Reasoning about vague concepts in the theory of property systems

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Abstract. We show that information relations derived from property systems are relative relations between properties determined by subsets of objects as well as relative relations between objects determined by subsets of properties. We introduce a modal logic PO made up of two sets of sentences: sentences about properties and sentences about objects. In its language, the modal operators are the abstract counterparts of indiscernibility relations between properties and indiscernibility relations between objects. We address the issue of its axiomatization/completeness as well as the issue of its decidability/complexity.

1 Introduction

Attribute systems, as a general purpose relational database not confined to a particular application, originated with the work of Pawlak [18] who proposed to model incompleteness of information by means of sets of values of attributes characterizing objects in a given universe. Following the simple idea that objects can be classified according to these sets of values, information systems of Pawlak has subsequently received much study motivated both by the mathematical attraction of the paradigm and by its potential usefulness for knowledge representation. A different approach to knowledge representation was taken by Vakarelov [22], who considered property systems. The information represented in property systems is given in terms of objects and properties. Information relations derived from attribute systems are determined by the sets of values of attributes that characterize objects whereas information relations between objects in property systems are determined by the properties that are possessed by these objects. Information relations can be traced back to the work of [17] for attribute systems and [22] for property systems whereas the concept of information logic took its shape in the fundamental work of [13–15] and [19–21] for attribute systems and [22] for property systems. Within the framework of attribute systems, the studies of Orłowska [13–15] revealed the existence of modal logics with relative accessibility relations $r(a)$ between objects based on information relations determined by sets a of attributes: Rare-logics for attribute systems. A lot of results are known concerning their proof theory and their complexity theory and we refer the reader to the detailed exposition of the subjects by [9]. Within the framework of property systems, Vakarelov [22] introduced modal logics providing a formal account for reasoning about objects in terms of properties. These logics are interpreted in structures where connections between objects are determined by the properties that are possessed by these objects. Property systems consist of two sets of beings: properties and objects. Consequently, it is only natural that we consider in this paper a new kind of modal language made up of two sets of sentences: sentences about properties and sentences about objects. We investigate a new kind of modal logic, Rare-logics for property systems, and we address the issue of its axiomatization/completeness as well as the issue of its decidability/complexity. We mainly study the modal logic with relative indiscernibility relations. Indiscernibility relations $R(A)$ between properties are determined by the objects in the set A of objects that have these properties whereas indiscernibility relations $r(a)$ between objects are determined by the properties in the set a of properties that are possessed by these objects. This paper has two major parts. The first, comprising sections 2–5, is an introduction to reasoning about properties in terms of objects as well as objects in terms of properties. In the second part of the paper, from section 6 to section 9, we turn to the following question: what is the modal logic of relative indiscernibility in property systems?

2 Property systems

Adapted from Vakarelov [22], a property system will be any structure of the form $S = (PAR, OBJ, f, F)$ where:

- PAR is a nonempty set of properties;
- OBJ is a nonempty set of objects;
- f is a function with domain PAR and range 2^{OBJ} — the power set of OBJ;
- F is a function with domain OBJ and range 2^{PAR} — the power set of PAR.

For every $x \in PAR$, $f(x)$ is the set of all objects that have property x whereas, for every $X \in OBJ$, $F(X)$ is the set of all properties that are possessed by object X. We want the functions f and F to reflect the intended meanings of sentences of the form "object X has property x" and "property x is possessed by object X". Thus a property system $S = (PAR, OBJ, f, F)$ will be defined to be standard when, for every $x \in PAR$ and for every $X \in OBJ$, $X \in f(x)$ iff $x \in F(X)$. As an illustrative example, let $S = (PAR, OBJ, f, F)$ be the property system where:

- PAR is $\{Arabic, Bulgarian, Castilian, Dutch, English, French, German, Hungarian\}$;
- OBJ is $\{Ann, Bob, Cindy, Daniel, Emma, Frederick, Gabrielle, Henry\}$;
- f is the function with domain PAR and range 2^{OBJ} defined by table 1;
- F is the function with domain OBJ and range 2^{PAR} defined by table 2.

In $S = (PAR, OBJ, f, F)$, the property *German* is possessed by the objects *Frederick* and *Gabrielle*, i.e. German is mastered by Frederick and Gabrielle, whereas the object *Bob* has the properties *Arabic*, *Bulgarian* and *Castilian*, i.e. Bob masters Arabic, Bulgarian and Castilian. One should note that this property system is standard.

