Constraint networks of lines in the euclidian plane and space.

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Abstract

In this paper, we address the general problem of determining the complexity of solving path-consistent constraint networks of lines. We first extend some of the known results that hold for a finite set of lines in a euclidian plane, in order to take into account other possible relations between two parallel lines, such as their orientation, or simply the distance that separates them. We then show that for a finite set of lines taken in the space, our problem appears to be NP-hard, even for an atomic network.

Keywords

Spatial reasoning - Constraint satisfaction problems - Qualitative geometry.

1 Introduction

The growing field of qualitative spatial reasoning finds its application in a variety of areas, such as Geographic Information Systems [9,10], Databases [19], Multimedia [24], Computer Vision and Artificial Intelligence. Most researchers have dealt with three main classes of spatial information. *Topological* relations describe how the interiors, the exteriors and the boundaries of two regions relate [3,7,9,21,22]. *Directional* relations describe the relative position of two regions to one another [11,14,15,25]. For instance *a south b* is a directional relation. *Distance* relations describe the relative distance of two regions [10,28]. In artificial intelligence, the variables usually hold for points [5,6] or mainly for regions [21]. The latter alternative is more often preferred because a region takes into account the space that might be occupied by an object, and thus enables us to study a wider class of common problems. Recently, among other problems, Balbiani [2] considered reasoning with lines in the euclidian plane, where the natural basic qualitative relations between two lines are the equality, the intersection and the parallelism. His aim was to provide a model for describing and representing objects in space, based on the study of networks of constraints. A constraint network describes the objects of interest, whereas the consistency problem of networks enlightens us on their spatial representation. In the current paper, we extend the relations of lines in a plane in order to account for properties like the distance separating two parallel lines. Then we complement those relations to reasoning with lines in the euclidian space. The article is organized as follows. In section 2, we introduce relevant notions concerning constraint networks in general, and recall some basic results about the consistency problem of lines in the euclidian plane. In section 3, we slightly modify the basic relations describing two parallel lines. For a given line, we wish to be able to say if a parallel line "precedes" or "follows" it. In section 4, our attention shifts to considering the distance separating two parallel lines and the consistency problem that arise. In section 5, we establish that the complexity problem of networks of lines in the euclidian space is NP-hard, even for atomic ones.

2 Lines in dimension 2.

Some of the results about reasoning with lines in a euclidian plane, established in [2], will be recalled.

From now on, by PO, EQ, DC we respectively mean: "having one point in common", "equal", "parallel and distinct". Two lines in the plane, denoted by d and d', are exactly in one of the following relations: $d\{PO\}d'$, $d\{EQ\}d'$, $d\{DC\}d'$. Let $E = \{PO, EQ, DC\}$ be the set of the jointly exhaustive and pairwise disjoint relations that compare the position of any couple of lines in a plane.

The definitions needed for describing a constraint satisfaction problem (CSP), are generally defined by Montanari [18]. They will be needed for the remaining of this paper. A network of linear constraints $\mathcal{R}$ is a couple (I, C), where $I \subseteq I\!N$ is a finite set of variables, and C is a mapping from I^2 to the set of the subsets of E (i.e. 2^E). The network $\mathcal{R}$ is atomic if for all $i, j \in I$, $Card(C(i,j)) > 1$ then $C(i,j) = E$. We say that $\mathcal{R}$ is path-consistent if for all $i, j, k \in I$, $C(i,j) \subseteq C(i,k) \circ C(k,j)$. A scenario (or an assignment) is a function V that maps I to a set of lines, in the euclidian plane or space. A scenario is consistent if for all $i, j \in I$, the relation that holds between the lines $V(i) = d_i$ and $V(j) = d_j$ is in $C(i,j)$. A relation $r \in C(i,j)$ is feasible if there exists a consistent scenario such that the lines d_i and d_j are in the relation r. A network of constraints is minimal if every relation in it is feasible. The notion of $k-consistency$, adapted from Freuder [12], leads naturally to strong k-consistency, as defined by Vilain and Kautz [27]. We say that $\mathcal{R}$ is k-consistent if, given any consistent assignment of $k-1$ of its variables to lines in a euclidian plane, there exists a consistent assignment of any k^{th} variable of the network. It is strongly k-consistent if it is j-consistent for all $j \leq k$. A network is said to be globally consistent if it is strongly n-consistent, for $n = Card(I)$. Given a network of constraints, the question that arises is whether or not there is a consistent scenario for it. The algorithm of path consistency is explored and analyzed in [16,17]. The constraints propagation algorithm due to Allen [1], that replaces each constraint $C(i,j)$ by $C(i,j) \cap (C(i,k) \circ C(k,j))$, transforms in polynomial time each network $\mathcal{R}$ into a path-consistent one, whose set of consistent scenarios is the same as for $\mathcal{R}$.

