Tableaux Systems

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What this tutorial is about

• in focus
  – the tableaux method
  – … for logics with possible worlds semantics
  – … and combinations thereof
  – … as a computerized proof system (LoTREC)

• not in focus:
  – tableaus
  – proof theory, sequent calculi (cf. course on LDS)
  – completeness proofs
  – efficiency issues
Overview

- possible worlds semantics: quickstart
- tableaux systems: basic ideas
- tableaux systems: basic definitions
- tableaux for simple modal logics
- tableaux for transitive modal logics
- tableaux for intuitionistic logic
- tableaux for other nonclassical logics
- tableaux for modal logics with transitive closure and other modal and description logics
- tableaux for 1st order logic
- some implemented tableaux theorem provers
Possible worlds

- possible world → valuation of classical logic

\[
\begin{align*}
  w \models P & \iff V_w(P) = 1, \text{ for } P \text{ in Atoms} \\
  w \models A \land B & \iff (w \models A \text{ and } w \models B)
\end{align*}
\]
Possible worlds models

• possible worlds model
  = labeled graph
  = transition system

• node = possible world
  – valuation of classical logic
  – not every valuation appears
    (some logically possible worlds are not actually possible)
  – $V_w = V_u$ does not imply $w = u$

• link = accessibility relation $R$
Possible worlds models: accessibility relations

- **temporal**
  \[ R_{wu} \text{ iff } u \text{ is in the future of } w \]

- **alethic**
  \[ R_{wu} \text{ iff } u \text{ is possible, given the actual world } w \]

- **epistemic**
  \[ R_{iwu} \text{ iff } u \text{ is possible for agent } i, \text{ given the actual world } w \]

- **deontic**
  \[ R_{wu} \text{ iff } u \text{ is an ideal version of } w \]

- **dynamic**
  \[ R_{awu} \text{ iff } u \text{ is a possible result of the execution of program/action } a \text{ in } w \]

- **comparative (preferential, …)**
  \[ R_{wu} \text{ iff } w \text{ is smaller than } u \]
  \[ R_{vwu} \text{ iff } w \text{ is smaller than } u, \text{ given } v \]

reading of $R \Rightarrow$ properties of $R$
Possible worlds models: properties of $R$

- **monomodal**
  - serial: for all $w$ exists $u$ $Rwu$
  - reflexive
  - transitive
  - Euclidian
  - confluent (Church-Rosser)
  - dense
  - ...
  - well-founded (not FO-definable!)
  - ...

- **multimodal**
  - $R_1$ included in $R_2$
  - $R_1 = R_2 \cup R_3$
  - $R_2 = (R_1)^{-1}$ (transitive closure)
  - $R_2 = (R_1)^*$ (transitive closure)
  - $R_1 \cdot R_2 = R_2 \cdot R_1$
  - Church-Rosser
  - ...

- well-founded (not FO-definable!)
Language:
modal operators

• express intensional concepts (belief, time, action, obligation, …)
• non truth functional
• schema: \( \text{op}(a_1, \ldots, a_n) \), where \( \text{op} \) is the name of the operator, and \( a_i \) some argument
• generic form:
  – \( []A = A \) is necessary (true in all possible worlds)
  – \( <>A = A \) is possible
• in general: \( []A \) same as \( ~<>~A \)
  – except in substructural logics (intuitionistic, …)
Language: modal operators

- **temporal**
  - \[A\] = henceforth A (true in all future time points)
  - <>A = eventually A

- **deontic**
  - \[A\] = A is obligatory (true in all ideal worlds)
  - <>A = A is permitted \( (~<>A = A \text{ is forbidden}) \)

- **epistemic**
  - \(i\)A = i believes A (true in all worlds possible for i)
  - <>|A = ..

- **dynamic**
  - [a]A = A is true after (every possible way of) executing a
  - <a>A = ...

- **conditional**
  - A => B = if A then B
    proof of A can be transformed into proof of B (intuitionistic)
    if A was true then B would be true (counterfactual)
Interpreting the language: truth conditions

- classical connectives
  
  \[ w \Vdash P \quad \text{iff} \quad V_w(P) = 1, \text{ for } P \text{ in Atoms} \]
  
  \[ w \Vdash A \land B \quad \text{iff} \quad (w \Vdash A \text{ and } w \Vdash B) \]

- interpretation of non-classical connectives
  – via accessibility relation \( R \)

- schema:
  
  \[ w \Vdash \text{op}(a_1,\ldots,a_n) \quad \text{iff} \quad \text{Cond}(\text{op},a_1,\ldots,a_n,w,R) \]

- the basic modal operators:
  
  \[ w \Vdash [\Box]A \quad \text{iff} \quad \text{for all } u: \text{Rwu implies } u \Vdash A \]
  
  \[ w \Vdash \langle \rangle A \quad \text{iff} \quad \text{exists } u: \text{Rwu and } u \Vdash A \]
Examples of truth conditions

• multimodal operators
  \[ w \models []_i A \iff \forall u: R_i w u \implies u \models A \]
  \[ w \models <>_i A \iff \ldots \]

• relation algebra operators
  \[ w \models [\cdot]^1_i A \iff \forall u: R^{-1} w u \implies u \models A \]
  \[ w \models [\cdot]_{i \cup j} A \iff \forall u: (R_i \cup R_j) w u \implies u \models A \]
  \[ w \models [\cdot]^* A \iff \forall u: R^* w u \implies u \models A \]

• non-normal operators
  \[ w \models <> A \iff \forall R_i \exists u: R_i w u \text{ and } u \models A \]
  \[ w \models [] A \iff \exists R_i \forall u \ldots \]
Examples of truth conditions: temporal operators