Arabic	Bulgarian	Castilian	Dutch
$\{Ann, Bob\}$	$\{Ann, Bob, Cindy\}$	$\{Ann, Bob\}$	$\{Ann\}$

English	French	German	Hungarian
$\{Henry\}$	$\{Gabrielle, Henry\}$	$\{Frederick, Gabrielle\}$	$\{Emma, Frederick, Gabrielle\}$

Table 1.

Ann	Bob	Cindy	Daniel
$\{Arabic, Bulgarian, Castilian, Dutch\}$	$\{Arabic, Bulgarian, Castilian\}$	$\{Bulgarian\}$	$\emptyset$

Emma	Frederick	Gabrielle	Henry
$\{Hungarian\}$	$\{German, Hungarian\}$	$\{French, German, Hungarian\}$	$\{English, French\}$

Table 2.

3 Informational relations

Property systems constitute the starting point for the formal examination of sentences of the form "property x is indistinguishable from property y with respect to a set A of objects" or "object X is distinguishable from object Y with respect to a set a of properties". Given a property system $S = (PAR, OBJ, f, F)$, connections between properties are determined by the sets of objects that have these properties whereas connections between objects are determined by the sets of properties that are possessed by these objects. These connections take the form of binary relations on PAR and binary relations on OBJ. Following the line of reasoning suggested by Orłowska within the framework of attribute systems, let $FIN_S(A)$, $IND_S(A)$, $BIN_S(A)$, $RORT_S(A)$, $COM_S(A)$ and $LORT_S(A)$ be the binary relations on PAR defined in the following way, for every $A \subseteq OBJ$ and for every $x, y \in PAR$:

Forward inclusion: $x\ FIN_S(A)\ y$ iff $A \cap f(x) \subseteq A \cap f(y)$;
Indiscernibility: $x\ IND_S(A)\ y$ iff $A \cap f(x) = A \cap f(y)$;
Backward inclusion: $x\ BIN_S(A)\ y$ iff $A \cap f(x) \supseteq A \cap f(y)$;
Right orthogonality: $x\ RORT_S(A)\ y$ iff $A \cap f(x) \subseteq A \cap \overline{f(y)}$;
Complementarity: $x\ COM_S(A)\ y$ iff $A \cap f(x) = A \cap \overline{f(y)}$;
Left orthogonality: $x\ LORT_S(A)\ y$ iff $A \cap f(x) \supseteq A \cap \overline{f(y)}$;

and let $fin_S(a)$, $ind_S(a)$, $bin_S(a)$, $rort_S(a)$, $com_S(a)$ and $lort_S(a)$ be the binary relations on OBJ defined in the following way, for every $a \subseteq PAR$ and for every $X, Y \in OBJ$:

Forward inclusion: $X\ fin_S(a)\ Y$ iff $a \cap F(X) \subseteq a \cap F(Y)$;
Indiscernibility: $X\ ind_S(a)\ Y$ iff $a \cap F(X) = a \cap F(Y)$;
Backward inclusion: $X\ bin_S(a)\ Y$ iff $a \cap F(X) \supseteq a \cap F(Y)$;
Right orthogonality: $X\ rort_S(a)\ Y$ iff $a \cap F(X) \subseteq a \cap \overline{F(Y)}$;
Complementarity: $X\ com_S(a)\ Y$ iff $a \cap F(X) = a \cap \overline{F(Y)}$;
Left orthogonality: $X\ lort_S(a)\ Y$ iff $a \cap F(X) \supseteq a \cap \overline{F(Y)}$.

$x\ FIN_S(A)\ y$ says that objects in A having property x have also property y whereas $X\ lort_S(a)\ Y$ says that properties in a not possessed by object Y are on the other hand possessed by object X. The property system $S = (PAR, OBJ, f, F)$ of tables 1 and 2 is such that if A is $\{Ann, Bob, Cindy, Daniel\}$ then $Arabic\ FIN_S(A)\ Bulgarian$, $Arabic\ IND_S(A)\ Castilian$ and $Arabic\ BIN_S(A)\ Dutch$ whereas if a is $\{English, French, German, Hungarian\}$ then, $Emma\ rort_S(a)\ Henry$, $Frederick\ com_S(a)\ Henry$ and $Gabrielle\ lort_S(a)\ Henry$. We list in the propositions below some useful, though simple, properties of our binary relations.

