Full primitive recursion on monotone inductive types

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Consider an extension of system F by monotone inductive types as follows:

Types: type variables, $\rho \to \sigma$, $\rho \times \sigma$, $\forall\alpha\rho$ and $\mu\alpha\rho$.

Typed terms in natural deduction formulation without contexts:

$$
\begin{array}{rl}
(\mathrm{V}) & x^{\rho} \\
(\to\text{-I}) & (\lambda x^{\rho} r^{\sigma})^{\rho\to\sigma} \\
(\to\text{-E}) & (r^{\rho\to\sigma} s^{\rho})^{\sigma} \\
(\times\text{-I}) & \langle r^{\rho}, s^{\sigma}\rangle^{\rho\times\sigma} \\
(\times\text{-E}) & (r^{\rho\times\sigma}\mathsf{L})^{\rho},\ (r^{\rho\times\sigma}\mathsf{R})^{\sigma} \\
(\forall\text{-I}) & (\Lambda\alpha r^{\rho})^{\forall\alpha\rho} \text{ under the usual proviso} \\
(\forall\text{-E}) & (r^{\forall\alpha\rho}\sigma)^{\rho[\alpha:=\sigma]}
\end{array}
$$

$$(\mu\text{-I})\quad (C_{\mu\alpha\rho} t^{\rho[\alpha:=\mu\alpha\rho]})^{\mu\alpha\rho}$$

$$(\mu\text{-E})\quad (r^{\mu\alpha\rho} \mathrm{E}_\mu \sigma m^{\forall\alpha\forall\beta.(\alpha\to\beta)\to\rho\to\rho[\alpha:=\beta]} s^{\rho[\alpha:=\sigma]\to\sigma})^\sigma$$

$$(\mu\text{-E}^+)\quad (r^{\mu\alpha\rho} \mathrm{E}^+_\mu \sigma m^{\forall\alpha\forall\beta.(\alpha\to\beta)\to\rho\to\rho[\alpha:=\beta]} s^{\rho[\alpha:=\mu\alpha\rho\times\sigma]\to\sigma})^\sigma$$

m is the witness of monotonicity of $\lambda\alpha\rho$. This witness is here used for the elimination rules. (μ-E) provides for iteration, (μ-E$^+$) gives (full) primitive recursion:

$$(\beta_\mu)\quad (C_{\mu\alpha\rho} t)\mathrm{E}_\mu \sigma m s \mapsto s\Big(m(\mu\alpha\rho)\sigma(\lambda x^{\mu\alpha\rho}.x\mathrm{E}_\mu\sigma m s)t\Big)$$

$$(\beta^+_\mu)\quad (C_{\mu\alpha\rho} t)\mathrm{E}^+_\mu \sigma m s \mapsto$$
$$s\Big(m(\mu\alpha\rho)(\mu\alpha\rho\times\sigma)\Big(\lambda x^{\mu\alpha\rho}.\langle x, (\lambda x^{\mu\alpha\rho}.x\mathrm{E}^+_\mu\sigma m s)x\rangle\Big)t\Big)$$

Theorem: β-reduction is strongly normalizing (incl. $\beta_\to$, $\beta_\times$, $\beta_\forall$).

Proof method: reduction-preserving embedding into an extension of Mendler's LICS '87 system:

Types as before ($\times$ not needed), (μ-I) as before.

$$(\mu\text{-E}) \quad (r^{\mu\alpha\rho}\mathrm{E}_\mu \sigma s^{\forall\alpha.(\alpha\to\sigma)\to\rho\to\sigma})^\sigma$$

$$(\mu\text{-E}^+) \quad (r^{\mu\alpha\rho}\mathrm{E}^+_\mu \sigma s^{\forall\alpha.(\alpha\to\mu\alpha\rho)\to(\alpha\to\sigma)\to\rho\to\sigma})^\sigma$$

The corresponding β-reduction rules:

$$(\beta_\mu) \qquad (C_{\mu\alpha\rho}t)\mathrm{E}_\mu\sigma s \mapsto s(\mu\alpha\rho)(\lambda x^{\mu\alpha\rho}.x\mathrm{E}_\mu\sigma s)t$$

$$(\beta^+_\mu) \qquad (C_{\mu\alpha\rho}t)\mathrm{E}^+_\mu\sigma s \mapsto s(\mu\alpha\rho)(\lambda y^{\mu\alpha\rho}y)(\lambda x^{\mu\alpha\rho}.x\mathrm{E}^+_\mu\sigma s)t$$

Essential difference to Mendler: No restriction on forming $\mu\alpha\rho$. Mendler required α to occur only (not necessarily strictly) positively in ρ and based his normalization proof on this assumption.

The reduction-preserving embedding $-'$ of monotone inductive types into this system: The types are preserved and the terms are translated homomorphically except for the elimination rules.