- branching time operators

\[ w \models \exists X A \iff \exists R \in \text{Paths}(w) : R(w) \models A \]

(\text{Paths}(w) = \text{the set of paths going through } w)
Examples of truth conditions: temporal operators

- branching time operators

  $w \models \exists X A$ iff $\exists R \text{ in } \text{Paths}(w): R(w) \models A$
  
  ($\text{Paths}(w) = \text{the set of paths going through } w$)

  $w \models \forall <> A$ iff $\forall R \text{ in } \text{Paths}(w) \exists n R^n(w) \models A$
Examples of truth conditions: temporal operators

• binary temporal operators
  \[ w \vdash A \text{ Until } B \iff \text{exists } u: R^*wu \text{ and } u \vdash B \text{ and } \forall u' (R^*wu' \text{ and } R^+vu' \implies u' \vdash A) \]

  \[ w \vdash A \text{ Since } B \iff \ldots \]

  \[ w \vdash \forall (A \text{ Until } B) \iff \forall u \in \text{Paths}(w) \ldots \]
Examples of truth conditions: implications

- **intuitionistic implication**
  \[ w \models A \rightarrow B \iff \forall u : Rwu \implies u \models A \rightarrow B \]

- **conditional operator**
  \[ w \models A \rightarrow B \iff \forall u : R_{[A]}wu \implies u \models B \]

- **relevant implication**
  \[ w \models A \rightarrow B \iff \forall u,u' : \text{Rwuuu' implies (u \models A \implies u' \models B)} \]
Models

- model $M = (W, R, V)$
  - $W$ nonempty set (possible worlds)
  - $R: \text{Ops} \rightarrow (W \times W)$ (accessibility relation)
  - $V: W \rightarrow (\text{Atoms} \rightarrow \{0, 1\})$ (valuation)

- pointed model $((W, R, V), w)$
  - $w$ in $W$ (actual world)

- extension of $A$ in $M$
  \[ [A]_M = \{ w \in W : w \models A \} \]
Validity and satisfiability

- $K =$ the set of all models (Kripke)

- $A$ is valid in $K$ iff $[A]_M = W$ for all $M$ in $K$ (|=$_K A$)

  examples:
  
  $\square (P \lor \neg P)$
  $\square (P \land Q) \rightarrow \square P \land \square Q$
  $\square P \land \square Q \rightarrow \square (P \land Q)$

- $A$ is satisfiable in $K$ iff $[A]_M$ nonempty for some $M$ in $K$

  examples:
  
  $P$
  $P \land \neg \square P$
  $P \land \square \neg P$
  $\square P \land \neg \square \square P$
Validity and satisfiability in a class of models C

- C is some subset of K

- A is valid in C if \([A]_M = W\) for all M in C \((|=_{C} A)\)

  examples:
  - \([P \rightarrow P]\) invalid in K
  - \([P \rightarrow P]\) valid in the class of reflexive models
  - \(<><>P\) valid in transitive models

- A is satisfiable in C if \([A]_M\) nonempty for some M in C

  examples:
  - \(P \land \neg[P]\) satisfiable in K
  - \(P \land \neg[P]\) unsatisfiable in reflexive models

  A is valid in C if \(\neg A\) is unsatisfiable in C
Classes of models: examples

- {M: \text{card}(W) = 1}
  \vdash_{\text{Cls}} <>A \rightarrow []A
- {M: \text{card}(W) = 2}
  \vdash_{\text{Cls}} <> (A \land B) \land <> (\neg A \land B) \rightarrow []B
- {M: \text{card}(W) \text{ finite}}
  ...
- {M: R(\[]) \text{ reflexive}} = \text{KT}
  \vdash_{\text{KT}} []A \rightarrow A
- {M: R(\[]) \text{ transitive}} = \text{K4}
  \vdash_{\text{K4}} <> <> A \rightarrow <> A
- {M: R(\[]) \text{ equivalence relation}} = \text{S5}
  \vdash_{\text{S5}} A \rightarrow [[]<>A
Reasoning problems

• model checking
  given A, M and w, do we have w \models A?

• validity
  given A and Cls, is A valid in Cls?

• satisfiability
  given A and Cls, does there exist M in Cls and w in M such that w \models A?

How can we solve them automatically?
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The basic idea for classical logic [Beth]

- try to find M and w by applying the truth conditions ("tableau rules")
  - $w \vdash A \land B \implies$ add $w \vdash A$, and add $w \vdash B$
  - $w \vdash A \lor B \implies$ add either $w \vdash A$, or add $w \vdash B$ (nondet.)
  - $w \vdash \neg A \implies$ "don’t add $w \vdash A"$ ???
    - $w \vdash \neg \neg A \implies$ add $w \vdash A$
    - $w \vdash \neg (A \lor B) \implies$ add $w \vdash \neg A$, and add $w \vdash \neg B$
    - $w \vdash \neg (A \land B) \implies$ add either $w \vdash \neg A$, or add $w \vdash \neg B$

- apply while possible ("downwards saturation")
- is this a model?
  - NO if both $w \vdash P$ and $w \vdash \neg P$ ("tableau is closed")
  - ELSE: for every $w$, if $w \vdash P$ put $V_w(P) = 1$, else put $V_w(P) = 0$
The basic idea:
example for classical logic

\[ A = P \land \neg(P \land Q) \]

- applying truth conditions:
  1. \( w \models P \land \neg(P \land Q) \)
  2. \( w \models P \land \neg(P \land Q), w \models P, w \models \neg(P \land Q) \)
  3. \( w \models P \land \neg(P \land Q), w \models P, w \models \neg(P \land Q), w \models \neg P \) (nondet.)
- no more truth condition applies
- can’t be a model:
  both \( w \models P \) and \( w \models \neg P \)
- backtrack on nondeterministic choices
The basic idea:
example for classical logic (ctd.)