Proposition 1. *Let $S = (PAR, OBJ, f, F)$ be a property system. For every $A \subseteq OBJ$ and for every $x, y \in PAR$:*

- $x\ FIN_S(A)\ x$;
- *If* $x\ FIN_S(A)\ y$ *then* $y\ BIN_S(A)\ x$;
- *If* $x\ FIN_S(A)\ y$ *and* $y\ FIN_S(A)\ z$ *then* $x\ FIN_S(A)\ z$;
- *If* $x\ FIN_S(A)\ y$ *and* $y\ RORT_S(A)\ z$ *then* $x\ RORT_S(A)\ z$;
- $x\ IND_S(A)\ y$ *iff* $x\ FIN_S(A)\ y$ *and* $x\ BIN_S(A)\ y$;
- $x\ BIN_S(A)\ x$;
- *If* $x\ BIN_S(A)\ y$ *then* $y\ FIN_S(A)\ x$;
- *If* $x\ BIN_S(A)\ y$ *and* $y\ BIN_S(A)\ z$ *then* $x\ BIN_S(A)\ z$;
- *If* $x\ BIN_S(A)\ y$ *and* $y\ LORT_S(A)\ z$ *then* $x\ LORT_S(A)\ z$;
- *If* $x\ RORT_S(A)\ y$ *then* $y\ RORT_S(A)\ x$;
- *If* $x\ RORT_S(A)\ x$ *then* $x\ FIN_S(A)\ y$;
- *If* $x\ RORT_S(A)\ x$ *then* $x\ RORT_S(A)\ y$;
- *If* $x\ RORT_S(A)\ y$ *and* $y\ BIN_S(A)\ z$ *then* $x\ RORT_S(A)\ z$;
- *If* $x\ RORT_S(A)\ y$ *and* $y\ LORT_S(A)\ z$ *then* $x\ FIN_S(A)\ z$;
- $x\ COM_S(A)\ y$ *iff* $x\ RORT_S(A)\ y$ *and* $x\ LORT_S(A)\ y$;
- *If* $x\ LORT_S(A)\ y$ *then* $y\ LORT_S(A)\ x$;
- *If* $x\ LORT_S(A)\ x$ *then* $x\ BIN_S(A)\ y$;
- *If* $x\ LORT_S(A)\ x$ *then* $x\ LORT_S(A)\ y$;
- *If* $x\ LORT_S(A)\ y$ *and* $y\ FIN_S(A)\ z$ *then* $x\ LORT_S(A)\ z$;
- *If* $x\ LORT_S(A)\ y$ *and* $y\ RORT_S(A)\ z$ *then* $x\ BIN_S(A)\ z$.

Proposition 2. *Let $S = (PAR, OBJ, f, F)$ be a property system. For every $a \subseteq PAR$ and for every $X, Y \in OBJ$:*

- $X \ fin_S(a) \ X$;
- If $X \ fin_S(a) \ Y$ then $Y \ bin_S(a) \ X$;
- If $X \ fin_S(a) \ Y$ and $Y \ fin_S(a) \ Z$ then $X \ fin_S(a) \ Z$;
- If $X \ fin_S(a) \ Y$ and $Y \ rort_S(a) \ Z$ then $X \ rort_S(a) \ Z$;
- $X \ ind_S(a) \ Y$ iff $X \ fin_S(a) \ Y$ and $X \ bin_S(a) \ Y$;
- $X \ bin_S(a) \ X$;
- If $X \ bin_S(a) \ Y$ then $Y \ fin_S(a) \ X$;
- If $X \ bin_S(a) \ Y$ and $Y \ bin_S(a) \ Z$ then $X \ bin_S(a) \ Z$;
- If $X \ bin_S(a) \ Y$ and $Y \ lort_S(a) \ Z$ then $X \ lort_S(a) \ Z$;
- If $X \ rort_S(a) \ Y$ then $Y \ rort_S(a) \ X$;
- If $X \ rort_S(a) \ X$ then $X \ fin_S(a) \ Y$;
- If $X \ rort_S(a) \ X$ then $X \ rort_S(a) \ Y$;
- If $X \ rort_S(a) \ Y$ and $Y \ bin_S(a) \ Z$ then $X \ rort_S(a) \ Z$;
- If $X \ rort_S(a) \ Y$ and $Y \ lort_S(a) \ Z$ then $X \ fin_S(a) \ Z$;
- $X \ com_S(a) \ Y$ iff $X \ rort_S(a) \ Y$ and $X \ lort_S(a) \ Y$;
- If $X \ lort_S(a) \ Y$ then $Y \ lort_S(a) \ X$;
- If $X \ lort_S(a) \ X$ then $X \ bin_S(a) \ Y$;
- If $X \ lort_S(a) \ X$ then $X \ lort_S(a) \ Y$;
- If $X \ lort_S(a) \ Y$ and $Y \ fin_S(a) \ Z$ then $X \ lort_S(a) \ Z$;
- If $X \ lort_S(a) \ Y$ and $Y \ rort_S(a) \ Z$ then $X \ bin_S(a) \ Z$.

