An Extension of SOL-resolution to Theories with Equality

Robert Demolombe and Pilar Pozos Parra*

ONERA-Toulouse
2, Avenue Edouard Belin BP 4025, 31055 Toulouse Cedex 4, France.

Abstract. Two inference rules called E-lit and E-equ, are defined in order to improve the efficiency of automated deduction with respect to Paramodulation, in the context of theories with equality. These inference rules are incorporated into SOL-resolution, an efficient Ordered Linear resolution that can be used to generate consequences in a given production field. It is proved that this extension, called SOLE-resolution, is sound and complete. The SOLE-resolution has been implemented, and we give an example of non trivial deduction obtained with the SOLE-resolution.

1 Introduction

Automated deduction in the context of theories with equality raises difficult problems with respect to efficiency. Indeed, when the Paramodulation inference rule is applied without restrictions, it can lead to a huge number of useless clauses (see [8, 1, 9]). The first objective of this paper is to adapt an existing efficient strategy to the treatment of equality in order to reduce the generation of useless clauses.

Another difficult problem in the field of automated deduction is to define efficient and complete strategies for the derivation of consequences. Most of the works have the objective to be complete for the derivation of the empty clause, but it is definitely a non trivial problem to transform a deduction procedure which is complete for the derivation of the empty clause into another one which is complete for the derivation of consequences. A concrete application of consequence generation is hypothesis generation [4].

In [5–7] Inoue has defined an efficient strategy, called SOL-resolution, which is of the type of Ordered Linear resolutions, and which is complete to derive consequences that are in a given production field (see Definition 6 in section 3). That is why our second objective was to extend the SOL-resolution to the treatment of equality.

The solution, which is presented in this paper, is based on the definition of two inference rules, called E-lit and E-equ, and on the extension of SOL-resolution with these inference rules. It is proved that with these extensions the soundness and completeness of SOL-resolution is extended to theories with equality.

In section 2, we briefly recall fundamental results by Chang and Lee [3]. In section 3 we prove that these results can be used to prove that SOL-resolution is complete for theories with equality if we add to the theories their corresponding sets of generalized equality axioms. Finally, in section 4, are defined the inference rules E-lit and E-equ. The SOL-resolution is extended with these inference rules, this extension is called SOLE-resolution. It is proved that with the SOLE-resolution

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we do not need any more the generalized equality axioms. The SOLE-resolution has been implemented and an example of SOLE-deduction that has been obtained with our implementation is presented in this section.

2 Equality theory

In this section the minimum background about equality theory is presented. In [3] Chang and Lee give the following definitions and theorem.

For convenience, if an expression E (a clause, a literal or a term) contains a term t, we denote E by $E[t]$, and, if one single occurrence of t is replaced by a term s, the result is denoted by $E[s]$. For instance, for the literal $P(f(a), x)$, $P[f(a)]$ (resp. $P[a]$) may be used to denote the fact that the term $f(a)$ (resp. a) occurs in $P(f(a), x)$.

Definition 1. *(Paramodulation inference rule).* Let C_1 and C_2 be two clauses (called parent clauses) with no variable in common. If C_1 is $r = s \vee C_1'$, and C_2 is $L[t] \vee C_2'$, where $L[t]$ is a literal containing the term t and C_1' and C_2' are clauses, and t and r have a most general unifier σ, then infer $L\sigma[s\sigma] \vee C_1'\sigma \vee C_2'\sigma$, where $L\sigma[s\sigma]$ denotes the result obtained by replacing one single occurrence of $t\sigma$ in $L\sigma$ by $s\sigma$.

