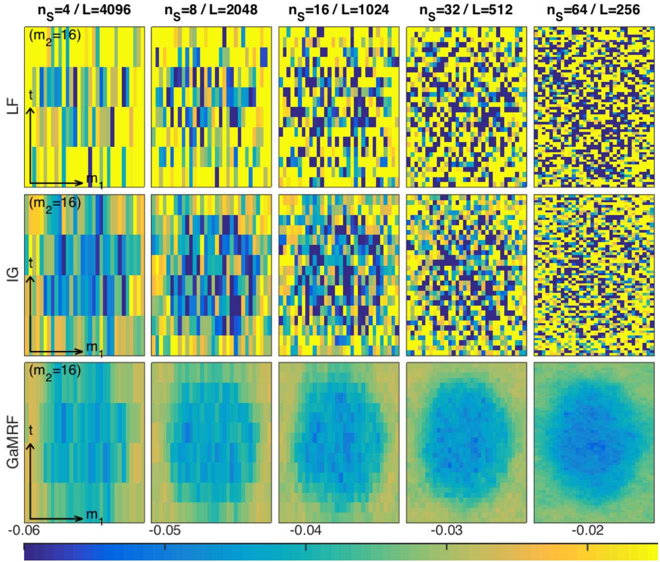
- Synthetic multifractal time series: Multifractal Random Walk
  $\sim$ Mandelbrot's celebrated multiplicative cascades
- collection of 32×32 time series of length $N = 2^{14}$
  - piece-wise constant $c_2 \in \{-0.02, -0.04\}$ along time



- Comparison of estimators for c_2
  - $n_S = 2^{2,\dots,6}$ windows of lengths $L = \{2^{12}, 2^{11}, 2^{10}, 2^9, 2^8\}$
  - LF – univariate linear regression based estimation
  - IG – univariate Bayesian estimation [TIP15,ICASSP16]
  - GaMRF – joint Bayesian estimator

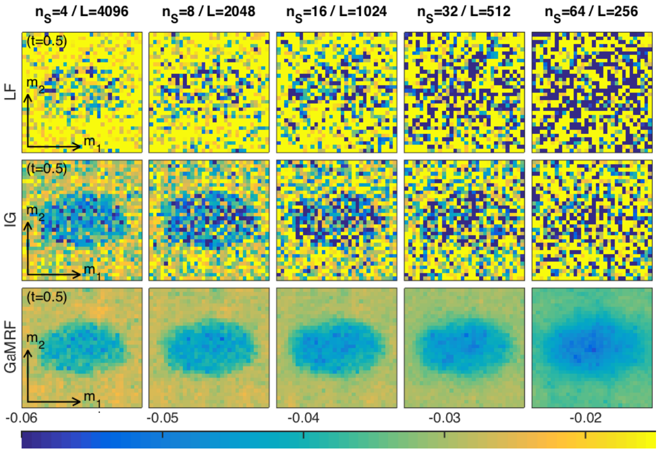
Time block wise estimation (2D+time)

estimates for c_2: temporal evolution at slice $m_2 = 16$



Time block wise estimation (2D+time)

estimates for c_2: spatial cross-section at $t = 0.5$



Time block wise estimation (2D+time)

RMSE (50 independent realizations)

n_S / L	4 / 2^{12}	8 / 2^{11}	16 / 2^{10}	32 / 2^{9}	64 / 2^{8}
LF	0.020	0.026	0.037	0.058	0.102
IG	0.011	0.013	0.018	0.024	0.036
GaMRF	**0.008**	**0.008**	**0.009**	**0.009**	**0.013**