***Exemple* 1** *Path-consistent but not globally consistent network of linear constraints:* $I = \{1,2,3,4,5\}$ *and* $C(1,2) = C(1,5) = C(3,4) = C(3,5) = \{PO, EQ\}$, $C(1,3) = C(1,4) = C(2,3) = C(2,4) = \{PO, DC\}$, $C(2,5) = \{PO, DC, EQ\}$, $C(4,5) = \{DC, EQ\}$. *Indeed, if we consider its subnetwork where* $C(1,2) = C(3,4) = \{EQ\}$ *and* $C(1,3) = C(1,4) = C(2,3) = C(2,4) = \{DC\}$, *we find it impossible* *to consistently assign the fifth variable. We can easily check that the above network is minimal.*

Proposition 1 *The consistency problem of networks of linear constraints is decidable in polynomial time, with mean of a deterministic algorithm.*

This result is due to Balbiani [2].
Later on, any given line d_i of the euclidian plane will be invariably referred to by the couple of reals (a_i, b_i), where $y = a_i x + b_i$ is its algebraic equation. Indeed, as the constraint networks we consider are finite, we take care during the construction of a scenario V not to assign to a variable i a parallel line to the the y axis, in order to avoid having equations of the form $x = c$, where $c \in I\!R$.

3 Oriented parallelism

Given two parallel lines in the euclidian plane, we give an orientation to their respective positions. Informally, for a line d_i, we wish to be able to distinguish the parallel lines that lie to its left from those that are to its right.

Definition 1 *For any two lines d_i and d_j of the euclidian plane:*

- $d_i\{<\}d_j$ *iff* $(a_i = a_j) \wedge (b_i < b_j)$.
- $d_i\{>\}d_j$ *iff* $(a_i = a_j) \wedge (b_i > b_j)$.
- *An oriented 2D network is a path-consistent linear constraint network of lines in a euclidian plane, where the relations are taken from the set $E = \{PO, EQ, <, >\}$.*

They are the inverse of each other (i.e. $d_i\{<\}d_j$ iff $d_j\{>\}d_i$). We have the equivalence $d_i\{DC\}d_j \Leftrightarrow ((d_i < d_j) \vee (d_i > d_j))$. The composition table of these relations is given in figure 1. We used the consistency-based composition of relations [4] to compute it: if $\alpha, \beta \in E$, $\alpha \circ \beta$ contains all the relations $\gamma \in E$ such that there exists lines d_i, d_j, d_k of the plane satisfying $d_i\{\alpha\}d_j$, $d_j\{\beta\}d_k$ and $d_i\{\gamma\}d_k$.

∘	EQ	<	>	PO
EQ	EQ	<	>	PO
<	<	<	<, >, EQ	PO
>	>	<, >, EQ	>	PO
PO	PO	PO	PO	<, >, EQ,PO

Figure 1. Composition table of oriented 2D networks.

***Exemple* 2** *Oriented 2D network which isn't minimal:* $I = \{1, 2, 3, 4\}$ *and* $C(2, 1) = C(3, 1) = C(4, 1) = C(4, 2) = C(4, 3) = \{>, EQ\}$, $C(2, 3) = \{<, >\}$. *Indeed, the relation between the first and the fourth variables can't be reduced to* $\{EQ\}$.