- 1st downward saturated graph for
  \[ A = P \land \neg(P \land Q) \]
  \[ \rightarrow \text{not a model} \]
  (contains \( P \) and \( \neg P \! )
The basic idea:
example for classical logic (ctd.)

- 1st downward saturated set for
  \( A = P \land \neg(P \land Q) \)
  \( \Rightarrow \) not a model
  (contains \( P \) and \( \neg P \! \))

- 2nd downward saturated set for
  \( A = P \land \neg(P \land Q) \)
  \( \Rightarrow \) is a model of \( A \)
The basic idea for modal logics

- apply truth conditions = build a graph
  - create nodes
  - add links between nodes
  - add formulas to nodes

- the basic cases
  \[ w \models []A \quad \Rightarrow \quad \text{forall } u \text{ such that } Rwu, \ add \ u \models A \]
  \[ w \models <>A \quad \Rightarrow \quad \text{add some new } u, \ add \ Rwu, \ add \ u \models A \]
  \[ w \models \neg []A \quad \Rightarrow \quad \text{add some new } u, \ add \ Rwu, \ add \ u \models \neg A \]
  \[ w \models \neg <>A \quad \Rightarrow \quad \text{...} \]

- “downwards saturated graph”: is this a model?
The basic idea:
example for modal logic

\[ A = P \land \neg \Box P \]

- applying tableau rules:
  1. \( w \vdash P \land \neg \Box P \)
  2. \( w \vdash P \land \neg \Box P, w \vdash P, w \vdash \neg \Box P \)
  3. \( w \vdash P \land \neg \Box P, w \vdash P, w \vdash \neg \Box P, R_{wu}, u \vdash \neg P \)
  no more tableau rule applies
  \( \Rightarrow \) never both \( w \vdash A \) and \( w \vdash \neg A \) ("open tableau")

- model can be built: \( M = (W,R,V) \)
  set of worlds \( W \): \( W = \{w,u\} \)
  accessibility relation \( R \): \( R_{wu} \)
  valuation \( V \): \( V_w(P) = 1, V_u(P) = 0 \)
The basic idea: example for modal logic (ctd.)

- premodel for
  \[ A = P \land \neg[]P \]

  ➔ not closed
  ➔ is a model of A
A remark on tableaux and truth tables

- Tableaux are a more convenient presentation of the familiar truth table analysis” [Beth]
- “Tableaux are more efficient than truth tables.” [folklore]
- … not exactly [d’Agostino]:
  \[ (P1 \lor P2 \lor P3) \land (P1 \lor P2 \lor \lnot P3) \land (P1 \lor \lnot P2 \lor P3) \land \ldots \]
  there are formulas with \( n \) atoms of length \( O(2^n) \)
  \[ \rightarrow \] truth tables have \( 2^n \) rows
  \[ \rightarrow \] at least \( n! \) closed tableaux, and \( n! \) grows faster than \( 2^n \)
Historical remarks

• the early days (1950-80): handwritten proofs
  – Beth, Gentzen
  – relation to sequent calculus
    “tableau proof = sequent proof backwards”
  – Kripke: explicit accessibility relation
  – Smullyan, Fitting: uniform notation

• today: mechanized systems
  – fast provers exist
    FaCT [Horrocks]
    K-SAT [Giunchiglia&Sebastiani]
    importance of strategies
  – applications exist: BDI logics, description logics
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Informal definition of tableau rules

- Tableau rules expand directed graphs by
  - adding formulas
  - adding nodes
  - adding links
  - duplicating the graph

- \( \text{rule}(G) = \{G_1, \ldots, G_n\} \)
Informal definition of tableau rules

- Tableau rules expand directed graphs by
  - adding formulas
  - adding nodes
  - adding links
  - duplicating the graph
- \( \text{rule}(G) = \{G_1, \ldots, G_n\} \)

- application of a rule to \( G \) = application to every formula in every node of \( G \).
- \( \text{rule}(\{G_1, \ldots, G_n\}) = \text{rule}(G_1) \cup \ldots \cup \text{rule}(G_n) \)
Tableau rules: syntax

• general form:
  
  rule ruleName
  if cond_1
  ...
  if cond_n
  do action_1
  ...
  do action_k

• example conditions:
  
  if hasElement node formula
  if isLinked node_1 node_2 R
  ...
  (more to come)

• example actions:
  
  do stop
  do addElement node formula
  do newNode node
  do link node_1 node_2 R
  do duplicate node_1 ...
  ...
  (more to come)
Example: tableau rules for classical logic

the LoTREC tableau prover
Example: tableau rules for classical logic

declaration of connectors: negation and conjunction only
Example: tableau rules for classical logic

rule Stop:
  if there is an explicit contradiction then stop exploring the tableau
Example: tableau rules for classical logic