(μ-E) $(r^{\mu\alpha\rho}\mathrm{E}_\mu \sigma m s)' := r'\mathrm{E}_\mu \sigma \hat{s}$ with
$\hat{s} := \Lambda\alpha\lambda z^{\alpha\to\sigma}\lambda y^\rho . s'(m'\alpha\sigma z y)$

(μ-E$^+$) $(r^{\mu\alpha\rho}\mathrm{E}^+_\mu \sigma m s)' := r'\mathrm{E}^+_\mu \sigma \hat{s}$ with $\hat{s} :=$
$\Lambda\alpha\lambda z_1^{\alpha\to\mu\alpha\rho}\lambda z_2^{\alpha\to\sigma}\lambda y^\rho . s'(m'\alpha(\mu\alpha\rho\times\sigma)(\lambda x^\alpha\langle z_1 x, z_2 x\rangle)y)$

This preserves β-reduction: easy calculation.

Theorem: This variant of Mendler's system is strongly normalizing.

Proof: First eliminate iteration by an easy embedding into the system without (μ-E) and (β_μ): Simply set $(r^{\mu\alpha\rho}\mathrm{E}_\mu \sigma s)' := r'\mathrm{E}^+_\mu \sigma \hat{s}$ with $\hat{s} := \Lambda\alpha\lambda z^{\alpha\to\mu\alpha\rho}.s'\alpha$ and everything else homomorphic. (In fact Mendler did not include (μ-E).)

Second step: Show strong normalization by extending a suitable proof of strong normalization for system F.

Essential tool: vector notation for multiple eliminations. Define $r\vec{s}$ for $\vec{s}$ being a list of terms, types, symbols L and R. An example: $r^{\mu\alpha\rho}s^{\forall\alpha.(\alpha\to\mu\alpha\rho)\to(\alpha\to\sigma_1\to(\sigma_2\times\sigma_3))\to\rho\to\sigma_1\to(\sigma_2\times\sigma_3)}t^{\sigma_1}\mathsf{R}$ shall denote a term of type σ_3.

(Lemma on head form) Every term has exactly one of the following forms:

(V)	$x^\rho \vec{s}$
($\to$-I)	$\lambda x^\rho r$
($\to$-R)	$(\lambda x^\rho r) s \vec{s}$
($\times$-I)	$\langle r, s \rangle$
($\times$-R)	$\langle r, s \rangle \mathsf{L} \vec{s}$, $\langle r, s \rangle \mathsf{R} \vec{s}$
($\forall$-I)	$\Lambda \alpha r$
($\forall$-R)	$(\Lambda \alpha r) \sigma \vec{s}$
(μ-I)	$C_{\mu\alpha\rho} t$
(μ-R)	$(C_{\mu\alpha\rho} t) \mathrm{E}^+_\mu \sigma s \vec{s}$

Definition of the set SN of terms by (strictly positive) induction:

(V) If $x^\rho \vec{s}$ is a term and the terms in $\vec{s}$ are in SN, then $x^\rho \vec{s} \in \mathsf{SN}$.

($\to$-I) If $r \in \mathsf{SN}$, then $\lambda x^\rho r \in \mathsf{SN}$.

($\beta_\to$) If $(\lambda x^\rho r) s \vec{s}$ is a term, $s \in \mathsf{SN}$ and $r[x^\rho := s]\vec{s} \in \mathsf{SN}$, then $(\lambda x^\rho r) s \vec{s} \in \mathsf{SN}$.

($\times$-I) If $r \in \mathsf{SN}$ and $s \in \mathsf{SN}$, then $\langle r, s \rangle \in \mathsf{SN}$.

($\beta_\times$) If $r\vec{s} \in \mathsf{SN}$ and $s \in \mathsf{SN}$, then $\langle r, s \rangle \mathsf{L} \vec{s} \in \mathsf{SN}$.
If $s\vec{s} \in \mathsf{SN}$ and $r \in \mathsf{SN}$, then $\langle r, s \rangle \mathsf{R} \vec{s} \in \mathsf{SN}$.

($\forall$-I) If $\Lambda\alpha r$ is a term and $r \in \mathsf{SN}$, then $\Lambda\alpha r \in \mathsf{SN}$.

($\beta_\forall$) If $(\Lambda\alpha r)\sigma\vec{s}$ is a term and $r[\alpha := \sigma]\vec{s} \in \mathsf{SN}$, then $(\Lambda\alpha r)\sigma\vec{s} \in \mathsf{SN}$.

(μ-I) If $C_{\mu\alpha\rho}t$ is a term and $t \in \mathsf{SN}$, then $C_{\mu\alpha\rho}t \in \mathsf{SN}$.