Proposition 2 *The consistency problem of oriented 2D networks is in* P.

Proof. In his work[15], and in order to compare the different positions of two couples of points (a_i, b_i), and (a_j, b_j), Ligozat defined the following nine relations: $(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, =), (=, <), (=, >)$. To each set Γ of oriented 2D relations we can associate a set Γ^C of cardinal relations such that an oriented 2D relation satisfied by the lines d_i and d_j is in Γ iff the cardinal relation satisfied by the couples (a_i, b_i) and (a_j, b_j) is in Γ^C:
$\emptyset^C = \emptyset$.
$\{PO\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >)\}$.
$\{EQ\}^C = \{(=, =)\}$.
$\{<\}^C = \{(=, <)\}$.
$\{>\}^C = \{(=, >)\}$.
$\{EQ, <\}^C = \{(=, =), (=, <)\}$.
$\{EQ, >\}^C = \{(=, =), (=, >)\}$.
$\{<, >\}^C = \{(=, <), (=, >)\}$.
$\{PO, EQ\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, =)\}$.
$\{PO, <\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, <)\}$.
$\{PO, >\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, >)\}$.
$\{PO, EQ, <\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, =), (=, <)\}$.
$\{PO, EQ, >\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, =), (=, >)\}$.
$\{PO, <, >\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, <), (=, >)\}$.
$\{EQ, <, >\}^C = \{(=, =), (=, <), (=, >)\}$.
$\{PO, EQ, <, >\}^C = \{(<, <), (<, =), (<, >), (>, <), (>, =), (>, >), (=, =), (=, <), (=, >)\}$.
By construction, the consistency problem of an oriented 2D network is polynomially reducible to the consistency problem of a network of cardinal constraints. Moreover, we easily check that the sets of cardinal relations defined above are preconvex. We conclude our proof by recalling that the consistency problem of networks of preconvex cardinal constraints is in P.

4 Metric constraints in dimension 2

Our aim in this section is to define quantitative relations that allow us to compare the distance that separates two parallel lines of the euclidian plane. We will first consider relations that enable us to tell if, for two parallel lines d_i and d_j, the distance between them (denoted later on by $d(d_i, d_j)$), is less or equal to a certain value. The latter relation will then be augmented by another one, in such a way to tell whether or not the distance from d_j to d_i is superior to a real number.

Definition 2 *Let* $h \geq 0$. *For any couple* (d_i, d_j) *of lines in the euclidian plane:*

- $d_i\{P_h^+\}d_j \Leftrightarrow d_i$ *is parallel to* d_j *and* $d(d_i, d_j) \leq h$.

– *$d_i\{P_h^-\}d_j \Leftrightarrow d_i$ is parallel to d_j and $d(d_i, d_j) \geq h$.*
– *A metric 2D network is a path-consistent linear constraint network of lines in a euclidian plane, where the relations are of the form $\{PO\}$, $\{P_h^+\}$ and $\{P_h^-\}$.*

Remarks. For $h > 0$, consider two parallel lines in the euclidian plane (d_i, d_j), distant of h, which equations are $y = a_i x + b_i$ and $y = a_j x + b_j$. These lines are identical iff $d_i\{P_0^+\}d_j$. Moreover, $d_i\{P_h^+\}d_j \Leftrightarrow ((a_i = a_j) \wedge |b_i - b_j| \leq D_a)$, where the real D_a satisfies the equation $h = \frac{D_a}{\sqrt{1+a_i^2}}$. Symmetrically, $d_i\{P_h^-\}d_j \Leftrightarrow ((a_i = a_j) \wedge |b_i - b_j| \geq D_a)$. We notice that the constraints $\{P_h^+\}$ and $\{P_h^-\}$ are strictly equivalent to $\{[-D_a, D_a]\}$ and $\{[-\infty, -D_a] \cup [D_a + \infty]\}$. This type of relations was introduced by Dechter *et al.* [8], for the characterization of a temporal constraint satisfaction problem ($TCSP$), where the variables represent time points, and the constraints sets of intervals. Thus we can invariably switch between $d_i\{P_h^+\}d_j$ and $(b_i - b_j) \in [-D_a, D_a]$ (the same reasoning applies to $d_i\{P_h^-\}d_j$). For the composition of relations of the form $\{P_h^+, P_h^-\}$ (showed in figure 2), we use the definition of its counterpart, given in [8]: for any two intervals T and S, the composition $T \circ S$ admits only values r for which there exist $t \in T$ and $s \in S$, such that $r = t + s$.