Example: tableau rules for classical logic

```plaintext
// CLASSICAL PROPOSITIONAL LOGIC with "not" and "and"
connector not  1  false "~_"  5
connector and  2  true "_ & _"  4

rule Stop
  if hasElement node0  (variable A)
  if hasElement node0  not (variable A)
  do add node0 FALSE
  do stop node0
end

rule NotNot
  if hasElement node0  not not (variable A)
  do add node0  (variable A)
end

rule And
  if hasElement node0  and (variable A) (variable B)
  do add node0  (variable A)
  do add node0  (variable B)
end

rule NotAnd
  if hasElement node0  not and (variable A) (variable B)
  do duplicate node0  begin node0 node1 end
  do add node0  not (variable A)
  do add node1  not (variable B)
end
```

rule NotNot: replaces ~~A by A
Example: tableau rules for classical logic

rule And:
if A & B is in a node then add A and B to node

```plaintext
// CLASSICAL propositional logic with "not" and "and"
connector not 1 false "_" 5
connector and 2 true "_ & _" 4

rule Stop
if hasElement node0 (variable A)
if hasElement node0 not (variable A)
do add node0 FALSE
do stop node0
end

rule NotNot
if hasElement node0 not not (variable A)
do add node0 (variable A)
end

rule And
if hasElement node0 and (variable A) (variable B)
do add node0 (variable A)
do add node0 (variable B)
end

rule NotAnd
if hasElement node0 not and (variable A) (variable B)
do duplicate node0 begin node0 node1 end
do add node0 not (variable A)
do add node1 not (variable B)
end
```
Example: tableau rules for classical logic

rule NotAnd:
if ~\((A\&B)\) is in a node then duplicate tableau, add ~A to the first tableau add ~B to the second tableau
Definition of strategies

• A *strategy* defines some order of application of the tableau rules:
  
  **firstrule** \( rule_1 \ldots rule_n \) **end**  
  “apply first applicable rule and stop”
  
  **allrules** \( rule_1 \ldots rule_n \) **end**  
  “apply all applicable rules in order”
  
  repeat **strategy** **end**  
  “repeat until no rule applicable”
  
• Strategy stops if no rule is applicable.
strategy CPLStrategy
repeat allRules
  Stop
  NotNot
  And
  NotAnd
end end

⇒ “fair strategy”
Strategy for classical logic: example

CPLStrategy(P & ~ (P & Q))
Strategy for classical logic: example (ctd.)

\[ \text{CPLStrategy}(P \& \neg(P \& Q)) = \{ T1, T2 \} \]
Definition of tableaux

The set of tableaux for A with strategy S is the set of graphs obtained by applying the strategy S to an initial single-node graph whose root contains only A.

- notation: S(A)

- Remark
  our tableau = “tableau branch” in the literature (sounds odd to call a graph a branch)
Tableaux: open or closed?

- A node is closed iff it contains FALSE.
- A tableau is closed iff it has a closed node.
- A set of tableaux is closed iff all its elements are.

An open tableau is a premodel:

→ build a model
Formal properties
to be proved for each strategy:

• **Termination**
  For every A, S(A) terminates.

• **Soundness**
  If S(A) is closed then A is unsatisfiable.

• **Completeness**
  If S(A) is open then A is satisfiable.
Termination

• For every A, CPLTableaux(A) terminates.

• Proof:
  – Every tableau rule only adds strict subformulas.
  – This can only be done a finite number of times, then the strategy stops.
Soundness

• If CPLTableaux(A) is closed then A is unsatisfiable.

• Proof:
  – Every tableau rule is “guaranteed” by the truth conditions:
    If G is CPL-satisfiable
    then there is G_i in rule(G) that is CPL-satisfiable
  – Hence if every graph is closed
    then the original A cannot be satisfiable.
Completeness

• If CPLTableaux(A) is open then A is satisfiable.
• Proof:
  – Take some open tableau G in CPLTableaux(A).
Completeness

• If CPLTableaux(A) is open then A is satisfiable.
• Proof:
  – Take some open tableau G in CPLTableaux(A).
  – G is a downwards closed set (“Hintikka set”):
    if ~~A in node then A in node
    if A&B in node then A in node and B in node
    if ~(A&B) in node then ~A in node or ~B in node
    (because allRules strategy is fair: every rule eventually applies)
Completeness

- If CPLTableaux(A) is open then A is satisfiable.
- Proof:
  - Take some open tableau G in CPLTableaux(A).
  - G is a downwards closed set (“Hintikka set”):
    if ~A in node then A in node
    if A&B in node then A in node and B in node
    if ~(A&B) in node then ~A in node or ~B in node
    (because allRules strategy is fair: every rule eventually applies)
  - Build a CPL model from G:
    $V_{node}(P) = 1$ iff P appears in node
Completeness

• If CPLTableaux(A) is open then A is satisfiable.
• Proof:
  – Take some open tableau G in CPLTableaux(A).
  – G is a downwards closed set (“Hintikka set”):
    if ~A in node then A in node
    if A&B in node then A in node and B in node
    if ~(A&B) in node then ~A in node or ~B in node
    (because allRules strategy is fair: every rule eventually applies)
  – Build a CPL model from G:
    $V_{\text{node}}(P) = 1$ iff P appears in node
  – Prove by induction on the form of A:
    for every A in node, $V_{\text{node}}(A) = 1$
    (“fundamental lemma”)

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In general ...

- soundness proof ... easy
- termination proof ... difficult
- completeness proof ... very difficult
In general …

- soundness proof: easy
- termination proof: difficult
- completeness proof: very difficult

…but soundness + termination of strategy is practically sufficient:

1. apply strategy to $A$
2. take an open tableau and build pointed model $(M, w)$
3. check if $M$ in model class
4. check if $M, w \models A$
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The basic modal logic $K$

- the basic modal operators:
  \[
  w \models []A \quad \text{iff} \quad \text{forall } u: Rwu \implies u \models A
  \]
  \[
  w \models <>A \quad \text{iff} \quad \exists u: Rwu \text{ and } u \models A
  \]
Tableau rules for K

connectors: not, and, nec

[some rules for classical logic...]
Tableau rules for K

connectors: not, and, nec

[some rules for classical logic…]

createSuccessor:
  if not nec A is in node0
  then create new node node1
    link it to node0
    add not A to node1
  end
Tableau rules for K

connectors: not, and, nec

[some rules for classical logic…]

propagateNec:
if nec A is in node0
    node0 is linked node1 R
then add node1 A
end
Tableaux for K

