

GENERALIZED LEGENDRE TRANSFORM MULTIFRACTAL FORMALISM FOR NONCONCAVE SPECTRUM ESTIMATION

Roberto Leonarduzzi¹, Hugo Touchette², Herwig Wendt³, Patrice Abry¹, Stéphane Jaffard⁴

¹ Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Lab de Physique, F-69342 Lyon, France

{roberto.leonarduzzi, patrice.abry}@ens-lyon.fr

² National Institute for Theoretical Physics, Dept. of Physics, Stellenbosch University, South Africa

htouchette@sun.ac.za

³ IRIT-ENSEEIH, Université de Toulouse, CNRS, Toulouse, France herwig.wendt@irit.fr

⁴ Université Paris Est, LAMA, UPEC, France jaffard@u-pec.fr

ABSTRACT

Despite widespread adoption of multifractal analysis as a signal processing tool, most practical multifractal formalisms suffer from a major drawback: since they are based on Legendre transforms, they can only yield concave estimates for multifractal spectrum that are, in most cases, only upper bounds on the (possibly nonconcave) true spectrum. Inspired by ideas borrowed from statistical physics, a procedure is devised for the estimation of not a priori concave spectra that retains the simple and efficient Legendre transform formalism structure. The potential and interest of the proposed procedure are illustrated and assessed on realizations of a synthetic multifractal process, with theoretically known nonconcave multifractal spectrum.

Index Terms— multifractal analysis; wavelet leaders; nonconcave multifractal spectrum; Legendre transform; generalized ensemble

1. INTRODUCTION

Multifractal analysis. Nowadays, multifractal analysis has become a standard signal processing tool, fruitfully used for the analysis of highly irregular signals in a wide range of applications, including biomedical [1], geophysics [2], finance [3], and art investigation [4], to name but a few. It essentially amounts to estimating the *multifractal spectrum* $D(h)$, which quantifies the distribution of points with a given *regularity exponent* h . This spectrum characterizes signals in terms of its regularity, and is thus used in standard classification or modeling tasks.

Multifractal formalism. In practice, $D(h)$ cannot be directly computed from its definition. For this reason, it is indirectly estimated by means of a procedure known as *multifractal formalism*. It is based on the scale-invariance properties of the data, measured by the so-called *scaling exponents* $\zeta(q)$. The

Legendre transform of $\zeta(q)$, which is called *Legendre spectrum* $\mathcal{L}(h)$, provides an upper bound $\mathcal{L}(h) \geq D(h)$.

Concavity of the Legendre spectrum. Despite its widespread use, a fundamental problem concerns the Legendre spectrum $\mathcal{L}(h)$: since it is defined through a Legendre transform, it will always be a concave function of h [5]. In those cases where $D(h)$ is concave to begin with, $\mathcal{L}(h)$ can provide an exact estimate. In this situation it is said that the multifractal formalism holds. If the true spectrum $D(h)$ is not a concave function, $\mathcal{L}(h)$ will only provide a concave upper bound for $D(h)$.

The vast majority of the multifractal analysis that has been performed on real data has made use of different variants of Legendre-transform-based multifractal formalisms (e.g. the *wavelet leader* method [6, 7], the *wavelet transform modulus maxima* method [8], and *multifractal detrended fluctuation analysis* [9]; see *a contrario* the *large deviation spectra* [10]). Therefore, it is not known whether concave spectra that have been observed in real data are actually correct, or if, on the contrary, they are just upper bounds that reflect limitations of the analysis tools, which hide their true nonconcave nature. In consequence, the development of methods that enable an efficient and practically feasible estimation of nonconcave spectra is an important open issue for practical multifractal analysis.

Related work. Attempts have been made to address the practical estimation of nonconcave spectra. A strategy based on the large deviation spectrum was proposed in [10]. More recently, a method relying on the estimation of the quantiles of wavelet leaders was proposed in [11–13], and shown to be able to estimate certain types of nonconcave spectra. However, these methods lack the simplicity and efficiency of the Legendre-transform-based multifractal formalism, which allows to compute all relevant estimates by simple linear regressions of straightforward quantities [7].