$\circ$	P_h^+	P_h^-	PO
P_r^+	P_{r+h}^+	P_{h-r}^-	PO
P_r^-	P_{r-h}^-	P_R	PO
PO	PO	PO	PO, P_R

Figure 2. Composition of metric 2D networks' relations.

Let $h \leq 0$. For any line of the euclidian plane d_i, the set of lines d such that $d\{P_h^-\}d_i$ represents by convention all the parallel lines that are equal or parallel to d_i. In that case, the relation $\{P_h^-\}$ will be denoted by $\{P_R\}$.

Proposition 3 *The consistency problem of metric 2D networks is NP-complete.*

Proof. Given a metric 2D network $\mathcal{R} = (I, C)$, for all $i, j \in I$, let $P^- = \{P_{h_1}^-, \ldots, P_{h_p}^-\}$ and $P^+ = \{P_{h_1}^+, \ldots, P_{h_q}^+\}$ be the sets of all corresponding parallel constraints appearing in $C(i, j)$. It's clear that the disjunction of constraints P^- (resp. P^+) can equivalently be reduced to $P_{h_{ij}^-}$ (resp. $P_{h_{ij}^+}$), where $h_{ij}^- = min\{h_k : 1 \leq k \leq p\}$ (resp. $h_{ij}^+ = max\{h_k : 1 \leq k \leq q\}$). Let $\mathcal{R}' = (I, C')$ be the constraint network so obtained. By replacing every occurrence of parallel constraint in $\mathcal{R}' = (I, C')$ with the relation $\{DC\}$, we can polynomially check if the resulting network of linear constraints is consistent or not (due to proposition 1). If it is consistent, let $\mathcal{R}'' = (I, C'')$ be a solution, which is an atomic linear constraint network. We then switch back to a subnetwork

$\mathcal{R}'_\alpha = (I, C_\alpha)$ of $\mathcal{R}'$, where every relation $\{DC\} = C''(i,j)$ is replaced by the original parallel one (i.e. $P_{h^+_{ij}}$ or/and $P_{h^-_{ij}}$). Thus, for all $i,j \in I$, either $C_\alpha(i,j) = \{PO\}$ or $C_\alpha(i,j) \in \{P_{h^+}, P_{r^-}\}$. Checking the consistency of $\mathcal{R}'_\alpha$ leads us to find a consistent scenario for the variables $i,j \in I$, where $C_\alpha(i,j) \in \{P_{h^+}, P_{r^-}\}$. Knowing that the consistency problem of a $TCSP$ is NP-complete [8], our proposition is proven.

Corollary 1 *The consistency problem of metric 2D networks, where the only allowed parallel constraints are of the form $\{P_h^+\}$, is in P.*

This result stems directly from the fact that the consistency problem of an STP (which is by definition an atomic $TCSP$) is in P [8].

5 Lines in dimension 3

We already know from [2] that the consistency problem of networks of linear constraints in the euclidian space is decidable. We next show that this problem is NP-hard, by polynomially reducing the problem "Not-all-equal SAT" to the problem of lines in the space. A "Not-all-equal SAT" formula is a 3SAT formula[1] satisfying: the three literals forming each clause haven't the same truth value (neither all true, nor all false). The proof of the NP-completeness of "Not-all-equal SAT" can be found in [23,20].