This inference rule can be graphically represented by[1]:

$$\frac{r = s \vee C_1' \qquad\qquad L[t] \vee C_2'}{L\sigma[s\sigma] \vee C_1'\sigma \vee C_2'\sigma}\; t\sigma \equiv r\sigma.$$

Definition 2. *(Set of equality axioms).* Let Σ be a set of clauses, then the *set of equality axioms* $K(\Sigma)$ for Σ is the set of clauses:

a. $x = x$
b. $\neg(x = y) \vee y = x$
c. $\neg(x = y) \vee \neg(y = z) \vee x = z$
d. $\neg(x_i = x_0) \vee \neg P(x_1, \ldots, x_i, \ldots, x_n) \vee P(x_1, \ldots, x_0, \ldots, x_n)$, for every n-place predicate symbol P occurring in Σ
e. $\neg(x_i = x_0) \vee f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, x_0, \ldots, x_n)$, for every n-place function symbol f occurring in Σ.

Definition 3. *(E-interpretation).* An *E-interpretation* I of a set of clauses Σ is an interpretation of Σ satisfying the following four conditions. Let α, β and γ be any terms in the Herbrand universe of Σ, and let L be a literal in I. Then

1. $(\alpha = \alpha) \in I$;
2. if $(\alpha = \beta) \in I$, then $(\beta = \alpha) \in I$;
3. if $(\alpha = \beta) \in I$ and $(\beta = \gamma) \in I$, then $(\alpha = \gamma) \in I$;
4. if $(\alpha = \beta) \in I$ and L' is the result of replacing some one occurrence of α in L by β, then $L' \in I$.

Definition 4. *(E-satisfiability).* A set Σ of clauses is called *E-satisfiable* if and only if there is an E-interpretation that satisfies all the clauses in Σ. Otherwise, Σ is called *E-unsatisfiable*.

Theorem 1. *(Chang and Lee)* Let Σ be a set of clauses and $K(\Sigma)$ be the set of equality axioms for Σ. Then Σ is E-unsatisfiable iff $\Sigma \cup K(\Sigma)$ is unsatisfiable.

[1] The notation $t\sigma \equiv r\sigma$ is used to express that the terms $t\sigma$ and $r\sigma$ are syntactically identical.

3 SOL-resolution for theories with equality

In this section we define a set of generalized equality axioms. We have proved the equivalence between the equality axioms and the generalized equality axioms, and we have proved the completeness of SOL-resolution for theories with equality that are completed with their generalized equality axioms.

Definition 5. *(Set of generalized equality axioms).* Let Σ be a set of clauses, then the *set of generalized equality axioms* $K^*(\Sigma)$ for Σ is the set of clauses[2]:

a*. $x = x$
b*. $\neg(x_i = x_0) \vee \neg P[x_i] \vee P[x_0]$, for every predicate symbol P occurring in Σ.

The following theorem shows the equivalence between $K(\Sigma)$ and $K^*(\Sigma)$.

Theorem 2. Let Σ be a set of clauses, $K(\Sigma)$ be the set of equality axioms for Σ, $K^*(\Sigma)$ be the set of generalized equality axioms for Σ. We have:

$$\vdash K^*(\Sigma) \leftrightarrow K(\Sigma)$$

Proof : The proof is by induction on the depth of terms[3]. □

Next lemma shows that E-unsatisfiability of a theory Σ is equivalent to its unsatisfiability provided Σ is extended by $K^*(\Sigma)$.

Lemma 3 Let Σ be a set of clauses and $K^*(\Sigma)$ be its set of generalized equality axioms. Then we have:

$$\Sigma \text{ is E-unsatisfiable iff } \Sigma \cup K^*(\Sigma) \text{ is unsatisfiable.}$$

Proof: Direct consequence of Theorem 1 and Theorem 2. □

Lemma 4 Let Σ and Σ_1 be two sets of clauses. We have:

$$\Sigma \cup K^*(\Sigma) \text{ is satisfiable iff } \Sigma \cup K^*(\Sigma) \cup K^*(\Sigma_1) \text{ is satisfiable.}$$

Proof :
($\Leftarrow$) Trivial.
($\Rightarrow$) The idea of the proof is to show that the Herbrand interpretation that satisfies $\Sigma \cup K^*(\Sigma)$ can be transformed into an Herbrand interpretation that satisfies $\Sigma \cup K^*(\Sigma) \cup K^*(\Sigma_1)$. □

The following Lemmas 5 and 6 respectively are the translations of Lemma 3 and Lemma 4 in terms of logical consequence.