(β^+_μ) If $s(\mu\alpha\rho)(\lambda y^{\mu\alpha\rho}y)(\lambda x^{\mu\alpha\rho}.x\mathrm{E}^+_\mu\sigma s)t\vec{s} \in \mathsf{SN}$, $x^{\mu\alpha\rho} \notin \mathsf{FV}(s)$ and $(C_{\mu\alpha\rho}t)\mathrm{E}^+_\mu\sigma s\vec{s}$ is a term, then $(C_{\mu\alpha\rho}t)\mathrm{E}^+_\mu\sigma s\vec{s} \in \mathsf{SN}$.

Lemma: SN is the set of strongly normalizing terms.

(The set sn of strongly normalizing terms is defined by (strictly positive) induction via

If for every term r' with $r \to r'$ we have $r' \in \mathsf{sn}$, then $r \in \mathsf{sn}$.

Hence, sn is simply the accessible part of $\to$.)

Definition of τ-saturated sets. Let $\mathcal{M}$ be a set of terms. $\mathcal{M}$ is τ-saturated, if the following holds

(SN) If $r \in \mathcal{M}$, then $r \in \mathsf{SN}_\tau$.

(V) If $x^\rho \vec{s}$ is a term of type τ and the terms in $\vec{s}$ are in SN, then $x^\rho \vec{s} \in \mathcal{M}$.

($\beta_\to$) If $(\lambda x^\rho r) s \vec{s}$ is a term, $s \in \mathsf{SN}$ and $r[x^\rho := s]\vec{s} \in \mathcal{M}$, then $(\lambda x^\rho r) s \vec{s} \in \mathcal{M}$.

($\beta_\times$) If $r\vec{s} \in \mathcal{M}$ and $s \in \mathsf{SN}$, then $\langle r, s\rangle \mathsf{L} \vec{s} \in \mathcal{M}$. If $s\vec{s} \in \mathcal{M}$ and $r \in \mathsf{SN}$, then $\langle r, s\rangle \mathsf{R} \vec{s} \in \mathcal{M}$.

($\beta_\forall$) If $(\Lambda \alpha r)\sigma \vec{s}$ is a term and $r[\alpha := \sigma]\vec{s} \in \mathcal{M}$, then $(\Lambda \alpha r)\sigma \vec{s} \in \mathcal{M}$ and

(β_μ^+) If $s(\mu\alpha\rho)(\lambda y^{\mu\alpha\rho} y)(\lambda x^{\mu\alpha\rho}.x\mathrm{E}_\mu^+ \sigma s)t\vec{s} \in \mathcal{M}$, $x^{\mu\alpha\rho} \notin \mathsf{FV}(s)$ and $(C_{\mu\alpha\rho}t)\mathrm{E}_\mu^+ \sigma s\vec{s}$ is a term, then $(C_{\mu\alpha\rho}t)\mathrm{E}_\mu^+ \sigma s\vec{s} \in \mathcal{M}$.

Trivially, SN_τ (the terms of type τ in SN) is τ-saturated.

Let M be a set of terms. Let $\mathsf{cl}_\tau(M)$ be the least τ-saturated set containing $M \cap \mathsf{SN}_\tau$. (Definable by strictly positive induction)

By means of saturated sets we define predicates of strong computability with respect to an assignment of saturated sets for type variables (a candidate assignment) and finally show that every term is strongly computable under substitution. This works if there are constructions for saturated sets corresponding to the type constructs of the system for which the introduction rules and the elimination rules are sound. For $\to$, $\times$ and $\forall$ this is standard. It will be sketched for μ.

Fix $\lambda\alpha\rho$. Let $\Phi := (\Phi_\tau)_\tau$ be a family of mappings $\Phi_\tau : \mathsf{SAT}_\tau \to \mathsf{SAT}_{\rho[\alpha:=\tau]}$. Define for $\mathcal{M} \in \mathsf{SAT}_{\mu\alpha\rho}$

$M_I(\mathcal{M}) := \{r \in \Lambda_{\mu\alpha\rho} | \exists t \in \Phi_{\mu\alpha\rho}(\mathcal{M}).r = C_{\mu\alpha\rho}t\}$

$$M_E(\mathcal{M}) := \{r \in \Lambda_{\mu\alpha\rho} | \forall\sigma\forall\mathcal{N} \in \mathsf{SAT}_\sigma \\ \forall s \in \forall\mathcal{P}.(\mathcal{P} \to \mathcal{M}) \to (\mathcal{P} \to \mathcal{N}) \to \Phi(\mathcal{P}) \to \mathcal{N}.rs \in \mathcal{N}\}$$

and $\Psi_I : \mathsf{SAT}_{\mu\alpha\rho} \to \mathsf{SAT}_{\mu\alpha\rho}$ by setting