Definition 3 *A spatial network is a path-consistent linear constraint network of lines in a euclidian space.*

To compare the different positions of lines in the space, we add a fourth relation representing the non-coplanarity (NC) between two lines. The set $E = \{PO, EQ, DC, NC\}$ contains relations that are jointly exhaustive and pairwise disjoint, where each one is equal to its inverse. As in section 3, we use the consistency-based composition of relations to compute the table in figure 3: if $\alpha, \beta \in E$, $\alpha \circ \beta$ contains all the relations $\gamma \in E$ such that there exists lines d_i, d_j, d_k of the space satisfying $d_i\{\alpha\}d_j$, $d_j\{\beta\}d_k$ and $d_i\{\gamma\}d_k$.

$\circ$	EQ	DC	PO	NC
EQ	EQ	DC	PO	NC
DC	DC	DC, EQ	PO, NC	PO, NC
PO	PO	PO, NC	PO, NC, DC,EQ	PO, NC, DC
NC	NC	PO, NC	PO, NC, DC	PO, NC, DC,EQ

Figure 3. Composition table of spatial relations.

[1] A 3SAT formula is a conjunction of disjunctions, $\phi = (\xi_1 \vee \xi_2 \vee \xi_3) \wedge \ldots \wedge (\xi_{n-2} \vee \xi_{n-1} \vee \xi_n)$, where $(\xi_i)_{1 \leq i \leq n}$ are literals.

***Exemple* 3** *Here are some examples of spatial networks, containing four elements.*

1. *Non-consistent atomic spatial network:* $C(1,2) = \{DC\}$, $C(1,3) = C(1,4) = C(2,3) = C(2,4) = \{PO\}$, $C(3,4) = \{NC\}$. *Indeed, we can't consistently assign the fourth variable. This shows that the polynomial method of path consistency (Allen's triangulation) doesn't provide a complete decision procedure for the consistency problem of spatial networks.*
2. *Non-globally consistent spatial network:* $C(1,2) = C(1,3) = C(2,3) = \{DC\}$, $C(1,4) = C(2,4) = \{PO\}$, $C(3,4) = \{NC\}$. *If we assign to the third variable a line* d_3 *included in the plane defined by the parallel lines* d_1 *and* d_2, *we find it impossible to consistently assign the fourth one.*

Proposition 4 *The consistency problem of spatial networks is NP-hard.*

Proof. Let $\phi = (C_1 \wedge \ldots \wedge C_n)$ be a "Not-all-equal SAT" formula. We next associate to ϕ a spatial network $\mathcal{R}_\phi$. For our purpose, $VAR = \{p, q, r, \ldots\}$ will designate the set of atoms appearing in ϕ, and $\{d_p, d_q, d_r, \ldots\}$ their associated lines in $\mathcal{R}_\phi$. The lines $\{d_0, d_1, d_0^{'}, d_1^{'}, d, d'\}$ that constitute our basic structure satisfy the conditions: $d_0\{PO\}d_1$, $d_0^{'}\{PO\}d_1^{'}$, $d_0\{DC\}d_0^{'}$, $d_1\{DC\}d_1^{'}$, $d_0\{NC\}d_1^{'}$, $d_1\{NC\}d_0^{'}$, $d\{PO\}d_0$, $d\{PO\}d_1$, $d\{PO\}d_0^{'}$, $d\{PO\}d_1^{'}$, $d'\{PO\}d_0^{'}$, $d'\{PO\}d_1^{'}$, $d'\{NC\}d$. For all $1 \leq i \leq n$, and to each atom p appearing in the clause $C_i = l_{i_1} \vee l_{i_2} \vee l_{i_3}$, we associate the lines d_p and $d_{\neg p}$ such that: $d_p\{PO\}d_0$, $d_p\{PO\}d_1$, $d_p\{DC, PO\}d'$, $d_{\neg p}\{PO\}d_0$, $d_{\neg p}\{PO\}d_1$, $d_{\neg p}\{DC, PO\}d'$, et $d_p\{NC\}d_{\neg p}$.
The interpretation of the literals in ϕ is: for each literal $l \in \{p, \neg p : p \in VAR\}$, l is true if the line d_l is included in the plane (P), defined by the lines (d_0, d_1), otherwise l is false.
Notice that all lines d_l are either parallel to d' and are included in (P), or they intersect d', d_0 and d_1. We easily check that for each atom appearing in a clause C_i, the construction of the lines d_p and $d_{\neg p}$ is mutually exclusive, in the sense that one of them is included in (P), whereas the other passes through it.
To successfully complete our reduction, for each clause $C_i = l_{i_1} \vee l_{i_2} \vee l_{i_3}$, at least one line $d_{l_{ij}}$ must be included in (P), and another one $d_{l_{ik}}$ must pass through it, where $(j, k \in \{1, 2, 3\})$. To achieve our aim, for all $1 \leq j \leq 3$, let $d_{l_{ij}}\{PO, NC\}d$. For each clause C_i, we represent the lines $d_{l_{ij}}$ in such a way that for every distinct $j, k, t \in \{1, 2, 3\}$, there exists d'' satisfying $d_{l_{ij}}\{PO\}d''$, $d_{l_{ik}}\{PO\}d''$ and $d_{l_{it}}\{NC\}d''$. This condition ensures that at most two lines are either parallel to d' or intersect it.
This proves that a formula ϕ is satisfiable iff $\mathcal{R}_\phi$ is consistent. As the reduction made above is done in polynomial time in the length of the formula ϕ, the proposition is established.