Goals and outline. In this work we propose a novel method to estimate nonconcave spectra that still relies on a (gener-

alized) Legendre transform structure, thus enabling a simple and efficient practical implementation. We borrow ideas that have been previously used for the estimation of nonconcave entropy functions in statistical mechanics [14–16]. After a review of the basic concepts of multifractal analysis in Sec. 2, we detail the generalized Legendre transform multifractal formalism in Sec. 3. Then, in Sec. 4 we show the method at work on a synthetic random process with nonconcave and multifractal spectrum which is known theoretically.

2. MULTIFRACTAL ANALYSIS

Hölder regularity. Let $X : \mathbb{R} \rightarrow \mathbb{R}$, denote the locally bounded signal to be analyzed; it belongs to the local Hölder space $C^\alpha(t_0)$ if there exist $K > 0$ and a polynomial P_{t_0} (with $\deg(P_{t_0}) < \alpha$) such that $|X(t) - P_{t_0}(t - t_0)| \leq K|t - t_0|^\alpha$ for $t \rightarrow t_0$. The Hölder exponent of X at t_0 is $h(t_0) = \sup\{\alpha : X \in C^\alpha(t_0)\}$. It quantifies the local regularity of X : the smaller $h(t_0)$, the “rougher” X is at t_0 .

Multifractal spectrum. For multifractal models, direct estimation of the function $h(t)$ is of little interest since it is a highly irregular function itself. Rather, one is interested in the distribution of the values it takes. This is achieved by the *multifractal spectrum* $D(h)$, which is defined as the Hausdorff dimension of the set of points t where $h(t) = h$. Since neither Hausdorff dimensions nor pointwise Hölder exponents can be estimated reliably from their definitions in practice, a procedure termed *multifractal formalism* has been proposed for its estimation [17].

Legendre-Fenchel transform. In the context of multifractal analysis, the Legendre-Fenchel (LF) transform¹ of $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as [5, 6]

$$f^*(y) = \inf_{x \in \mathbb{R}} \{1 + xy - f(x)\}. \quad (1)$$

We denote the double LF transform as $f^{**}(x) = \inf_{y \in \mathbb{R}} \{1 + xy - f^*(y)\}$. The following properties, which we state without proof (cf. e.g. [5] for a detailed and rigorous analysis), are of interest to us.

Property 1. $f^*(y)$ is always a concave function of y .

Property 2. $f^{**}(x) = f(x)$ if and only if $f(x)$ is concave at x .

Property 3. $f^*(y)$ is differentiable at all y if and only if $f(x)$ is strictly convex for all x .

Property 4. If $f^*(y)$ is not differentiable at y_c , then $f(x)$ is nonconcave or affine over the open interval (x_l, x_h) , with $x_l = f^{*'}(y_c^-)$ and $x_h = f^{*'}(y_c^+)$.

Wavelet coefficients and leaders. Let ψ denote the *mother wavelet*, a zero-average compactly supported function characterized by a number $N_\psi \in \mathbb{N}$, which encapsulates both its order of Hölder regularity and vanishing moments, i.e. such

that $f \in C^{N_\psi}$ and $\int_{\mathbb{R}} t^k \psi(t) dt = 0, \forall k = 0, \dots, N_\psi - 1$. Further, let $\{\psi_{j,k}(t) = 2^{-j} \psi(2^{-j}t - k)\}_{(j,k) \in \mathbb{N}^2}$ be the orthonormal basis of $L^2(\mathbb{R})$ formed by dilations and translations of ψ . The (L^1 -normalized) discrete wavelet transform coefficients are defined as $c_{j,k} = \langle \psi_{j,k} | X \rangle$ (cf., e.g., [18], for more details on wavelet transforms).

