Monotone Fixed-Point Types and Strong Normalization

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Why extending F?

Impredicative encoding of inductive types only models iteration. This is unsatisfactory in two respects (in the example nat):

- Full primitive recursion derivable only for numerals
- Predecessor function takes a linear number of steps

Remedy: Extensions of F by full primitive recursion on inductive types.
But: Anomalies already show up for the predecessor.
Therefore: For simplicity only postulate the existence of an efficient predecessor, i. e. study extensions by fixed-point types.

Geuvers (1992) studies them ("retract types") for "positive type schemes". Our aim is to be free from positivity requirements.

Consider an extension of system F by fixed-point types as follows:

Types: type variables, $\rho \to \sigma$, $\rho \times \sigma$, $\forall \alpha \rho$ and $\mu \alpha \rho$.

Typed terms in natural deduction formulation without contexts:

(V) x^{ρ}

($\to$-I) $(\lambda x^{\rho} r^{\sigma})^{\rho \to \sigma}$

($\to$-E) $(r^{\rho \to \sigma} s^{\rho})^{\sigma}$

($\times$-I) $\langle r^{\rho}, s^{\sigma} \rangle^{\rho \times \sigma}$

($\times$-E) $(r^{\rho \times \sigma} \mathsf{L})^{\rho}$, $(r^{\rho \times \sigma} \mathsf{R})^{\sigma}$

($\forall$-I) $(\Lambda \alpha r^{\rho})^{\forall \alpha \rho}$ under the usual proviso

($\forall$-E) $(r^{\forall \alpha \rho} \sigma)^{\rho[\alpha := \sigma]}$

Abadi and Fiore (1996) study "recursive types with explicit isomorphism". Folding and unfolding may be done by term rules

(μ-I) If $t^{\rho[\alpha:=\mu\alpha\rho]}$ is a term, then $(\mathrm{C}_{\mu\alpha\rho}t)^{\mu\alpha\rho}$ is a term.

(μ-E) If $r^{\mu\alpha\rho}$ is a term, then $(r\mathrm{E}_{\mu})^{\rho[\alpha:=\mu\alpha\rho]}$ is a term.

The obvious beta rule for μ would be

$$(\beta_\mu) \qquad (\mathrm{C}_{\mu\alpha\rho}t)\mathrm{E}_{\mu} \mapsto t$$

Together with $\beta_{\to}$, $\beta_{\times}$ and $\beta_{\forall}$ normalization fails. We show this even for a naive restriction to monotone fixed-point types: Only use $\mu\alpha\rho$ if there is a monotonicity witness for $\lambda\alpha\rho$, i. e. if there is a term m of type $\forall\alpha\forall\beta.(\alpha \to \beta) \to \rho \to \rho[\alpha := \beta]$ (with $\beta \notin \{\alpha\} \cup \mathsf{FV}(\rho)$).

Two possibilities:

- Elimination-based, i. e. elimination rule as easy as before but witness m required in introduction rule: System $\mathsf{EMF}_{\mathsf{naive}}$
- Introduction-based, i. e. introduction rule unchanged but witness m required for elimination: System $\mathsf{IMF}_{\mathsf{naive}}$

More precisely: Terms of $\mathsf{EMF}_{\mathsf{naive}}$ are defined by extending the definition for F by

(μ-I) If $t^{\rho[\alpha:=\mu\alpha\rho]}$ is a term and $m^{\forall\alpha\forall\beta.(\alpha\to\beta)\to\rho\to\rho[\alpha:=\beta]}$ is a term, then $(\mathrm{C}_{\mu\alpha\rho} m t)^{\mu\alpha\rho}$ is a term.

(μ-E) If $r^{\mu\alpha\rho}$ is a term, then $(r\mathrm{E}_\mu)^{\rho[\alpha:=\mu\alpha\rho]}$ is a term.

Relation $\mapsto$ is extended for $\mathsf{EMF}_{\mathsf{naive}}$ by (β_μ) $\quad (\mathrm{C}_{\mu\alpha\rho} m t)\mathrm{E}_\mu \mapsto t$

For $\mathsf{IMF}_{\mathsf{naive}}$:

(μ-I) If $t^{\rho[\alpha:=\mu\alpha\rho]}$ is a term, then $(\mathrm{C}_{\mu\alpha\rho}t)^{\mu\alpha\rho}$ is a term.