Proposition 5 *Any spatial network is polynomially reducible to an atomic one.*

Proof. Let (I, C) be a spatial network. In order to prove this result, we need to check that for each $i, j \in I$ such that $C(i, j)$ is a disjunction of constraints of the set $E = \{PO, EQ, DC, NC\}$, the variables i, j and the constraint $C(i, j)$ can be equivalently replaced by an atomic spatial network. Ten cases are to be considered.

1. $d_i\{EQ, DC\}d_j$.
   There exist lines d_u, d_v such that $((d_u\{PO\}d_v) \wedge (d_i\{PO\}d_u) \wedge (d_i\{DC\}d_v) \wedge (d_j\{PO\}d_u) \wedge (d_j\{DC\}d_v))$.
2. $d_i\{EQ, PO\}d_j$.
   There exist lines d_u, d_v, d_w such that $((d_u\{PO\}d_v) \wedge (d_u\{DC\}d_w) \wedge (d_v\{NC\}d_w) \wedge (d_i\{PO\}d_u) \wedge (d_i\{PO\}d_v) \wedge (d_i\{PO\}d_w) \wedge (d_j\{PO\}d_u) \wedge (d_j\{PO\}d_v) \wedge (d_j\{PO\}d_w))$.
3. $d_i\{EQ, NC\}d_j$.
   There exist lines d_u, d_v, d_w such that $((d_u\{NC\}d_v) \wedge (d_u\{NC\}d_w) \wedge (d_v\{NC\}d_w) \wedge (d_i\{PO\}d_u) \wedge (d_i\{PO\}d_v) \wedge (d_i\{PO\}d_w) \wedge (d_j\{PO\}d_u) \wedge (d_j\{PO\}d_v) \wedge (d_j\{PO\}d_w))$.
   With such conditions, we can't have $d_i\{PO\}d_j$. Indeed, if we choose a point A on one line (e.g. d_u), the latter defines with one of the remaining lines (e.g. d_v) a plane that the third one necessarily cuts in a point B. Thus, the line (AB) intersects (d_u, d_v, d_w). As the latter lines are two by two non coplanar, it's easy to notice that no line can, at the same time, intersect (AB), d_u, d_v, and d_w.
4. $d_i\{PO, NC\}d_j$.
   There exist lines d_u, d_v, d_w such that $((d_i\{PO\}d_w) \wedge (d_j\{PO\}d_w) \wedge (d_w\{NC\}d_u) \wedge (d_i\{PO\}d_u) \wedge (d_j\{PO\}d_u) \wedge (d_i\{DC\}d_v) \wedge (d_j\{PO\}d_v))$.
5. $d_i\{DC, NC\}d_j$.
   There exist lines d_t, d_u, d_v, d_w, d_z that respectively satisfy the same constraints as those satisfied by $d_0, d_1, d_0^{'}, d_1^{'}, d$, defined during the proof of proposition 4. Moreover, let the conditions $((d_i\{PO\}d_u) \wedge (d_i\{PO\}d_v) \wedge (d_i\{NC\}d_z) \wedge (d_j\{PO\}d_t) \wedge (d_j\{PO\}d_w) \wedge (d_j\{NC\}d_z))$ be true.
6. $d_i\{DC, PO\}d_j$.
   There exist lines d_u, d_v, d_w such that $((d_u\{DC\}d_w) \wedge (d_i\{PO\}d_w) \wedge (d_i\{PO\}d_u) \wedge (d_j\{PO\}d_w) \wedge (d_j\{PO\}d_u) \wedge (d_v\{NC\}d_i) \wedge (d_v\{PO\}d_j))$.
7. $d_i\{EQ, DC, PO\}d_j$.
   There exist lines d_u, d_v such that $((d_i\{PO\}d_u) \wedge (d_i\{PO\}d_v) \wedge (d_j\{PO\}d_u) \wedge (d_j\{PO\}d_v) \wedge (d_u\{DC\}d_v))$.
8. $d_i\{EQ, NC, PO\}d_j$.