• … plus rules for the definable connectives

• KStrategy(<>P & <>Q & [](R v <>S))
Modal logic KT

- accessibility relation is *reflexive*
- idea: integrate this into truth condition
  - $w \models \Box A$ iff $w \models A$ and forall $u$: $Rwu$ implies $u \models A$
Tableaux for modal logic KT

[connectors as for K…]

[rules as for K…]
Tableaux for modal logic KT

[connectors as for K…] [rules as for K…]

plus: “when []A is in a node then add A to it”

- KTStrategy(P & [][][~P])
Tableaux
for modal logic S5

accessibility relation is
equivalence relation

can be supposed to be
a single equivalence
class

optimized tableau rules

...
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Tableau rules for S4

accessibility relation is reflexive and transitive

tableau rules for S4:
• [connectors as for KT…]
• [rules as for KT…]
• … and take into account transitivity:
  “when []A is in a node
  then add []A to all children”
Tableau rules for S4

accessibility relation is reflexive and transitive

tableau rules for S4:
• [connectors as for KT…]
• [rules as for KT…]
• … and take into account transitivity:
  “if []A is in a node
  then add []A to all children”

problem: find a terminating strategy
Tableau rules for S4

- Example: $w \models [\diamondsuit \neg \diamondsuit]P$
  - add $w \models \neg \diamondsuit P$ (by rule for reflexivity)
Tableau rules for S4

- Example: $w \models []\neg []P$
  - add $w \models \neg []P$ (by rule for reflexivity)
  - create $u$, add $Rwu$, add $u \models \neg P$ (by createSuccessor)
Tableau rules for S4

• Example: \( w \models [\neg \neg P] \)
  - add \( w \models \neg \neg P \) (by rule for reflexivity)
  - create \( u \), add \( Rwu \), add \( u \models \neg P \) (by createSuccessor)
  - add \( u \models \neg \neg P \) (by rule for transitivity)
Tableau rules for S4

- Example: $w \models [\neg ]_P$
  - add $w \models \neg[P]$ (by rule for reflexivity)
  - create $u$, add $Ru$, add $u \models \neg P$ (by createSuccessor)
  - add $u \models [\neg ]_P$ (by rule for transitivity)
  - add $u \models \neg[P]$ (by rule for reflexivity)
Tableau rules for S4

- Example: $w \models [\neg] \neg [\neg] P$
  - add $w \models \neg [\neg] P$ (by rule for reflexivity)
  - create $u$, add $Rwu$, add $u \models \neg P$ (by createSuccessor)
    - add $u \models [\neg] \neg [\neg] P$ (by rule for transitivity)
    - add $u \models \neg [\neg] P$ (by rule for reflexivity)
    - create $u'$
    - ...

Tableau rules for S4

- Example: $w \models [\neg []P$
  - add $w \models \neg []P$ (by rule for reflexivity)
  - create $u$, add $Rw\,u$, add $u \models \neg P$ (by createSuccessor)
  - add $u \models [\neg []P$ (by rule for transitivity)
  - add $u \models \neg []P$ (by rule for reflexivity)
  - create $u'$
  - ...

  put a looptest into the rules!
principle:
• if a node is included in an ancestor then mark it.
Tableau rules for S4 (ctd.)

principle:
• if a node is included in an ancestor then mark it.

• if a node is marked then block the createSuccessor rule

• \textbf{S4Strategy}([\_\_\_] \sim [\_\_\_]P)
S4Strategy

$$\langle [] \leftrightarrow [] \rangle (P \lor Q) \land \langle [] \leftrightarrow \neg P \rangle \land \langle \rangle \land \neg Q)$$
Overview

- possible worlds semantics: quickstart
- tableaux systems: basic ideas
- tableaux systems: basic definitions
- tableaux for simple modal logics
- tableaux for transitive modal logics
- **tableaux for intuitionistic logic**
- tableaux for other nonclassical logics
- tableaux for modal logics with transitive closure and other modal and description logics
- tableaux for 1st order logic
- some implemented tableaux theorem provers
Intuitionistic logic

- no modal operators, but different semantics for implication and negation
- aim: invalidate
  \[ (\neg P \Rightarrow \text{FALSE}) \Rightarrow P \]
  \[ P \lor \neg P \]
  \[ (\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q) \]
- \( R \) is reflexive, transitive and hereditary:
  \[ \text{if } Rwu \text{ and } V_w(P) = 1 \text{ then } V_u(P) = 1 \]
- similar to S4
- truth condition
  \[ w \models A \Rightarrow B \text{ iff for all } u: Rwu \text{ implies } u \models A \Rightarrow B \]
Tableaux rules for intuitionistic logic

- follow translation from LJ to S4:

  \[
  \begin{align*}
  P' & = \Box P & \text{(inheritance)} \\
  (A \Rightarrow B)' & = \Box (A' \Rightarrow B') \\
  (~A)' & = \Box ~(A')
  \end{align*}
  \]

- tableaux similar to S4

- signed formulas

  \( T(P) \) “P is true”
  \( F(P) \) “P is false”
  \( F(P) \neq \sim P \)
Tableaux rules for intuitionistic logic

- create successor
- make $A \Rightarrow B$ false in $w$:
  - create $u$, add link $Rwu$,
  - make $A$ false in $u$,
  - make $B$ true in $u$
Tableaux rules for intuitionistic logic

- create successor
  make \( A \Rightarrow B \) false in \( w \):
  create \( u \), add link \( R_{wu} \),
  make \( A \) false in \( u \),
  make \( B \) true in \( u \)

- inheritance
  if \( w \models P \) and \( R_{wu} \) then add \( u \models P \)
Tableaux rules for intuitionistic logic: ~~P=>P

\[ \text{LJStrategy}(((P=>\text{False})=>\text{False})=>P) \rightarrow 4 \text{ tableaux, 1 open} \]
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Relevant logics

• ...