(μ-E) If $r^{\mu\alpha\rho}$ and $m^{\forall\alpha\forall\beta.(\alpha\to\beta)\to\rho\to\rho[\alpha:=\beta]}$ are terms, then $(r\mathrm{E}_\mu m)^{\rho[\alpha:=\mu\alpha\rho]}$ is a term.

(β_μ) $(\mathrm{C}_{\mu\alpha\rho}t)\mathrm{E}_\mu m \mapsto t$

Lemma: $\mathsf{EMF}_{\mathsf{naive}}$ and $\mathsf{IMF}_{\mathsf{naive}}$ are not normalizing.

Proof (for $\mathsf{IMF}_{\mathsf{naive}}$): Set $1 := \forall\alpha.\alpha \to \alpha$, $\mathsf{IN1} := \Lambda\alpha\lambda x^\alpha.x$ (of type 1) and $\mathsf{p} := \mu\alpha.\alpha \to 1$. We get the trivial closed monotonicity witness $m := \Lambda\alpha\Lambda\beta\lambda f^{\alpha\to\beta}\lambda x^{\alpha\to 1}\lambda y^\beta.\mathsf{IN1}$ for $\lambda\alpha.\alpha \to 1$. Define $\omega := \lambda x^{\mathsf{p}}.(x\mathrm{E}_\mu m)x$ of type $\mathsf{p} \to 1$. Then $C_{\mathsf{p}}\omega : \mathsf{p}$ and therefore $\Omega := \omega(C_{\mathsf{p}}\omega) : 1$. We calculate

$$\Omega \mapsto_{\beta_\to} (\mathrm{C}_{\mathsf{p}}\omega)\mathrm{E}_\mu m(\mathrm{C}_{\mathsf{p}}\omega) \to_{\beta_\mu} \omega(\mathrm{C}_{\mathsf{p}}\omega) = \Omega$$

Observation: Monotonicity witness does not enter the definition of beta reduction and therefore cannot contribute to normalization in case of trivially monotone fixed-point types.

In a type assignment system the above proof exhibits that $(\lambda x.xx)(\lambda x.xx)$ of untyped lambda calculus is typable with (monotone) recursive types.

Mendler (1987) showed that non-positivity always leads to the typability of non-normalizing terms.

Proposal: Change the definition of beta reduction for $\mu\alpha\rho$. Variant EMF of $\mathsf{EMF}_{\mathsf{naive}}$ has instead of (β_μ) the rule

$$(\beta_\mu)' \qquad (\mathrm{C}_{\mu\alpha\rho} m t)\mathrm{E}_\mu \mapsto m(\mu\alpha\rho)(\mu\alpha\rho)(\lambda y^{\mu\alpha\rho} y)t$$

IMF similarly has

$$(\beta_\mu)' \qquad (\mathrm{C}_{\mu\alpha\rho} t)\mathrm{E}_\mu m \mapsto m(\mu\alpha\rho)(\mu\alpha\rho)(\lambda y^{\mu\alpha\rho} y)t$$

The argument for failure of normalization no longer works in EMF and IMF. We show it for IMF:

$$\Omega \mapsto_{\beta_\rightarrow} (\mathrm{C}_\mathsf{p}\omega)\mathrm{E}_\mu m(\mathrm{C}_\mathsf{p}\omega) \rightarrow_{\beta_\mu{}'} m\mathsf{pp}(\lambda y^\mathsf{p} y)\omega(\mathrm{C}_\mathsf{p}\omega) \rightarrow^2_{\beta_\forall} \rightarrow^3_{\beta_\rightarrow} \mathsf{IN1}$$

Theorem: The systems EMF and IMF are strongly normalizing.

Proof by reduction-preserving embedding into the system NPF of non-interleaved positive fixed-point types described next.

NPF has the standard beta rule but restricted formation of $\mu\alpha\rho$: $\mu\alpha\rho$ is only allowed if all free occurrences of α in ρ are at positive positions and not in some subexpression of the form $\mu\beta\sigma$. The definition has to speak about negative positions because we do not only consider strictly-positive fixed-point types. The second condition forbids interleaving and hence a real use of free parameters.