We shall use the following notations. $\Sigma \vdash T$ denotes the fact that T is a logical consequence of Σ. $Th(\Sigma)$ denotes the set of clauses T such that $\Sigma \vdash T$. $\Sigma \vdash_E T$ denotes the fact that T is a logical consequence of $\Sigma \cup K(\Sigma) \cup K(\neg T)$, i.e. $\Sigma, K(\Sigma), K(\neg T) \vdash T$. $Th\_E(\Sigma)$ denotes the set of clauses T such that $\Sigma \vdash_E T$.

[2] When there is no risk of misinterpretation we use the notation $K^*(C)$ instead of $K^*(\{C\})$, where C is a clause or the negation of a clause.

[3] An extended version of this paper with detailed proof of all the lemmas and theorems is available on request.

Definition 6. *(Production field).* A production field P (see Inoue [6] Definition 2.1) is represented by a pair $< L, Cond >$, where L is a set of literals closed under instantiation and $Cond$ is a certain condition. A clause C belongs to the production field P if every literal in C is in L and C satisfies $Cond$.

$Th_P(\Sigma)$ denotes the set of clauses in $Th(\Sigma)$ that are in the production field P. $Th\_E_P(\Sigma)$ denotes the set of clauses in $Th\_E(\Sigma)$ that are in the production field P.

Lemma 5 Let Σ be a set of clauses and T be a clause. We have:

$$T \in Th\_E(\Sigma) \text{ iff } T \in Th(\Sigma \cup K^*(\Sigma) \cup K^*(\neg T)).$$

Proof: From Theorem 1 and Lemma 3. □

Lemma 6 Let Σ and Σ_1 be two set of clauses and T be a clause. We have:

$$T \in Th(\Sigma \cup K^*(\Sigma) \cup K^*(\neg T)) \text{ iff } T \in Th(\Sigma \cup K^*(\Sigma) \cup K^*(\neg T) \cup K^*(\Sigma_1)).$$

Proof: From Lemma 4. □

A *SOL-deduction* is a sequence of structured clauses (see Definition 7) that satisfies some conditions. To avoid redundancies we have not explicited here the definition of a SOL-deduction because it is very close to the definition of a SOLE-deduction (see Definition 9). In a SOL-deduction the E-inference rules are not allowed. The process of finding a SOL-deduction is called a *SOL-resolution*.

Theorem 7. (Completeness of SOL-resolution for theories with equality) Let P be a production field, Σ be a set of clauses and C be a clause. If T is a clause such that $T \notin Th\_E_P(\Sigma)$ and $T \in Th\_E_P(\Sigma \cup \{C\})$, then there is an SOL-deduction of a clause S from $\{\Sigma, K^*(\Sigma), K^*(\neg T), K^*(C)\} + C$[4] and P such that S subsumes T.

Proof: The proof is based on Lemma 5 and Theorem 4.7 by Inoue in [6]. □

This theorem shows that the SOL-resolution can be used for theories with equality, provided appropriate generalized equality axioms are added. With these additional axioms, the equality predicate can considered like any other predicate.

Notice that when the SOL-resolution is used to derive consequences (not only the empty clause), the set of axioms $K^*(\neg T)$ is unknown as long as T is unknown. However, if we are looking for consequences S formed with the predicates that occur in Σ, $K^*(\neg T)$ is included in $K^*(\Sigma)$, and $K^*(\neg T)$ can be ignored.

4 SOLE-resolution

In this section two inference rules are defined for the treatment of equality in the context of SOL-deductions. These inference rules play the same role as the K^* axioms presented in the previous section (except for the axiom $x = x$).

Definition 7. *(Structured clause).* A *structured clause* is a pair $< P, \boldsymbol{Q} >$, such that P is a clause and $\boldsymbol{Q}$ is an ordered clause (a sequence of distinct literals).