Now let $\lambda = \lambda_{j,k} = [k2^j, (k+1)2^j)$ denote a dyadic interval and $3\lambda = \bigcup_{m \in \{-1, 0, 1\}} \lambda_{j,k+m}$ the union of λ and its two neighbours. Further, let $\lambda_j(x)$ denote the only interval at scale j that contains x . *Wavelet leaders* are defined as $L_\lambda = L_{j,k} \triangleq \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$, where the supremum involves all wavelet coefficients in a narrow time neighbourhood of $t = 2^{-j}k$ for all finer scales $j' \geq j$. A key property of wavelet leaders is that their local decay reproduces exactly the Hölder exponent, in the limit of fine scales [6]:

$$h(x) = \liminf_{j \rightarrow -\infty} \frac{\log_2 L_{\lambda_j(x)}}{j}. \quad (2)$$

Multifractal formalism. The wavelet leader multifractal formalism allows for the estimation of $D(h)$ from easily computable quantities. It is based on the power-law decay at fine scales of the sample moments of $L_{j,k}$, which are called *structure functions* $S(q, j)$:

$$S(q, j) = \frac{1}{n_j} \sum_{k=1}^{2^j} \left(L_{\lambda_{j,k}}^{(p)} \right)^q \sim 2^{j\zeta(q)}, \quad j \rightarrow -\infty, \quad (3)$$

where n_j denotes the number of $L_{\lambda_{j,k}}$ at scale j . The *scaling function* $\zeta(q)$ is computed in practice by means of linear regressions of $\log_2 S(q, j)$ versus j [7]. One can show (see [6] and Sec. 3) that $\zeta(q) \leq D^*(q)$. Thus, inversion of the LF transform yields a concave upper bound of D :

$$D(h) \leq \mathcal{L}(h) = \zeta^*(h) = D^{**}(h). \quad (4)$$

Note that, by Property 1 above, \mathcal{L} will always be concave, regardless of the shape of D . Further, equality between D and \mathcal{L} can only follow when the former is concave (property 2). Otherwise \mathcal{L} will only provide a concave upper-bound for D .

3. GENERALIZED LEGENDRE TRANSFORM MULTIFRACTAL FORMALISM

Principle. As discussed in the previous section, if the spectrum D of a function X is not concave, then it will not coincide with its Legendre spectrum \mathcal{L} . Inspired by a method proposed in the context of nonconcave entropy functions in statistical mechanics [14, 15, 19], we propose the following method which allows to estimate a nonconcave spectrum while still using the Legendre transform of a proper scaling function.

Let $D : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the (possibly nonconcave) multifractal spectrum of $X \in L^\infty(\mathbb{R})$, and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function

¹In the literature on multifractal analysis the LF transform is simply referred to as the “Legendre transform”. In fact, the latter is simply a special case of the former for the case of differentiable functions.

such that $D+g$ is strictly concave. Suppose that, given X , we can build another function X_g such that its multifractal spectrum is $D+g$. Then, we can apply the multifractal formalism to X_g and expect to get a Legendre spectrum \mathcal{L}_g that coincides with $D+g$ (because of concavity of $D+g$ and Property 2). Finally, we obtain a sharper and nonconcave upper bound for the original spectrum as: $\mathcal{L}_g(h) - g(h)$.

An explicit construction of the function X_g can be advantageously replaced by a construction of the corresponding wavelet leaders, which can be obtained by a simple transformation from those of X . This will be detailed further ahead.

Following [14], in this work we will concentrate on the choice $g(h) = \gamma h^2$, which corresponds to the Gaussian Ensemble in statistical physics.