Interleaving μ-types are ruled out in NPF but not nesting! Define $\mathsf{nat} := \mu\alpha.1+\alpha$ (Use $\rho+\sigma := \forall\alpha.(\rho\to\alpha)\to(\sigma\to\alpha)\to\alpha$ with $\alpha\notin\mathsf{FV}(\rho)\cup\mathsf{FV}(\sigma)$) and

$$\mathsf{tree} := \mu\alpha.1+(\mathsf{nat}\to\alpha)$$

tree is clearly nested. The following type (due to Ulrich Berger) is not allowed although it is positive:

$$\mathsf{Tree} := \mu\alpha.1+((\mu\beta.1+(\alpha\to\beta))\to\alpha)$$

NPF lies between F_{ret} of Geuvers (1992) and M2Lμ of Leivant (1990). Both argue that the systems are strongly normalizing by reference to Mendler's 1987 system of some kind of inductive types which is strongly normalizing.

Nevertheless in the paper to this conference I give a direct proof of strong normalization. Reasons:

- I easily deduce strong normalization of Mendler's system from it and even cover Mendler's system without Mendler's positivity requirement which he essentially uses for his proof.
- I claim that my proof is a very easy extension of some proof of normalization for F (method: saturated sets). It is so easy because we only deal with fixed-points and not with inductive types which stand for least pre-fixed points.
- And also the eta rule

$$(\eta_\mu) \qquad \mathrm{C}_{\mu\alpha\rho}(r^{\mu\alpha\rho}\mathrm{E}_\mu) \mapsto r$$

is covered without any effort.

Before embedding EMF and IMF into NPF we study the opposite embeddings: Monotone is more general than (non-interleaving) positive. Nearly: One has to add the eta rules for system F. Then we may prove for the canonical monotonicity witnesses $\mathsf{map}_{\lambda\alpha\rho} : \forall\alpha\forall\beta.(\alpha \to \beta) \to \rho \to \rho[\alpha := \beta]$ and their comrades $\mathsf{comap}_{\lambda\alpha\rho} : \forall\alpha\forall\beta.(\alpha \to \beta) \to \rho[\alpha := \beta] \to \rho$ (for α negative in ρ) the first functor law, more precisely

$$\mathsf{map}_{\lambda\alpha\rho}\sigma\sigma(\lambda x^\sigma x)t^{\rho[\alpha:=\sigma]} \to^* t \text{ and } \mathsf{comap}_{\lambda\alpha\rho}\sigma\sigma(\lambda x^\sigma x)t^{\rho[\alpha:=\sigma]} \to^* t$$

An embedding $-'$ of NPF in EMF is now immediately defined by induction on the terms of NPF setting

$$(\mu\text{-I}) \quad (\mathrm{C}_{\mu\alpha\rho}t)' := \mathrm{C}_{\mu\alpha\rho}\mathsf{map}_{\lambda\alpha\rho}t'.$$

and homomorphic rules for all the other term formers.

Embedding EMF **into** NPF

Define for every type ρ of EMF a type ρ' of NPF with the only non-homomorphic rule:

$$(\mu\alpha\rho)' := \mu\alpha\forall\beta.(\alpha \to \beta) \to \rho'[\alpha := \beta]$$

Easy: we indeed get non-interleaving (non-strictly) positive fixed-point types

Lemma: $(\rho[\vec{\alpha} := \vec{\sigma}])' = \rho'[\vec{\alpha} := \vec{\sigma}']$.

Define for every term r^ρ of EMF a term r' of type ρ' of NPF (the other clauses are homomorphic):

$$(\mu\text{-I}) \quad (\mathrm{C}_{\mu\alpha\rho} m t)' := \mathrm{C}_{(\mu\alpha\rho)'}\Big(\Lambda\beta\lambda y^{(\mu\alpha\rho)'\to\beta}.m'(\mu\alpha\rho)'\beta y t'\Big)$$

$$(\mu\text{-E}) \quad (r^{\mu\alpha\rho}\mathrm{E}_\mu)' := r'\mathrm{E}_\mu(\mu\alpha\rho)'(\lambda x^{(\mu\alpha\rho)'}x)$$

[More precisely, we simultaneously have to prove that r' has the same type variables as r and that x^τ is a free variable of r' iff $\tau = \sigma'$ and x^σ is free in r.]

Lemma: $(r[\vec{x}^{\vec{\rho}} := \vec{s}^{\vec{\rho}}])' = r'[\vec{x}^{\vec{\rho}'} := \vec{s}'] $ and $(r[\vec{\alpha} := \vec{\sigma}])' = r'[\vec{\alpha} := \vec{\sigma}']$.

Embedding Lemma: If $r \to \hat{r}$, then $r' \to^+ \hat{r}'$.