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Paraconsistent logics

• ...
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Linear Temporal Logic

- two modal operators:
  - \( [\] \) = always
  - \( X \) = next
- \( R(X) \) is serial and deterministic
- \( R([]) = R(X))^* \)
  - \( R([]) \) linear order
- mix axioms:
  - \( []A \leftrightarrow A \land X[]A \)
  - \( <>A \leftrightarrow A \lor X<>A \)
- induction axiom:
  - \( A \land [](A \rightarrowXA) \rightarrow []A \)
- decidable, EXPTIME complete
Tableau rules for Linear Temporal Logic

how take induction into account?

- solution: don’t care, and only apply the mix axioms:
  rewrite []A to A ∧ X[]A
  rewrite <>A to A ∨ X<>A
- only create successors for X, never for <>
- termination: use the looptest from transitive modal logics
  - nodes only contain subformulas of orig. formula
  - looptest succeeds at most at polynomial depth
Tableau rules for Linear Temporal Logic: example

• Example: w ⊩ []P

  add w ⊩ P ∧ X[]P (by mix axioms)
  add w ⊩ P, w ⊩ X[]P
  create u, add R_X w u, add u ⊩ []P

    (by propagation rule for X)

  add u ⊩ P ∧ X[]P (by mix axioms)
  add u ⊩ P, u ⊩ X[]P

w contains u: mark u “contained”
Tableau rules for Linear Temporal Logic (ctd.)

\[ \sim P \rightarrow \sim P \rightarrow \sim P \rightarrow \sim P \rightarrow \ldots \]

- may result in ‘nonstandard’ models of \( <>P \)
  - “\( P \) never fulfilled”
  - check if all \( <> \) are fulfilled!
Tableau rules for Linear Temporal Logic: example

- Example: $\text{LTLStrategy}(<>P)$
  
  \[
  w \models <>P
  \]
  
  \[
  \]
  
  \[
  \]
Tableau rules for Linear Temporal Logic

- Example: \texttt{LTLStrategy}(<>P)
  
  \[
  \begin{align*}
  w \models & <>P \\
  w \models & P \lor X<>P \\
  \end{align*}
  \]

  (by mix)
Tableau rules for Linear Temporal Logic

- Example: LTLStrategy(<>P)

\[
\begin{align*}
\text{w} & \models <>P \\
\text{w} & \models P \lor X<>P \\
\text{w} & \models <>P, \text{w} & \models P, \quad \text{w'} & \models <>P, \text{w'} & \models X<>P
\end{align*}
\]

(by mix)

\[
\begin{align*}
\text{w} & \models <>P, \text{w} & \models P \\
\text{w'} & \models <>P, \text{w'} & \models X<>P
\end{align*}
\]

(nothing applies)
Tableau rules for Linear Temporal Logic

- Example: \textbf{LTLStrategy(<>P)}

\[
\begin{align*}
  w ||- &<>P \\
  w ||- &P \lor X<>P & \text{(by mix)} \\
  \end{align*}
\]

\[
\begin{align*}
  w ||- &<>P, w ||- P & w' ||- <>P, w' ||- X<>P \\
  (nothing applies) & R_x w'u', u' ||- <>P \\
  \end{align*}
\]
Tableau rules for Linear Temporal Logic

- **Example:** \texttt{LTLStrategy(<>P)}

\[
\begin{align*}
    w &\vdash <>P \\
    w &\vdash P \lor X<>P \quad \text{(by mix)} \\
    w &\vdash <>P, w &\vdash P \\
    w' &\vdash <>P, w' &\vdash X<>P \\
    (\text{nothing applies}) \\
    R_xw'u', u' &\vdash <>P \\
    u' &\vdash P \lor X<>P \quad \text{(by mix)}
\end{align*}
\]
Tableau rules for Linear Temporal Logic

- Example: \texttt{LTLStrategy(<>P)}

\[
\begin{align*}
  w & \models <>P \\
  w & \models P \lor X<>P & \text{(by mix)} \\
  w & \models <>P, w & \models P & w' & \models <>P, w' & \models X<>P & \text{(nothing applies)} \\
  & \quad & & R_xw'u', u' & \models <>P \\
  & \quad & & u' & \models P & u'' & \models X<>P & \text{(by mix)} \\
  & \quad & & u' & \models P & u'' & \models X<>P
\end{align*}
\]
Tableau rules for Linear Temporal Logic

- Example: $\text{LTLStrategy}(<>P)$

\[
\begin{align*}
    w & \models <>P \\
    w & \models P \lor X<>P \\
    (nothing applies) & \\
    w & \models <>P, w & \models P \\
    w' & \models <>P, w' & \models X<>P \\
    (nothing applies) & \\
    R_X w' & \models u', u' & \models <>P \\
    u' & \models P \lor X<>P \\
    (nothing applies) & \\
    u' & \models P \\
    u'' & \models X<>P \\
    \text{contained in } w'
\end{align*}
\]
Tableau rules for Linear Temporal Logic

Example: \texttt{LTLStrategy(<>P)}

\[
\begin{align*}
  w |- <>P \\
  w |- P v X<>P
\end{align*}
\]
(by mix)

\[
\begin{align*}
  w |- <>P, w |- P \\
  w |- <>P, w |- P v X<>P
\end{align*}
\]
(nothing applies)

\[
\begin{align*}
  w' |- <>P, w' |- X<>P \\
  R_x w'u', u' |- <>P \\
  u' |- P v X<>P
\end{align*}
\]
(by mix)

\[
\begin{align*}
  u' |- P \\
  u'' |- X<>P
\end{align*}
\]
(nothing applies)

\[
\begin{align*}
  u'' |- X<>P \\
  u'' contained in w' \\
  <>P not fulfilled
\end{align*}
\]
Propositional dynamic logic (PDL)