Quadratic Multifractal Formalism. Let us denote $\phi_{\lambda_j(x)} = \log_2(L_{\lambda_j(x)})/j$ and define new multiresolution coefficients $P_{\lambda}^{(\gamma,q)}$ with quadratic perturbation, such that

$$\log_2 P_{\lambda_j(x)}^{(\gamma,q)} = j \left(q\phi_{\lambda_j(x)} + \gamma\phi_{\lambda_j(x)}^2 \right). \quad (5)$$

It follows from (2) that

$$\liminf_{j \rightarrow -\infty} \frac{\log_2 P_{\lambda_j(x)}^{(\gamma,q)}}{j} = qh(x) + \gamma h^2(x). \quad (6)$$

The generalized structure functions are given by

$$S_{\gamma}(q, j) = \frac{1}{n_j} \sum_k P_{\lambda_{j,k}}^{(\gamma,q)} \sim 2^{j\zeta_{\gamma}(q)}, \quad (7)$$

where $\zeta_{\gamma}(q)$ is the generalized scaling exponent.

Following [6], we adapt the heuristic argument for the multifractal formalism. By definition of the fractional dimension, there are $\sim 2^{-jD(h)}$ cubes λ_j which cover points where $h(x) = h$. According to (6), each one contributes $\sim 2^{qh(x) + \gamma h^2(x)}$ to $S_{\gamma}(q, j)$. Also, $n_j \sim 2^{-j}$. Therefore $S_{\gamma}(q, j) \sim 2^{j(1+qh(x) + \gamma h^2(x) - D(h))}$. In the limit of fine scales, the smallest exponent dominates and thus

$$\begin{aligned} \zeta_{\gamma}(q) &= \inf_h \{1 + qh + \gamma h^2 - D(h)\} \\ &= \inf_h \{1 + qh - \tilde{D}_{\gamma}(h)\}, \end{aligned} \quad (8)$$

where $\tilde{D}_{\gamma}(h) = D(h) - \gamma h^2$. Eq. (8) shows that $\zeta_{\gamma}(q)$ is the LF transform of the modified spectrum $\tilde{D}_{\gamma}(h)$. Inversion of (8) thus yields:

$$D(h) - \gamma h^2 = \tilde{D}_{\gamma}(h) \leq \mathcal{L}_{\gamma}(h) = \zeta_{\gamma}^*(h). \quad (9)$$

In practice, for a large enough $\gamma > \gamma_c$, \tilde{D}_{γ} will be strictly concave and equality in (9) will hold. In this case $\tilde{D}_{\gamma}(h) = \mathcal{L}_{\gamma}(h)$, and the spectrum D can be recovered from \mathcal{L}_{γ} . In general, from (9) we can define an estimate of D parametrized by γ : $D_{\gamma}(h) = \mathcal{L}_{\gamma}(h) + \gamma h^2$, such that

$$D_{\gamma}(h) = D(h) \quad \text{for } \gamma \geq \gamma_c. \quad (10)$$

Minimization over γ . In principle, the use of a sufficiently large value of γ should be enough to recover a C^2 nonconcave spectrum D using (10). In practice, however, such large values are numerically unstable and prevent the estimation of the full spectrum (this issue will be further discussed in Sec. 4). The estimate of a nonconcave spectrum in (10) can be improved by selecting the optimal parabola for each possible value of h [15]:

$$D_{min}(h) = \inf_{\gamma \geq 0} \{ \mathcal{L}_{\gamma}(h) + \gamma h^2 \}. \quad (11)$$

This procedure allows for the use of small values of γ in the concave regions of the spectrum, and then switch to larger values only when needed. The quality of this estimate will be discussed in Sec. 4.

4. NUMERICAL ILLUSTRATIONS

α -stable Lévy process. Let $M(dx)$ be a symmetric α -stable random measure, with $0 \leq \alpha \leq 2$. Linear α -stable Lévy motion L_{α} is defined by the stochastic integral: $L_{\alpha}(x) = \int_{\mathbb{R}} f(x, u) M(du)$, with kernel $f(x, u) = \mathbb{1}_{x-u \geq 0}(u) - \mathbb{1}_{-u \geq 0}(u)$ [20]. Its multifractal spectrum is given by [21]:

$$D(h) = \begin{cases} h\alpha & 0 \leq h \leq \frac{1}{\alpha}, \\ -\infty & \text{otherwise.} \end{cases} \quad (12)$$

The computation of the Legendre transform from (12) yields:

$$\zeta(q) = \begin{cases} \frac{1}{\alpha}q & -\infty < q \leq \alpha, \\ 1 & \text{otherwise.} \end{cases} \quad (13)$$

Note that $\zeta(q)$ has a nondifferentiable point at $q = \alpha$ since $D(h)$ is affine, as indicated by Properties 3 and 4 in Sec. 3.