We shall state two more elementary propositions which say that if we have more objects then the connections between properties are smaller whereas if we have more properties then the connections between objects are smaller.

Proposition 3. *Let* $S = (PAR, OBJ, f, F)$ *be a property system and* $R \in \{FIN_S, IND_S, BIN_S, RORT_S, COM_S, LORT_S\}$. *For every* $A, B \subseteq OBJ$:

- *If* $A \subseteq B$ *then* $R(A) \supseteq R(B)$;
- $R(A \cup B) = R(A) \cap R(B)$.

Proposition 4. *Let* $S = (PAR, OBJ, f, F)$ *be a property system and* $r \in \{fin_S, ind_S, bin_S, rort_S, com_S, lort_S\}$. *For every* $a, b \subseteq PAR$:

- *If* $a \subseteq b$ *then* $r(a) \supseteq r(b)$;
- $r(a \cup b) = r(a) \cap r(b)$.

4 Informational representability of indiscernibility

From now on, we shall concentrate our attention on the connections of indiscernibility between properties and the connections of indiscernibility between objects. The starting point of our discussion is the notion of an indiscernibility frame which is any structure of the form $F = (PAR, OBJ, IND, ind)$ where PAR is a nonempty set of properties, OBJ is a nonempty set of objects, IND is a function with domain 2^{OBJ} and range the set of all binary relations on PAR such that, for every $A, B \subseteq OBJ$ and for every $x, y \in PAR$:

- $x \ IND(A) \ x$;

- If $x\ IND(A)\ y$ then $y\ IND(A)\ x$;
- If $x\ IND(A)\ y$ and $y\ IND(A)\ z$ then $x\ IND(A)\ z$;
- If $A \subseteq B$ then $IND(A) \supseteq IND(B)$;

and ind is a function with domain 2^{PAR} and range the set of all binary relations on OBJ such that, for every $a, b \subseteq PAR$ and for every $X, Y \in OBJ$:

- $X\ ind(a)\ X$;
- If $X\ ind(a)\ Y$ then $Y\ ind(a)\ X$;
- If $X\ ind(a)\ Y$ and $Y\ ind(a)\ Z$ then $X\ ind(a)\ Z$;
- If $a \subseteq b$ then $ind(a) \supseteq ind(b)$.

Normal indiscernibility frames are indiscernibility frames $F = (PAR, OBJ, IND, ind)$ such that, for every $A, B \subseteq OBJ$, $IND(A \cup B) = IND(A) \cap IND(B)$ and, for every $a, b \subseteq PAR$, $ind(a \cup b) = ind(a) \cap ind(b)$. Given a property system $S = (PAR, OBJ, f, F)$, it is obvious from the above definitions that the structure $F_S = (PAR, OBJ, IND_S, ind_S)$ is a normal indiscernibility frame. Before moving on to our next topic, we wish to say a few words about the following question that remains completely unsolved:

- Given a normal indiscernibility frame $F = (PAR, OBJ, IND, ind)$, does there exists a property system $S' = (PAR', OBJ', f', F')$ such that $F = (PAR, OBJ, IND, ind)$ and $F_{S'} = (PAR', OBJ', IND_{S'}, ind_{S'})$ are isomorphic, i.e. there is a bijective function γ with domain PAR and range PAR' and there is a bijective function Γ with domain OBJ and range OBJ' such that, for every $A \subseteq OBJ$ and for every $x, y \in PAR$, $x\ IND(A)\ y$ iff $\gamma(x)\ IND'(\Gamma(A))\ \gamma(y)$ and, for every $a \subseteq PAR$ and for every $X, Y \in OBJ$, $X\ ind(a)\ Y$ iff $\Gamma(X)\ ind'(\gamma(a))\ \Gamma(Y)$?

A positive answer to this question would provide a foundation for the development of reasoning systems that are concerned with incomplete information, seeing that every normal frame of the modal logic for indiscernibility defined in section 6 would be informationally representable.

5 Reasoning about indiscernibility

Given an indiscernibility frame (PAR, OBJ, IND, ind), let $\langle IND(A)\rangle$ and $\langle ind(a)\rangle$ be the operators defined in the following way, for every $A \subseteq OBJ$ and for every $a \subseteq PAR$:

- $\langle IND(A)\rangle a$ is $\{x \in PAR$: there exists $y \in PAR$ such that $xIND(A)y$ and $y \in a\}$;
- $\langle ind(a)\rangle A$ is $\{X \in OBJ$: there exists $Y \in OBJ$ such that $Xind(a)Y$ and $Y \in A\}$.