[4] If E is a set of clauses, $E + C$ is used to denote the set of clauses $E \cup C$, with the top clause C.

In the following it is assumed that clauses are ordered clauses. The inference rules E-lit and E-equ are to be used in function of the first literal that occur in $\boldsymbol{Q}$. Depending on the fact that this literal is formed with equality predicate or not, we use the appropriate rule.

Definition 8. *(Inference rules for equality).* The inference rules E-lit and E-equ are defined as follows.

E-lit Let $L[t] \vee C$ be a clause such that t is a term that occurs in L and the variable x does not occur in $L[t] \vee C$. From $L[t] \vee C$ we can infer the clause $L[x] \vee \neg(t = x) \vee \boxed{L[t]} \vee C$[5], where $L[x]$ denotes the result of replacing one single occurrence of t by x in L.
The rule can be graphically represented as follows:

$$\frac{L[t] \vee C}{L[x] \vee \neg(t = x) \vee \boxed{\mathrm{L[t]}} \vee C} \qquad \text{(E-lit)}$$

E-equ Let $r = s \vee C$ be a clause such that the first literal is formed with equality predicate. From $r = s \vee C$ we can infer the clause[6] $\neg L[r] \vee L[s] \vee \boxed{r = s} \vee C$.
The rule can be graphically represented as follows.

$$\frac{r = s \vee C}{\neg L[r] \vee L[s] \vee \boxed{\mathrm{r{=}s}} \vee C} \qquad \text{(E-equ).}$$

The next definition extends the definition of an SOL-deduction given by Inoue in [6] in order to take into account properties of equality. The only difference with the original Inoue's definition is that Inoue's definition does not involve the E-inference rules.

Definition 9. *(SOLE-deduction).* Let Σ be a set of clauses such that $x = x$ in Σ, C be a clause and P be a production field. An SOLE-deduction of a clause S from $\Sigma + C$ and P is a sequence of structured clauses $D_0, D_1, \ldots, D_n$ such that:

1. $D_0 =< \Box, \boldsymbol{C} >$.
2. $D_n =< S, \Box >$.
3. For each $D_i =< P_i, \boldsymbol{Q}_i >$, $P_i \cup Q_i$ is not a tautology.
4. For each $D_i =< P_i, \boldsymbol{Q}_i >$, Q_i is not subsumed by any Q_j with the empty substitution, where $D_j =< P_j, \boldsymbol{Q}_j >$ is a previous structured clause and $j < i$.
5. For each $D_i =< P_i, \boldsymbol{Q}_i >$, P_i belongs to the production field P.
6. $D_{i+1} =< P_{i+1}, \boldsymbol{Q}_{i+1} >$ is generated from $D_i =< P_i, \boldsymbol{Q}_i >$ according to the following steps:
   (a) Let l be the left-most literal of $\boldsymbol{Q}_i$. P_{i+1} and $\boldsymbol{R}_{i+1}$ are obtained by applying one of the following rules:
      i. **Skip:** If $P_i \cup \{l\}$ belongs to P, then $P_{i+1} = P_i \cup \{l\}$ and $\boldsymbol{R}_{i+1}$ is the ordered clause obtained by removing l from $\boldsymbol{Q}_i$.

[5] Framed literals are used to keep memory of eliminated literals.
[6] If L is a negative literal of the form $\neg P(t_1, \ldots, t_n)$, $\neg L$ is used to denote $P(t_1, \ldots, t_n)$.

ii. **Resolve:** If there is a clause B_i in $\Sigma \cup \{C\}$ such that $\neg k \in B_i$ and l and k are unifiable with the mgu σ, then $P_{i+1} = P_i\sigma$ and $\boldsymbol{R}_{i+1}$ is an ordered clause obtained by concatenating $\boldsymbol{B}_i\sigma$ and $\boldsymbol{Q}_i\sigma$, framing $l\sigma$, and removing $\neg k\sigma$ (*extension*).

iii. **E-inference:**

A. If t is a term in l and the variable x does not occur in D_i, then $P_{i+1} = P_i$ and $\boldsymbol{R}_{i+1}$ is an ordered clause obtained by **E-lit** by concatenating $l[x] \vee \neg(t = x)$ and $\boldsymbol{Q}_i$, framing $l[t]$.