   There exist lines d_u, d_v such that $((d_i\{PO\}d_u) \wedge (d_i\{PO\}d_v) \wedge (d_j\{PO\}d_u) \wedge (d_j\{PO\}d_v) \wedge (d_u\{NC\}d_v))$.
9. $d_i\{DC, PO, NC\}d_j$.
   There exist lines d_t, d_u, d_v, d_w such that $((d_u\{PO\}d_v) \wedge (d_v\{DC\}d_w) \wedge (d_u\{NC\}d_w) \wedge (d_t\{PO\}d_u) \wedge (d_t\{PO\}d_v) \wedge (d_t\{PO\}d_w) \wedge (d_i\{PO\}d_u) \wedge (d_i\{PO\}d_v) \wedge (d_i\{NC\}d_t) \wedge (d_j\{PO\}d_t) \wedge (d_j\{PO\}d_w))$.
10. $d_i\{DC, EQ, NC\}d_j$.
    There exist lines d_u, d_v, d_w such that $((d_v\{DC\}d_u) \wedge (d_u\{PO\}d_w) \wedge (d_w\{PO\}d_v) \wedge (d_i\{DC\}d_u) \wedge (d_i\{NC\}d_w) \wedge (d_j\{DC, PO\}d_u) \wedge (d_j\{DC, PO\}d_v))$.

The path consistency property enables us to check the truthfulness of each case. Also note that for a spatial network of n elements, there are at most n^2 non atomic constraints. We conclude that our reduction is polynomial in the size of the network (I, C), and is done in time $O(n^2)$.

Corollary 2 *The consistency problem of an atomic spatial network is NP-hard.*

In the proof of proposition 4, it suffices to transform the relations $\{PO, NC\}$ and $\{PO, DC\}$ into atomic ones.

6 Conclusion and perspectives.

In this paper we studied several path-consistent constraint networks of lines in the euclidian plane and space. Our main goal was to determine the complexity of the consistency problem of a network of linear lines. Balbiani [2] has already showed that for a network of linear lines, this problem is in P. Considering some special relations between two parallel lines in a plane, we established that the complexity problem of an oriented 2D linear network (i.e. when the orientation of parallel lines is taken into account) is still in P, whereas the one concerning a metric 2D network (i.e. when the distance separating two parallel lines is considered) is NP-complete. We then showed that the consistency problem of atomic networks of lines in the euclidian space is NP-hard, by polynomially reducing the problem "Not-all-equal SAT" to the problem of lines in the space.
Our next step is to determine the complexity of oriented and metric 2D networks in the euclidian space. But first, an important question that arises is whether the consistency problem of spatial networks is NP-complete. Given such a network $\mathcal{R}$, we need to exhibit a non deterministic program running in polynomial time in the length of the spatial network's variables, that computes a consistent scenario for $\mathcal{R}$.

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