The dual operators $[IND(A)]$ and $[ind(a)]$ are introduced by the following abbreviations:

- $[IND(A)]a$ is $PAR \setminus \langle IND(A)\rangle(PAR \setminus a)$;
- $[ind(a)]A$ is $OBJ \setminus \langle ind(a)\rangle(OBJ \setminus A)$.

Proposition 5. *Let (PAR, OBJ, IND, ind) be an indiscernibility frame. For every $A, B \subseteq OBJ$ and for every $a, b \subseteq PAR$:*

- $\langle IND(A)\rangle(a \cup b) = \langle IND(A)\rangle a \cup \langle IND(A)\rangle b$;
- *If* $A \subseteq B$ *then* $[IND(A)]a \subseteq [IND(B)]a$;
- $[IND(A)]PAR = PAR$;
- $\langle ind(a)\rangle(A \cup B) = \langle ind(a)\rangle A \cup \langle ind(a)\rangle B$;
- *If* $a \subseteq b$ *then* $[ind(a)]A \subseteq [ind(b)]A$;
- $[ind(a)]OBJ = OBJ$.

Moreover, for every $A \subseteq OBJ$ and for every $a \subseteq PAR$:

- $[IND(A)]a \subseteq a$;
- $a \subseteq [IND(A)]\langle IND(A)\rangle a$;
- $[IND(A)]a \subseteq [IND(A)][IND(A)]a$;
- $[ind(a)]A \subseteq A$;
- $A \subseteq [ind(a)]\langle ind(a)\rangle A$;
- $[ind(a)]A \subseteq [ind(a)][ind(a)]A$.

The operators $\langle IND(A)\rangle$ and $\langle ind(a)\rangle$ defined above are also used as modal operators in a logical system which semantics is defined in terms of indiscernibility frames.

6 Modal logic for indiscernibility: syntax and semantics

We assume some familiarity with propositional modal logic. Readers wanting more details may refer, for example, to [4] or [5]. Let Π_a be a countable set of atomic properties and let Φ_a be a countable set of atomic objects. The set Π_c of the complex properties and the set Φ_c of the complex objects are defined by induction in the following way:

- $a ::= \pi \mid \neg a \mid (a \wedge b) \mid \langle A\rangle a$;
- $A ::= \phi \mid \neg A \mid (A \wedge B) \mid \langle a\rangle A$.

In order to make complex properties and complex objects more readable, we shall introduce abbreviations to our language in the usual way. In particular, we let $[A]a = \neg\langle A\rangle\neg a$ and $[a]A = \neg\langle a\rangle\neg A$. Another abbreviation which we shall adopt is to leave out unnecessary parentheses. The indiscernibility frame is a relational structure consisting of two sets of beings: properties and objects. Consequently, it is only natural that the decision should have been reached to consider a language made up of two sets of sentences: complex properties and complex objects. Hereafter, we use θ to denote a complex property or to denote a complex object. Intuitively, complex properties stand for sentences about properties whereas complex objects stand for sentences about objects. In particular:

- For every $A \in \Phi_c$ and for every $a \in \Pi_c$, the complex property $\langle A \rangle a$ signifies that in some property A-indiscernible with the current property, it is the case that a;
- For every $a \in \Pi_c$ and for every $A \in \Phi_c$, the complex object $\langle a \rangle A$ signifies that in some object a-indiscernible with the current object, it is the case that A.

Let PAR be a nonempty set of properties and OBJ be a nonempty set of objects. Let R be a function with domain 2^{OBJ} and range the set of all the reflexive, symmetrical and transitive relations on PAR such that, for every $A, B \subseteq OBJ$:

- $R(A \cup B) \subseteq R(A) \cap R(B)$;

and let r be a function with domain 2^{PAR} and range the set of all the reflexive, symmetrical and transitive relations on OBJ such that, for every $a, b \subseteq PAR$:

- $r(a \cup b) \subseteq r(a) \cap r(b)$.

(PAR, OBJ, R, r) is called indiscernibility frame. (PAR, OBJ, R, r) is normal when, for every $A, B \subseteq OBJ$, $R(A \cup B) = R(A) \cap R(B)$ and, for every $a, b \subseteq PAR$, $r(a \cup b) = r(a) \cap r(b)$. Let m be a function with domain Π_c and range 2^{PAR} and let M be a function with domain Φ_c and range 2^{OBJ} such that, for every $a, b \in \Pi_c$ and for every $A \in \Phi_c$:

- $m(\neg a) = PAR \setminus m(a)$;
- $m(a \wedge b) = m(a) \cap m(b)$;
- $m(\langle A \rangle a) = \{x \in PAR$: there exists $y \in PAR$ such that $xR(M(A))y$ and $y \in m(a)\}$;

and, for every $A, B \in \Phi_c$ and for every $a \in \Pi_c$:

- $M(\neg A) = OBJ \setminus M(A)$;
- $M(A \wedge B) = M(A) \cap M(B)$;
- $M(\langle a \rangle A) = \{X \in OBJ$: there exists $Y \in OBJ$ such that $Xr(m(a))Y$ and $Y \in M(A)\}$.