Proof Induction on $r \to \hat{r}$: Cases $(\beta_\to)$ and $(\beta_\forall)$ follow immediately from the preceding lemma. $(\beta_\times)$ is trivial. (β_μ):

$$\begin{array}{rcl}
((\mathrm{C}_{\mu\alpha\rho} m t)\mathrm{E}_\mu)' & = & \mathrm{C}_{(\mu\alpha\rho)'}\left(\Lambda\beta\lambda y^{(\mu\alpha\rho)'\to\beta}.m'(\mu\alpha\rho)'\beta y t'\right)\mathrm{E}_\mu(\mu\alpha\rho)'(\lambda x^{(\mu\alpha\rho)'}x) \\
& \to & \left(\Lambda\beta\lambda y^{(\mu\alpha\rho)'\to\beta}.m'(\mu\alpha\rho)'\beta y t'\right)(\mu\alpha\rho)'(\lambda x^{(\mu\alpha\rho)'}x) \\
& \to & \left(\lambda y^{(\mu\alpha\rho)'\to(\mu\alpha\rho)'}.m'(\mu\alpha\rho)'(\mu\alpha\rho)' y t'\right)(\lambda x^{(\mu\alpha\rho)'}x) \\
& \mapsto & m'(\mu\alpha\rho)'(\mu\alpha\rho)'(\lambda x^{(\mu\alpha\rho)'}x)t' \\
& = & \left(m(\mu\alpha\rho)(\mu\alpha\rho)(\lambda x^{\mu\alpha\rho}x)t\right)'
\end{array}$$

Term closure clauses are easy. $\square$

Embedding IMF **into** NPF

We use second-order existential types $\exists\alpha\rho$ which may be encoded in a reduction-preserving way. Rules:

($\exists$-I) If $t^{\rho[\alpha:=\tau]}$ is a term, then $(\mathrm{C}_{\exists\alpha\rho,\tau}t)^{\exists\alpha\rho}$ is a term.

($\exists$-E) If $r^{\exists\alpha\rho}$ is a term, $s^{\forall\alpha.\rho\to\sigma}$ is a term and $\alpha \notin \mathsf{FV}(\sigma)$, then $(r\mathrm{E}_\exists s)^\sigma$ is a term.

($\beta_\exists$) $(C_{\exists\alpha\rho,\tau}t)\mathrm{E}_\exists \mapsto s\tau t$

Define for every type ρ of IMF a type ρ' of $\mathsf{NPF}_{\mathsf{ex}}$ with the only non-trivial rule:

$$(\mu\alpha\rho)' := \mu\alpha\exists\beta.(\beta \to \alpha) \times \rho'[\alpha := \beta]$$

We even get strictly-positive non-interleaving fixed-point types.

Define for every term r^ρ of IMF a term r' of type ρ' of $\mathsf{NPF}_{\mathsf{ex}}$ as follows (the other clauses are homomorphic):

$$
\begin{array}{rl}
(\mu\text{-I}) & (\mathrm{C}_{\mu\alpha\rho}t)' := \\
& \mathrm{C}_{(\mu\alpha\rho)'}\mathrm{C}_{\exists\beta.(\beta\to(\mu\alpha\rho)')\times\rho'[\alpha:=\beta],(\mu\alpha\rho)'}\langle\lambda x^{(\mu\alpha\rho)'}x, t'\rangle \\
(\mu\text{-E}) & (r^{\mu\alpha\rho}\mathrm{E}_\mu m)' := \\
& r'\mathrm{E}_\mu\mathrm{E}_\exists\Big(\Lambda\beta\lambda z^{(\beta\to(\mu\alpha\rho)')\times\rho'[\alpha:=\beta]}.m'\beta(\mu\alpha\rho)'(z\mathsf{L})(z\mathsf{R})\Big)
\end{array}
$$

Proving the lemmas on compatibility with type and term substitution is again trivial. The proof that $-'$ is indeed an embedding is again simply some calculation in case of (β_μ).

Conclusion

- Normalization fails for monotone fixed-point types with the standard reduction rule.
- Monotone fixed-point types with the new reduction rules have strong normalization (and confluence).
- If one has "good" monotonicity witnesses also an efficient predecessor may be defined.
- Non-interleaving positive fixed-point types have "good" monotonicity witnesses.

Hence, concerns are separated: Normalization on one hand, properties of monotonicity witnesses on the other.

Finis