$$\Psi_I(\mathcal{M}) := \mathsf{cl}_{\mu\alpha\rho}(M_I(\mathcal{M}))$$

and $\mu_E(\Phi) :=$ any fixed-point of $\Psi_E : \mathsf{SAT}_{\mu\alpha\rho} \to \mathsf{SAT}_{\mu\alpha\rho}$, defined by

$$\Psi_E(\mathcal{M}) := \mathsf{cl}_{\mu\alpha\rho}(M_E(\mathcal{M})).$$

Because Ψ_E is monotone (it is even non-strictly positive) and $\mathsf{SAT}_{\mu\alpha\rho}$ is a complete lattice, such a fixed-point exists. Hence, $\mu_E(\Phi)$ is $\mu\alpha\rho$-saturated.

Lemma 1: $M_E(\mathcal{M}) \cap \mathsf{SN} \in \mathsf{SAT}$ (which implies $\Psi_E(\mathcal{M}) = M_E(\mathcal{M}) \cap \mathsf{SN}$).

Lemma 2: $M_I(\mathcal{M}) \subseteq \mathsf{SN}$ (which implies $\Psi_I(\mathcal{M}) \supseteq M_I(\mathcal{M})$).

Lemma 3: A $\mu\alpha\rho$-saturated set $\mathcal{M}$ is a pre-fixed-point of Ψ_I iff for all $t \in \Phi_{\mu\alpha\rho}(\mathcal{M})$, we have that $C_{\mu\alpha\rho}t \in \mathcal{M}$. A $\mu\alpha\rho$-saturated set $\mathcal{M}$ is a post-fixed-point of Ψ_E iff for all $r \in \mathcal{M}$, types σ, $\mathcal{N} \in \mathsf{SAT}_\sigma$ and $s \in \forall\mathcal{P}.(\mathcal{P} \to \mathcal{M}) \to (\mathcal{P} \to \mathcal{N}) \to \Phi(\mathcal{P}) \to \mathcal{N}$, we have $r\mathrm{E}_\mu^+\sigma s \in \mathcal{N}$.

Theorem: $\mu_E(\Phi)$ is a pre-fixed-point of Ψ_I (i.e. $\Psi_I(\mu_E(\Phi)) \subseteq \mu_E(\Phi)$).

We therefore have the required rules for calculating with $\mu_E(\Phi)$:

(μ-I) If $t \in \Phi_{\mu\alpha\rho}(\mu_E(\Phi))$, then $C_{\mu\alpha\rho}t \in \mu_E(\Phi)$.

(μ-E$^+$) If $r \in \mu_E(\Phi)$, σ type, $\mathcal{N} \in \mathsf{SAT}_\sigma$ and $s \in \forall\mathcal{P}.(\mathcal{P} \to \mu_E(\Phi)) \to (\mathcal{P} \to \mathcal{N}) \to \Phi(\mathcal{P}) \to \mathcal{N}$, then $r\mathrm{E}_\mu^+\sigma s \in \mathcal{N}$.

We thus get a direct proof of strong normalization of the system à la Mendler without iteration.

Analysis of the proof gives even more: We only used "any fixed-point of Ψ_E" and Ψ_E is (non-strictly) positive and does not apply Φ to its argument. It is now easy to define a term rewrite system of non-interleaving positive fixed-point types with some appropriate notion of β-reduction into which our variant of Mendler's system may be embedded in a reduction preserving way. The clauses for the definition of the embedding on terms may be simply read off the proofs of the above lemmas and the theorem by using modified realizability. The proof of strong normalization for this system of fixed-points goes along the same lines as before but is much more perspicuous. The new proof only allows for the trivial embedding in itself being read off. Hence, the direct proof is faithful to it in the sense of only reflecting the concepts modelled by the system.

Other results:

- There is also a direct proof of strong normalization for the system of monotone inductive types (following an idea of Thorsten Altenkirch written down in an ASCII note in 1993).
- Also corecursion (present in Mendler's original system) embeds into non-interleaving fixed-point types.
- Mendler's system embeds into non-interleaving strictly positive types including the second-order existential quantifier.
- One may define a system of monotone inductive types by requiring the monotonicity witness to appear in the introduction rule. This system embeds into some "dual" of Mendler's system and in turn embeds into non-interleaving positive inductive types.

- One may also require only special instances of the monotonicity witness leading to seemingly stronger systems (but being reduction equivalent because there is also an embedding of positive inductive types into monotone inductive types).

- Of course, the iterative versions of all of the systems readily embed into system F.

- Embedding non-interleaving fixed-point types into positive inductive types is possible after adding some η-rules. The interleaving of fixed-point types calls for permutative conversions for μ in order to embed them into systems of inductive types. Establishing strong normalization of this extended system is one of my major goals in the next future.

Finis