(m, M) is called valuation on (PAR, OBJ, R, r) and (PAR, OBJ, R, r, m, M) is called indiscernibility model on (PAR, OBJ, R, r) defined from (m, M). Especially important kinds of sentences are valid sentences and satisfiable sentences. Consider a complex property a and a complex object A. We shall say that a is valid in (PAR, OBJ, R, r, m, M) when $m(a) = PAR$ whereas we shall say that A is valid in (PAR, OBJ, R, r, m, M) when $M(A) = OBJ$. a is said to be satisfiable in (PAR, OBJ, R, r, m, M) when $m(a) \neq \emptyset$ whereas A is said to be satisfiable in (PAR, OBJ, R, r, m, M) when $M(A) \neq \emptyset$.

7 Axiomatical presentation

In this section, we provide a syntactic characterization of our modal logic for indiscernibility. By a syntactic characterization, we mean a deducibility predicate to be defined inductively by operations on sentences that depend only on their syntactic structure and not on any reference to the semantical notions of validity and satisfiability. Seeing that our language is made up of two sets of sentences, the Hilbert-style deductive system of our modal logic consists of two sets of axioms:

PC for complex properties	PC for complex objects
$\langle A\rangle(a \vee b) \leftrightarrow \langle A\rangle a \vee \langle A\rangle b$	$\langle a\rangle(A \vee B) \leftrightarrow \langle a\rangle A \vee \langle a\rangle B$
$[A]a \rightarrow a$	$[a]A \rightarrow A$
$\langle A\rangle a \rightarrow [A][A]a$	$\langle a\rangle A \rightarrow [a][a]A$

and two sets of rules of inference:

From $A \rightarrow B$ infer $[A]a \rightarrow [B]a$	From $a \rightarrow b$ infer $[a]A \rightarrow [b]A$
From a infer $[A]a$	From A infer $[a]A$

It is well worth noting that the use of the propositional connective $\rightarrow$ in the premisses of the rules "From $A \rightarrow B$ infer $[A]a \rightarrow [B]a$" and "From $a \rightarrow b$ infer $[a]A \rightarrow [b]A$" has a lot to do with the use of the binary relation $\subseteq$ in the deductive system of BML, Boolean modal logic, brought in by Gargov, Passy and Tinchev [11] and furthered by Gargov and Passy [10]. It follows immediately from the definition of our deductive system that the axioms of PO are valid in every indiscernibility model and the inference rules of PO preserve validity in every indiscernibility model. Consequently, a proof by induction will show that:

Theorem 1. *Let θ be a sentence. If θ is a theorem of PO then θ is valid in every indiscernibility model.*

8 Canonical model

A set x of complex properties is consistent iff, for every $a \in \Pi_c$, a is not deducible from x in PO or $\neg a$ is not deducible from x in PO whereas a set X of complex objects is consistent iff, for every $A \in \Phi_c$, A is not deducible from X in PO or $\neg A$ is not deducible from X in PO. We say that a set x of complex properties is maximal iff it is consistent and, for every $a \in \Pi_c$, $a \in x$ or $\neg a \in x$ whereas we say that a set X of complex objects is maximal iff it is consistent and, for every $A \in \Phi_c$, $A \in X$ or $\neg A \in X$. The presence of PC in PO suffices to establish the result known as Lindenbaum's lemma:

- Every consistent set of complex properties has a maximal extension;
- Every consistent set of complex objects has a maximal extension.

Let PAR_C be the set of all the maximal sets of complex properties and let OBJ_C be the set of all the maximal sets of complex objects. Let R_C be the function with domain 2^{OBJ_C} and range the set of all the binary relations on PAR_C defined in the following way, for every $A \subseteq OBJ_C$ and for every $x, y \in PAR_C$:

- $xR_C(A)y$ iff, for every $A' \in \Phi_c$ and for every $a' \in \Pi_c$, if $\{X \in OBJ_C : A' \in X\} \subseteq A$ and $[A']a' \in x$ then $a' \in y$;

and let r_C be the function with domain 2^{PAR_C} and range the set of all the binary relations on OBJ_C defined in the following way, for every $a \subseteq PAR_C$ and for every $X, Y \in OBJ_C$:

- $X r_C(a) Y$ iff, for every $a' \in \Pi_c$ and for every $A' \in \Phi_c$, if $\{x \in PAR_C\colon a' \in x\} \subseteq a$ and $[a']A' \in X$ then $A' \in Y$.