B. If l is of the form $r = s$, where r and s are terms, and k is a literal formed with any predicate symbol, then $P_{i+1} = P_i$ and $\boldsymbol{R}_{i+1}$ is an ordered clause obtained by **E-equ** by concatenating $\neg k[r] \vee k[s]$ and $\boldsymbol{Q}_i$, framing $r = s$.

iv. **Reduce:** If either

A. P_i or $\boldsymbol{Q}_i$ contains an unframed literal k that is either different from l (*factoring*) or another occurrence of l (*merge*), or

B. $\boldsymbol{Q}_i$ contains a framed literal $\boxed{\neg k}$ (*ancestry*),

and l and k are unifiable with mgu σ, then $P_{i+1} = P_i\sigma$ and $\boldsymbol{R}_{i+1}$ is obtained from $\boldsymbol{Q}_i\sigma$ by deleting $l\sigma$.

(b) $\boldsymbol{Q}_{i+1}$ is obtained from $\boldsymbol{R}_{i+1}$ by deleting every framed literal not preceded by an unframed literal in the remainder (*truncation*).

The process of finding SOLE-deductions is called SOLE-resolution. The proof of completeness of SOLE-resolution is based on Theorem 7.

Theorem 8. (Soundness and completeness of SOLE-resolution) Let P be a production field, C be a clause and Σ be a set of clauses that contains $x = x$.
(1) Soundness of SOLE-resolution. If a clause S is derived using an SOLE-deduction from $\Sigma + \{C\}$ and P, then S belongs to $Th\_E_P(\Sigma \cup \{C\})$.
(2) Completeness of SOLE-resolution. If $T \notin Th\_E_P(\Sigma)$ and $T \in Th\_E_P(\Sigma \cup \{C\})$, then there is an SOLE-deduction of a clause S from $\Sigma + C$ and P such that S subsumes T.

Proof:
Completeness. The proof is based on the completeness of SOL-resolution for theories with equality (Theorem 7). Then, it is shown that SOL-deductions can be transformed into SOLE-deductions.
Soundness. Direct consequence of Theorem 2 and Theorem 4.7 by Inoue in [6]. □
Example: Let us consider an example in the context of robotics, which is intended to show how SOLE-resolution can be used in theories with equality to infer consequences that are in a given production field.

The predicates $R(x)$ and $O(x)$ respectively mean that a robot is at the position x and that there is an obstacle at the position x. It is assumed that there is an obstacle at the distance of one unit in front of the robot (formula (a)), and that if there is an obstacle it is necessarily at the position 3 or 7 (formula (b)).

(a) $\exists x \exists y \exists z (R(x) \wedge O(y) \wedge Plus(x, 1, z) \wedge z = y)$.
(b) $\forall x (O(y) \rightarrow y = 3 \vee y = 7)$.

We want to know the position of the robot.

To avoid to deal with all the machinery of arithmetics, instead of the formula $\exists x \exists y \exists z(R(x) \wedge O(y) \wedge y = x + 1)$ we have represented the information in (a) with the predicate $Plus(x, y, z)$ whose intuitive meaning is $z = x + y$. Then, we can use SOLE-resolution to "postpone" the treatment of arithmetic constraints. For that purpose we define a production field $< L, Cond >$, where L is the set of instances of the literals $Plus(x, y, z)$ and $\neg Plus(x, y, z)$, and the condition $Cond$ is $True$ (i.e. no condition).