Nonconcave spectrum. To simulate a process with a nonconcave spectrum, we analyze the concatenation of two α -stable processes. Let L_{α_1} and L_{α_2} , with spectra D_1 and D_2 , respectively, be defined on $[0, 1]$. Let us also impose the restriction $L_{\alpha}(0.5^-) = L_{\alpha}(0.5^+)$. Then, we define

$$\tilde{L}_{\alpha_1, \alpha_2}(x) = \begin{cases} L_{\alpha_1}(x) & 0 \leq x \leq 0.5 \\ L_{\alpha_2}(x) & 0.5 < x \leq 1 \end{cases}. \quad (14)$$

In this case, the multifractal spectrum of the concatenation $\tilde{L}_{\alpha_1, \alpha_2}$ is $D(h) = \sup(D_1(h), D_2(h))$.

Simulation setup. We analyzed $N_{MC} = 50$ independent realizations of length $N = 2^{18}$, and hereafter report the mean values of the estimates computed from these realizations. We used a Daubechies wavelet with $N_{\psi} = 3$ vanishing moments, and computed scaling exponents using unweighted linear regressions (cf. [7]), in the scaling range $3 \leq j \leq 10$. We computed structure functions for orders $q \in [-50, 50]$.

Logscale diagrams. Fig. 1 shows the logscale diagrams for $q = -2$ (left) and $q = 2$ (right), for several values of γ . It can

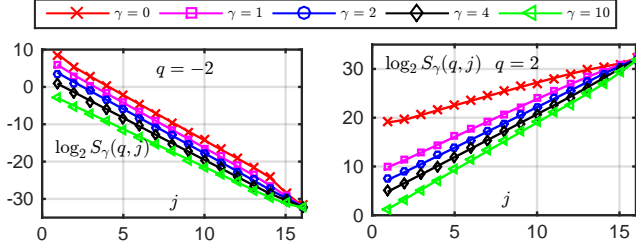


Fig. 1. Logscale diagrams. $\log_2 S_\gamma(q, j)$ for $q = -2$ (left) and $q = 2$ (right), and for several values of γ .

be seen that the scaling behavior of structure functions computed from the quantity P_γ is excellent, with linear behavior over a wide range of scales for all values of γ . This evidence indicates that scaling exponents can be readily estimated by linear regressions on these logscale diagrams, and the generalized Legendre transform multifractal formalism applied for the estimation of the spectrum.

Multifractal spectra. Fig. 2 shows the generalized scaling functions (left) and multifractal spectra (right) computed for different values of γ . Regarding Fig. 2 (left), it can be seen that increasing the value of γ has the effect of “smoothing” the point of nondifferentiability at $q = \alpha$. In fact, for large enough values of γ (e.g. red and black lines) the function ζ_γ appears to be differentiable in all its domain.

In concordance with the smoothing effect of the scaling function, Fig. 2 (right) shows that, as γ increases, $D_\gamma(h)$ is more and more able to “dig into the hole” of the multifractal spectrum, and satisfactorily estimate the nonconcave region. This can be understood in light of Property 3 in Sec. 2. Since ζ_γ is differentiable for large γ , the corresponding “modified” spectrum \tilde{D}_γ is strictly concave, and therefore exactly recovered by the Legendre spectra: $\mathcal{L}_\gamma(h) \equiv \tilde{D}_\gamma(h)$. Thus, the addition of the parabola produces a more precise estimate of $D(h)$.