• two kinds of expressions
  – formulas:
    \[ A ::= \mathbf{P} \mid \neg A \mid A \land B \mid [\pi]A \]
  – programs:
    \[ \pi ::= a \mid \pi_1;\pi_2 \mid \pi_1 \cup \pi_2 \mid \pi^* \mid A? \]

• in the models: \( R \) interprets programs
  \[
  \begin{align*}
  R(\pi_1;\pi_2) &= R(\pi_1);R(\pi_2) \\
  R(\pi_1 \cup \pi_2) &= R(\pi_1) \cup R(\pi_2) \\
  R(\pi^*) &= (R(\pi))^* \\
  R(A?) &= \{<w,w>: w \models A\}
  \end{align*}
  \]
Tableaux for PDL

• similar to LTL:
  – expand $[\pi^*]A$ to $A \land [\pi][\pi^*]A$
  – don’t apply createSuccessor to formulas $\neg[\pi^*]A$
  – mark nodes that are included in some ancestor
  – don’t apply createSuccessor to formulas $\neg[\pi]A$ if node is marked
  – expand the other program expressions:
    $[\pi_1;\pi_2]A \leftrightarrow [\pi_1][\pi_2]A$
    $[\pi_1\cup\pi_2]A \leftrightarrow [\pi_1]A \land [\pi_2]A$
    $[A?]B \leftrightarrow A\rightarrow B$
Description logics

- “roles” and “concepts”
  - more expressive than classical propositional logic
  - less expressive than 1st order logic
- focus on decidable logics
- applications:
  - databases
  - software engineering
  - web-based information systems
    - description of medical terminology
  - ontology of the semantic web
    - standards: DAML+OIL, OWL
  - description of web services
    - WSDL, OWL-S
Description logics: concepts and roles

- roles = binary relations
  - hasChild
  - hasHusband

- concepts = unary relations = properties
  - Person
  - Female
  - Parent ∩ Female
  - Father U Mother
  - ~Parent
  - ∃ hasChild.Female "individuals having a female child"
  - ∀ hasChild.Female "…"
  - >1 hasChild.T "individuals having more than 1 child"

- set of concepts ⇒ “assertion box” (ABox)
Description logics: TBoxes

• set of relations between concepts and roles

→ “terminological box” (TBox)
  – restricted to concept abbreviations (sometimes: fixpoint definitions)

Mother = Person ∩ Female

→ are expanded away → TBox = ∅
Description logics: reasoning tasks

- satisfiability of a concept $C$
- subsumption of $C_1$ by $C_2$
  - same as: $C_1 \cap \neg C_2$ unsatisfiable
- equivalence of $C_1$ by $C_2$
  - same as: $C_1$ subsumes $C_2$ and $C_1$ subsumes $C_2$
- disjointness of $C_1$ and $C_2$
  - $\bot$ subsumes $C_1 \cap C_2$

$\Rightarrow$ all reasoning tasks reduce to concept satisfiability
Description logics

• translation of concepts into modal logics

\[ \exists \text{hasChild}. \text{Female} \quad = \quad \langle \text{hasChild} \rangle \text{Female} \]
\[ \forall \text{hasChild}. \text{Female} \quad = \quad [\text{hasChild}. \text{Female}] \]
\[ \text{Parent} \cap \text{Female} \quad = \quad \text{Parent} \land \text{Female} \]
\[ \text{Father} \cup \text{Mother} \quad = \quad \text{Father} \lor \text{Mother} \]
\[ <2 \text{hasChild}. \text{T} \quad = \quad [\text{hasChild}]_2 \text{T} \]
\[ \geq 2 \text{hasChild}. \text{T} \quad = \quad \langle \text{hasChild} \rangle_2 \text{T} \]

…modal logics with number restrictions

[Fattorosi&Barnaba, van der Hoek]
Description logics

- description logic $\text{ALC}$:
  - $\neg C$
  - $C_1 \cap C_2$
  - $C_1 \cup C_2$
  - $\exists R.C$
  - $\forall R.C$
  - $= \text{multimodal } K$
- description logic $\text{ALC}_{\text{reg}} =$
  - $\text{ALC} + \text{regular expressions on roles}$
  - $= \text{PDL}$
- all description logic reasoning tasks reduce to satisfiability checking in modal logics
- tableaux used as optimal decision procedures
Logics of action and knowledge

- 2 modal operators
  \( \text{Knw}_i A \) “agent i knows that A”
  \([a] A \) “after execution of action a, A holds”

- “product logics”:
  \( R_{\text{Knw}_i} \circ R_a = R_a \circ R_{\text{Knw}_i} \) (permutation)
  if \( wR_{\text{Knw}_i} u \) and \( wR_a v \) then exists \( t \) such that \( uR_a t \) and \( vR_{\text{Knw}_i} t \) (confluence)

- axiomatically:
  \( \text{Knw}_i[a]A \leftrightarrow [a]\text{Knw}_iA \)
  \( <a>\text{Knw}_iA \rightarrow \text{Knw}_i<a>A \)

  tableaux: ...
  ➔ problem: combination with transitivity
Belief-Desire-Intention logics

• [Bratman, Rao&Georgeff]
• 3 modal operators

  Bel_i A  “agent i believes that A”
  Desire_i A  “agent i desires that A”
  Intend_i A  “agent i intends that A”

• plus branching time logic
Modal logics with density

• accessibility relation is dense
  if Rwu then exists v : Rwv and Rvu

• ...
Non-normal modal logics

- no accessibility relation, but neighborhood functions: $N: W \rightarrow 2^2 W$
  
  $w \models []A$ iff exists $U$ in $N(w)$ forall $u$ in $U$: $u \models A$

  non-normal modal logic EM

- can be represented by a set of relations
  
  $w \models []A$ iff exists $R_i$ forall $u$ ($R_iwu$ implies $u \models A$)

  logic EM: “non-normal”

  not valid: $[]P \land []Q \rightarrow [](P \land Q)$
  
  but valid: $[](P \land Q) \rightarrow []P \land []Q$
Tableau rules for EM

• ...