It is easy to verify that (PAR_C, OBJ_C, R_C, r_C) is an indiscernibility frame. Let (m_C, M_C) be the valuation on (PAR_C, OBJ_C, R_C, r_C) such that, for every $\pi \in \Pi_a$:

- $m_C(\pi) = \{x \in PAR_C\colon \pi \in x\}$;

and, for every $\phi \in \Phi_a$:

- $M_C(\phi) = \{X \in OBJ_C\colon \phi \in X\}$.

A proof by induction will show that, for every $a \in \Pi_c$, $m_C(a) = \{x \in PAR_C\colon a \in x\}$ and, for every $A \in \Phi_c$, $M_C(A) = \{X \in OBJ_C\colon A \in X\}$. Consequently:

Theorem 2. *Let θ be a sentence. If θ is valid in every indiscernibility model then θ is a theorem of PO.*

In section 9, we prove that sentences valid in every normal indiscernibility model are also valid in every indiscernibility model. In section 11, we prove that sentences valid in every finite indiscernibility model are also valid in every indiscernibility model.

9 Normal completeness

Consider an indiscernibility model (PAR, OBJ, R, r, m, M). Let i be the set of all the functions f with domain $2^{OBJ} \times OBJ$ and range 2^{PAR} such that, for every $A \subseteq OBJ$, the set $\{X \in OBJ\colon f(A, X) \neq \emptyset\}$ is finite and let I be the set of all the functions F with domain $2^{PAR} \times PAR$ and range 2^{OBJ} such that, for every $a \subseteq PAR$, the set $\{x \in PAR\colon F(a, x) \neq \emptyset\}$ is finite. Hereafter we shall use, for every $a, b \subseteq PAR$, $a \uplus b$ to denote $(a \setminus b) \cup (b \setminus a)$ whereas we shall use, for every $A, B \subseteq OBJ$, $A \biguplus B$ to denote $(A \setminus B) \cup (B \setminus A)$. From the definition of $\uplus$, we see that $\uplus$ is commutative, $\uplus$ is associative and, for every $a \subseteq PAR$, $a \uplus a = \emptyset$ whereas, from the definition of $\biguplus$, we see that $\biguplus$ is commutative, $\biguplus$ is associative and, for every $A \subseteq OBJ$, $A \biguplus A = \emptyset$. Let $PAR' = PAR \times i$ and $OBJ' = OBJ \times I$. Let R' be the function with domain $2^{OBJ'}$ and range the set of all the binary relations on PAR' defined in the following way, for every $A' \subseteq OBJ'$, for every $x, y \in PAR$ and for every $f, g \in i$, $(x, f) R'(A') (y, g)$ iff, for every $A \subseteq OBJ$:

- For every $X \in OBJ$, if $X \in A$ and there exists $F \in I$ such that $(X, F) \in A'$ then $f(A, X) = g(A, X)$;
- $R(A)(x) \uplus (\uplus\{f(A, X)\colon X \in A\}) = R(A)(y) \uplus (\uplus\{g(A, X)\colon X \in A\})$;

and let r' be the function with domain $2^{PAR'}$ and range the set of all the binary relations on OBJ' defined in the following way, for every $a' \subseteq PAR'$, for every $X, Y \in OBJ$ and for every $F, G \in I$, $(X, F)r'(a')(Y, G)$ iff, for every $a \subseteq PAR$:

- For every $x \in PAR$, if $x \in a$ and there exists $f \in i$ such that $(x, f) \in a'$ then $F(a, x) = G(a, x)$;
- $r(a)(X) \biguplus (\biguplus \{F(a, x)\colon x \in a\}) = r(a)(Y) \biguplus (\biguplus \{G(a, x)\colon x \in a\})$.

It is easy to verify that (PAR', OBJ', R', r') is a normal indiscernibility frame. Let (m', M') be the valuation on (PAR', OBJ', R', r') such that, for every $\pi \in \Pi_a$:

- $m'(\pi) = m(\pi) \times i$;

and, for every $\phi \in \Phi_a$:

- $M'(\phi) = m(\phi) \times I$.