It is also noteworthy that the support of $D_\gamma(h)$ is smaller as γ increases (note that the same values of q were used for all γ). Because of the construction of the multiresolution quantity in (5), the effective value of q is modified by gamma. In consequence, estimation of the endpoints of D_γ for large γ would require huge values of q . However, as q and γ increase, the computation of $P^{(\gamma, q)}$ becomes numerically difficult for finite size data. Therefore, estimation of the full nonconcave multifractal spectrum must rely on the joint use of several values of γ , as proposed in (11).

Fig. 3 shows the estimated spectrum $D_{min}(h)$, which achieves a good agreement with the theoretical spectrum, over the entire support. For comparison, the standard concave Legendre spectrum, i.e. the case corresponding to $\gamma = 0$, is also shown. Besides the fact that it misses the nonconcave region, its performance is even slightly worse in the linear region around $h = 0.5$.

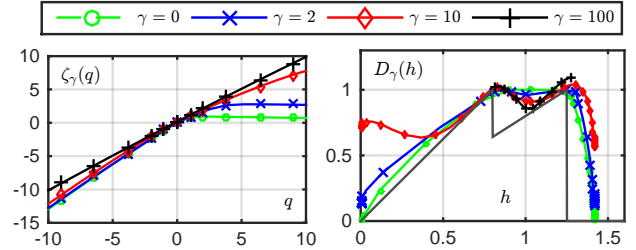


Fig. 2. Dependence on γ . Scaling exponents $\zeta_\gamma(q)$ (left) and multifractal spectra $D_\gamma(h)$ for several values of γ . The theoretical multifractal spectrum is showed in gray solid lines.

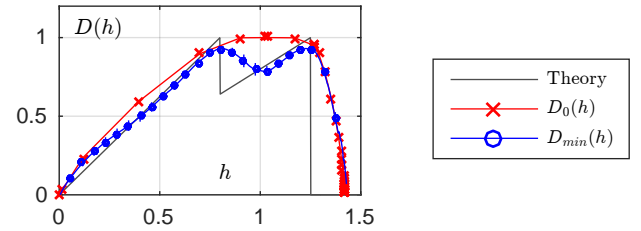


Fig. 3. Infimum over γ . Theoretical spectrum $D(h)$, traditional Legendre spectrum $D_0(h)$, and nonconcave estimate $D_{min}(h)$, computed from all values of γ as in (11). 95% confidence intervals for $D_{min}(h)$ are of similar size to the markers and barely visible.

5. CONCLUSIONS

In this contribution we have proposed an efficient method for the estimation of nonconcave multifractal data. Inspired by ideas from statistical mechanics, our method relies on a modification of the multiresolution quantities upon which the analysis is based to compute instead a “modified” spectrum. Then, the original spectrum is computed from this modified version by a simple deterministic correction to the Legendre transform. Thus, the proposed algorithm maintains the simple nature of the Legendre transform multifractal formalism, and can be efficiently implemented in practice. We have shown its good performance through numerical simulations on a synthetic multifractal process. This contribution only presented preliminary results; a forthcoming paper will deal with a thorough validation of the method for a larger class of multifractal processes, the comparison with other estimation methods, the use of other functions g and its applications to real life data.

6. REFERENCES

- [1] J. Spilka, J. Frecon, R. Leonarduzzi, N. Pustelnik, P. Abry, and M. Doret, “Intrapartum fetal heart rate classification from trajectory in sparse svm feature space,” in *IEEE Int. Conf. Engineering in Medicine and Biology Society (EMBC)*, 2015, pp. 2335–2338.