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1st order logic

- How should we handle the quantifiers?
  \[ \forall x \ p(x) \land \neg p(a) \text{ is unsatisfiable} \]
  \[ \forall x \ p(x) \land \exists x \ \neg p(x) \text{ is unsatisfiable} \]

- naïve implementation [Beth, Smullyan]:
  
  ```
  if hasElement node0 forall x A(x)
  do createTerm t 
  do add node0 A(t)

  if hasElement node exists x A(x)
  do createNewConstant c
  do add node A(c)
  ```

  *(doesn’t exist in LoTREC yet)*

  ➔ problem: loops for satisfiable formulas
Herbrand Tableaux for 1st order logic

• 1st solution: restrict instantiation to Herbrand universe
  if hasElement node0 forall x A(x)
  do createHerbrandTerm t
  do add node0 A(t)
  
  (doesn't exist in LoTREC yet)

• ex.: $\exists x \ p(x,x) \land \exists x \forall y \neg p(x,y)$ satisfiable

  1. $\exists x \ p(x,x)$
  2. $\exists x \forall y \neg p(x,y)$
  3. $\forall y \neg p(a,y)$ (2), new constant
  4. $\neg p(a,a)$ (3), only Herbrand term
  5. $p(b,b)$ (1), new constant
  6. $\neg p(a,b)$ (3), Herbrand term

  no further instantiation of (3) is possible

• decision procedure for formulas without positive $\forall \ldots \exists$
Herbrand Tableaux for 1st order logic

- counterexample: $\forall x \exists y \ p(x,y)$ satisfiable
  1. $\forall x \exists y \ p(x,y)$
  2. $\exists y \ p(a,y)$ (1), Herbrand term
  3. $p(a,b)$ (2), new constant
  4. $\exists y \ p(b,y)$ (1), Herbrand term
  5. $p(b,c)$ (4), new constant
  6. ...

$\Rightarrow$ loops
Free-variable tableaux with unification

• 2nd solution: don’t instantiate at all
  – work with free variables
  – runtime skolemization of existential quantifiers
  – term unification

• ex.: $\forall x \exists y p(x,y) \land \forall x \exists y \neg p(x,y)$ satisfiable
  1. $\forall x \exists y p(x,y)$
  2. $\forall x \exists y \neg p(x,y)$
  3. $\exists y p(x_1,y)$ from (1), replace $x$ by free $x_1$
  4. $\exists y \neg p(x_2,y)$ from (2), replace $x$ by free $x_2$
  5. $p(x_1,f(x_1))$ from (3), Skolem function $f(x_1)$
  6. $\neg p(x_2,g(x_2))$ from (4), Skolem function $g(x_2)$

  stops: (5) and (6) don’t unify

• … but does not terminate in all cases (sure)
  else 1st order logic would be decidable
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LoTREC

- IRIT-CNRS Toulouse (Sahade, Gasquet, Herzig); accessible through www
- general theorem prover
- explicit accessibility relations
- easy to implement logics with symmetric accessibility relations etc.
  - back-and-forth rules
- inefficient
TableauxWorkBench (TWB)

- Australian National U. (Abate, Goré)
- general theorem prover
- close to Gentzen sequents
- accessibility relations remain implicit
- hard to implement logics with symmetric accessibility relations
  - temporal logic with future and past
  - converse of programs
LogicWorkBench (LWB)

- U. Bern (Jäger, Heuerding); accessible through www
- efficient algorithms for all the basic modal and temporal logics
- hard to implement a new logic
FaCT

- U. Manchester (Horrocks); open source
- fast decision procedure for description logics with inverse roles and qualified number restrictions
  = multimodal K + converse + number restrictions
- optimized backtracking: “backjumping”
KSAT

• U. Trento (Giunchiglia, Sebastiani)
• combines tableaux method with fast SAT solvers for classical propositional logic
  – call a SAT solver, where subformulas []A, <>B are viewed as atomic
  – SAT solver returns a tentative valuation
  – use modal tableau rules to generate children
    if inconsistent then there is no model
    else iterate
• very efficient
• exists for all basic modal logics
KSAT (ctd.)

- **KSAT([)](P\&Q) \& \langle\rangle\neg P)**
  - call SAT with set of clauses {[)](P\&Q), \langle\rangle\neg P}
  - SAT returns:
    - $V([)](P\&Q)) = 1$
    - $V(\langle\rangle\neg P) = 1$
  - apply createOneSuccessor and propagateNec:
    - $w \vdash [)](P\&Q), w \vdash \langle\rangle\neg P, Rw, u \vdash \neg P, u \vdash P\&Q$
  - call SAT with set of clauses {P,Q,\neg P}
  - SAT returns:
    - *set of clauses unsatisfiable*
  - [)](P\&Q) \& \langle\rangle\neg P is unsatisfiable in K
Conclusion

• search for models = exploit the truth conditions
• tableaux work both ways:
  – finding a model
  – refuting
• termination = decidability
• tableaux as optimal decision procedures
  ➔ description logics
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