A proof by induction will show that, for every $a \in \Pi_c$, $m'(a) = m(a) \times i$ and, for every $A \in \Phi_c$, $M'(A) = M(A) \times I$. Consequently:

Theorem 3. *Let θ be a sentence. If θ is valid in every normal indiscernibility model then θ is valid in every indiscernibility model.*

10 Finite completeness

Consider an indiscernibility model (PAR, OBJ, R, r, m, M). Let Π be a set of complex properties and let Φ be a set of complex objects such that for every $a, b \in \Pi_c$ and for every $A \in \Phi_c$:

- If $\neg a \in \Pi$ then $a \in \Pi$;
- If $a \wedge b \in \Pi$ then $a \in \Pi$ and $b \in \Pi$;
- If $\langle A \rangle a \in \Pi$ then $A \in \Phi$ and $a \in \Pi$;

and, for every $A, B \in \Phi_c$ and for every $a \in \Pi_c$:

- If $\neg A \in \Phi$ then $A \in \Phi$;
- If $A \wedge B \in \Phi$ then $A \in \Phi$ and $B \in \Phi$;
- If $\langle a \rangle A \in \Phi$ then $a \in \Pi$ and $A \in \Phi$.

Let $\equiv_\Pi$ be the relation of equivalence on PAR defined in the following way, for every $x, y \in PAR$:

- $x \equiv_\Pi y$ iff, for every $a \in \Pi_c$, if $a \in \Pi$ then $x \in m(a)$ iff $y \in m(a)$;

and let $\equiv_\Phi$ be the relation of equivalence on OBJ defined in the following way, for every $X, Y \in OBJ$:

- $X \equiv_\Phi Y$ iff, for every $A \in \Phi_c$, if $A \in \Phi$ then $X \in M(A)$ iff $Y \in M(A)$.

For every $x \in PAR$, the equivalence class of x modulo $\equiv_{\Pi}$ is denoted $\mid x \mid$ whereas, for every $X \in OBJ$, the equivalence class of X modulo $\equiv_{\Phi}$ is denoted $\mid X \mid$. Let $PAR' =\mid PAR \mid$ and $OBJ' =\mid OBJ \mid$. Let R' be the function with domain $2^{OBJ'}$ and range the set of all the binary relations on PAR' defined in the following way, for every $A' \subseteq OBJ'$ and for every $x, y \in PAR$:

- $\mid x \mid R'(A') \mid y \mid$ iff, for every $A \in \Phi_c$ and for every $a \in \Pi_c$, if $[A]a \in \Pi$ and $\mid M(A) \mid \subseteq A'$ then $x \in m([A]a)$ iff $y \in m([A]a)$;

and let r' be the function with domain $2^{PAR'}$ and range the set of all the binary relations on OBJ' defined in the following way, for every $a' \subseteq PAR'$ and for every $X, Y \in OBJ$:

- $\mid X \mid r'(a') \mid Y \mid$ iff, for every $a \in \Pi_c$ and for every $A \in \Phi_c$, if $[a]A \in \Phi$ and $\mid m(a) \mid \subseteq a'$ then $X \in M([a]A)$ iff $Y \in M([a]A)$.

It is easy to verify that (PAR', OBJ', R', r') is an indiscernibility frame. Let (m', M') be a valuation on (PAR', OBJ', R', r') such that, for every $\pi \in \Pi_a$, if $\pi \in \Pi$ then $m'(\pi) =\mid m(\pi) \mid$ and, for every $\phi \in \Phi_a$, if $\phi \in \Phi$ then $M'(\phi) = \mid M(\phi) \mid$. A proof by induction will show that, for every $a \in \Pi_c$, if $a \in \Pi$ then $m'(a) =\mid m(a) \mid$ and, for every $A \in \Phi_c$, if $A \in \Phi$ then $M'(A) =\mid M(A) \mid$. Consequently:

Theorem 4. *Let θ be a sentence. If θ is valid in every finite indiscernibility model then θ is valid in every indiscernibility model.*

11 Conclusion

It is well worth noting that the normal indiscernibility frame (PAR', OBJ', R', r') defined in section 9 is finite if the indiscernibility model (PAR, OBJ, R, r, m, M) considered there is finite. As a result, the following conditions are equivalent, for every sentence θ:

- θ is a theorem of PO;
- θ is valid in every indiscernibility model;
- θ is valid in every normal indiscernibility model;
- θ is valid in every finite indiscernibility model;
- θ is valid in every normal finite indiscernibility model.

In other respects, the finite indiscernibility frame (PAR', OBJ', R', r') defined in section 10 contains less than $2^{Card(\Pi)}$ properties as well as less than $2^{Card(\Phi)}$ objects. Consequently, PO-satisfiability is in NEXPTIME whereas the exact complexity of the PO-satisfiability problem remains open.

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