- [2] L. Telesca and M. Lovallo, “Analysis of the time dynamics in wind records by means of multifractal detrended fluctuation analysis and the Fisher–Shannon information plane,” *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2011, no. 07, pp. P07001, 2011.
- [3] L. Calvet and A. Fisher, “Forecasting multifractal volatility,” *Journal of Econometrics*, vol. 105, no. 1, pp. 27–58, 2001.
- [4] P. Abry, S. Jaffard, and H. Wendt, “When Van Gogh meets Mandelbrot: Multifractal classification of painting’s texture,” *Signal Proces.*, vol. 93, no. 3, pp. 554–572, 2013.
- [5] R. Tyrrell Rockafellar, *Convex Analysis. Princeton landmarks in mathematics*, Princeton University Press, Princeton, NJ, 1997.
- [6] S. Jaffard, “Wavelet techniques in multifractal analysis,” in *Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, M. Lapidus and M. van Frankenhuijzen, Eds., Proc. Symposia in Pure Mathematics*. 2004, vol. 72(2), pp. 91–152, AMS.
- [7] H. Wendt, P. Abry, and S. Jaffard, “Bootstrap for empirical multifractal analysis,” *IEEE Signal Proc. Mag.*, vol. 24, no. 4, pp. 38–48, 2007.
- [8] J. F. Muzy, E. Bacry, and A. Arneodo, “Multifractal formalism for fractal signals: The structure-function approach versus the wavelet-transform modulus-maxima method,” *Phys. Rev. E*, vol. 47, no. 2, pp. 875, 1993.
- [9] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. E. Stanley, “Multifractal detrended fluctuation analysis of nonstationary time series,” *Physica A*, vol. 316, no. 1, pp. 87–114, 2002.
- [10] J. Barral and P. Gonçalves, “On the estimation of the large deviations spectrum,” *J. Stat. Phys.*, vol. 144, no. 6, pp. 1256–1283, 2011.
- [11] P. Abry, S. Jaffard, and H. Wendt, “A bridge between geometric measure theory and signal processing: Multifractal analysis,” *Operator-Related Function Theory and Time-Frequency Analysis, The Abel Symposium 2012, K. Gröchenig, Y. Lyubarskii and K. Seip, Eds.*, vol. 9, pp. 1–56, 2015.
- [12] C. Esser, T. KleynstSENS, and S. Nicolay, “A multifractal formalism for non-concave and non-increasing spectra: The leaders profile method,” *Applied and Computational Harmonic Analysis (in press)*, 2016.
- [13] F. Bastin, C. Esser, and S. Jaffard, “Large deviation spectra based on wavelet leaders,” *Rev. Mat. Iberoam. (in press)*, 2016.
- [14] M. Costeniuc, R. S. Ellis, H. Touchette, and B. Turkington, “Generalized canonical ensembles and ensemble equivalence,” *Phys. Rev. E*, vol. 73, no. 2, pp. 026–105, 2006.
- [15] M. Costeniuc, R. S. Ellis, H. Touchette, and B. Turkington, “The generalized canonical ensemble and its universal equivalence with the microcanonical ensemble,” *Journal of Statistical Physics*, vol. 119, no. 5-6, pp. 1283–1329, 2005.
- [16] H. Touchette and C. Beck, “Nonconcave entropies in multifractals and the thermodynamic formalism,” *J. Stat. Phys.*, vol. 125, pp. 455–471, 2006.
- [17] G. Parisi and U. Frisch, “Fully developed turbulence and intermittency,” in *Turbulence and Predictability in geophysical Fluid Dynamics and Climate Dynamics*, M. Ghil, R. Benzi, and G. Parisi, Eds., Amsterdam, 1985, Proc. of Int. School, p. 84, North-Holland.
- [18] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, San Diego, CA, 1998.
- [19] H. Touchette, “Methods for calculating nonconcave entropies,” *J. Stat. Mech.*, vol. 2010, no. 05, pp. P05008, 2010.
- [20] G. Samorodnitsky and M. Taqqu, *Stable non-Gaussian random processes*, Chapman and Hall, New York, 1994.
- [21] S. Jaffard, “The multifractal nature of Levy processes,” *Probability Theory and Related Fields*, vol. 114, no. 2, pp. 207–227, 1999.