Then, we have:

$\Sigma \cup C = \{(\mathbf{1})\ R(\alpha),\ (\mathbf{2})\ O(\beta),\ (\mathbf{3})\ Plus(\alpha, 1, \gamma),\ (\mathbf{4})\ \gamma = \beta,$
$(\mathbf{5})\ \neg O(y) \vee y = 3 \vee 7 = y,$ [7] $(\mathbf{6})\ x = x\}.$

If the top clause C is (5), we can generate the following SOLE-deduction.

$D_0. < \Box, \neg O(y) \vee y = 3 \vee 7 = y >$ (top)

$D_1. < \Box, \neg O(x) \vee \neg(y = x) \vee \boxed{\neg O(y)} \vee y = 3 \vee 7 = y >$ (E-lit)

$D_2. < \Box, \neg(y = \beta) \vee \boxed{\neg O(y)} \vee y = 3 \vee 7 = y >$ (res D_1,2)

$D_3. < \Box, \gamma = 3 \vee 7 = \gamma >$ (res D_2,4)

$D_4. < \Box, \neg Plus(x, y, \gamma) \vee Plus(x, y, 3) \vee \boxed{\gamma = 3} \vee 7 = \gamma >$ (E-equ)

$D_5. < \Box, Plus(\alpha, 1, 3) \vee \boxed{\gamma = 3} \vee 7 = \gamma >$ (res D_4,3)

$D_6. < Plus(\alpha, 1, 3), 7 = \gamma >$ (skip)

$D_7. < Plus(\alpha, 1, 3), Plus(x, y, 7) \vee \neg Plus(x, y, \gamma) \vee \boxed{7 = \gamma} >$ (E-equ)

$D_8. < Plus(\alpha, 1, 3) \vee Plus(x, y, 7), \neg Plus(x, y, \gamma) \vee \boxed{7 = \gamma} >$ (skip)

$D_9. < Plus(\alpha, 1, 3) \vee Plus(\alpha, 1, 7), \Box >$ (res D_8,3)

The intuitive meaning of the derived clause is $\alpha + 1 = 3 \vee \alpha + 1 = 7$. From that clause a specific program for the treatment of arithmetic constraints could be called to infer that the position α of the robot satisfies $\alpha = 2 \vee \alpha = 6$.

5 Conclusions

We have presented an extension of SOL-resolution which is complete for theories with equality. It has been proved that this extension is sound and complete. It is worth noting that the SOLE-resolution has been implemented, since there are not so many goal-directed equality strategies that have been implemented (see [9]).

In [2] Brand has presented a method which is based on the same intuitive idea as the E-lit inference rule. He defines a transformation of the initial set of clauses Σ into a new set of clauses Σ' such that, for example, the clause $\neg(t = x) \vee \neg(f(x) = y) \vee C(y)$ is in Σ' iff the clause $C(f(t))$ is in Σ, where $f(t)$ is the unique term that occurs in C. Therefore, the set of clauses generated by E-lit in a SOLE-deduction is a subset of Σ'. Moreover, the set of clauses generated by E-lit is smaller than Σ'

[7] To show how the symmetry of equality can be managed in SOLE-deductions we have commuted the operands in $y = 7$.

since the clauses generated by E-lit depend on the top clause while Σ' is independent of the top clause. For these reasons SOLE-resolution is more efficient.

According to Definition 8, the E-lit inference rule can be applied to a variable that occurs in a literal. For instance, $\neg(y = x) \vee L[y] \vee C$ can be generated from $L[x] \vee C$. Brand has prevented such application of his transformation, and he has proved that this restriction preserves the completeness. We plan to prove in the same way that this restriction preserves the completeness of SOLE-resolution, and we shall implement this restriction.

Another significant difference with Brand's method is that the SOLE-resolution has been proved to be complete for consequence generation, while Brand only considers the generation of the empty clause.

Finally, the SOLE-resolution takes benefit of Inoue's idea of restricting consequence generation to clauses that are in a given production field, and this restriction preserves completeness.

Then, even if E-lit is based on the same intuitive idea as Brand's method, SOLE-resolution is more efficient and offers new functionalities.

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