# Line-based affine reasoning in Euclidean plane*

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**Abstract.** We consider the binary relations of parallelism and convergence between lines in a 2-dimensional affine space. Associating with parallelism and convergence the binary predicates $P$ and $C$ and the modal connectives $[P]$ and $[C]$, we consider a first-order theory based on these predicates and a modal logic based on these modal connectives. We investigate the axiomatization/completeness and the decidability/complexity of this first-order theory and this modal logic.

## 1 Introduction

In recent years, there has been an increasing interest in spatial reasoning and important applications to practical issues such as geographical information systems have made the field even more attractive [15]. Historically, topological spaces were among the first mathematical models of space applied to spatial information processing and they occupy the central position in the subject. The work of Randell, Cui and Cohn [12], who brought in the region connection calculus, was influential at the early stages. A major impetus for studying topological spaces in general and the region connection calculus in particular was the fact that, within the framework of constraint satisfaction problems, qualitative spatial reasoning can be easily automated [3, 13]. In the second half of the 1990's, this work was continued by others and their efforts generated many results concerning different kinds of spatial relationships between different types of spatial entities [1, 7, 8, 10, 11].

Plane affine geometry, one of the most prominent mathematical models of space, arises from the study of points and lines by means of properties stated in terms of incidence. In plane coordinate geometry, lines are sets of points satisfying linear equations. Completely determined by two of their points, they can also be considered as abstract entities. They have certain mutual relations like parallelism and convergence: two lines are parallel iff they never meet whereas they are convergent iff they have exactly one common point. Lines are to be found in many axiomatizations of plane affine geometry — however we had great difficulty finding any examples of qualitative forms of spatial reasoning based solely on them. To confirm this claim, we have not been able to find any explicit reference to a first-order language or to a modal language devoted to the study

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of parallelism and convergence, the only possible relationships between lines in plane affine geometry.

Associating with parallelism and convergence the binary predicates $P$ and $C$, this paper is about the completeness and the complexity of a first-order theory based on these predicates. Linking the modal connectives $[P]$ and $[C]$ with parallelism and convergence, this paper is interested in the completeness and the complexity of a modal logic founded on these modal connectives. The paper has two major parts. The first introduces a first-order theory of lines based on the binary predicates $P$ and $C$ and applies the technique of model theory to it. It mainly proves that the first-order theory of parallelism and incidence in plane affine geometry is a complete first-order theory whose membership problem can be decided in polynomial space. In the second part of the paper we turn to the following question: what is the modal logic of lines with the modal connectives $[P]$ and $[C]$? Our modal logic of parallelism and convergence is a complete modal logic whose membership problem can be decided in nondeterministic polynomial time. In all parts, completeness means completeness with respect to the Euclidean plane.

## 2 First-order theory

It is now time to meet the first-order languages we will be working with. What we would like to do in this section is study structures consisting of lines in a space of dimension 2. We assume some familiarity with model theory. Readers wanting more details may refer, for example, to [6]. Our line-based first-order theory is based on the idea of associating with parallelism and convergence the binary predicates $P$ and $C$, with the formulas $P(x, y)$ and $C(x, y)$ being read "$x$ is parallel to $y$" and "$x$ is convergent with $y$". The *formulas* are given by the rule:

$$\phi ::= P(x, y) \mid C(x, y) \mid x = y \mid \neg\phi \mid \phi \vee \psi \mid \forall x\phi,$$

where $x$ and $y$ range over a countable set of *individual variables*. Let the *size of* $\phi$, denoted $|\phi|$, be the number of symbols occurring in $\phi$. We adopt the standard definitions for the remaining Boolean operations and for the existential quantifier.

### 2.1 Parallelism and convergence

A *line-based affine plane* is a relational structure of the form $\mathcal{F} = (L, P, C)$ where $L$ is a nonempty set of *lines* and $P$ and $C$ are binary relations on $L$. We shall say that an affine plane $\mathcal{F} = (L, P, C)$ is *standard* iff it satisfies the following sentences:

| | |
|---|---|
| *IRREF* | $\forall x \neg P(x, x)$ |
| | $\forall x \neg C(x, x)$ |
| *TRANS* | $\forall x \forall y \forall z (P(x, y) \wedge C(y, z) \rightarrow C(x, z))$ |
| *UNIV* | $\forall x \forall y (x = y \vee P(x, y) \vee C(x, y))$ |

$$
\begin{aligned}
DENS_n \quad & \forall x \forall y_1 \ldots \forall y_n (P(x,y_1) \wedge \ldots \wedge P(x,y_n) \to \exists z (P(x,z) \wedge P(z,y_1) \wedge \ldots \\
& \qquad\qquad \wedge P(z,y_n))), \quad n \geq 0 \\
& \forall x \forall y_1 \ldots \forall y_n (C(x,y_1) \wedge \ldots \wedge C(x,y_n) \to \exists z (C(x,z) \wedge C(z,y_1) \wedge \ldots \\
& \qquad\qquad \wedge C(z,y_n))), \quad n \geq 0
\end{aligned}
$$

Notice first that:

**Proposition 1.** *Equality and convergence are first-order definable with $P$ in any standard affine plane.*

*Proof.* It suffices to observe that the sentences $\forall x \forall y (x = y \leftrightarrow \forall z (P(x,z) \leftrightarrow P(y,z)))$ and $\forall x \forall y (C(x,y) \leftrightarrow \neg P(x,y) \wedge x \neq y)$ are true in all standard affine planes. $\dashv$

Let $\mathcal{F} = (L, P, C)$ be a standard affine plane. It is a simple matter to check that the following sentences $\forall x \forall y (P(x,y) \to P(y,x))$, $\forall x \forall y (C(x,y) \to C(y,x))$, $\forall x \forall y \forall z (\neg C(x,y) \wedge C(y,z) \to C(x,z))$ and $\forall x \forall y \exists z (C(x,z) \wedge C(z,y))$ are true in $\mathcal{F}$. Consequently $\neg C$ is an equivalence relation on $L$ such that every equivalence class in $L$ modulo $\neg C$ is made up of infinitely many lines whereas the partition of $L$ modulo $\neg C$ consists of infinitely many equivalence classes. In the sequel the following notation will be used, for all lines $a$ in $L$: $[a] = \{b\colon \neg C(a,b)\}$. Obviously, the axioms as above have models in each infinite power. We should consider, for instance, the affine plane $\mathcal{F}^2_{\mathbb{R}}$. Its set of lines consists of all lines in the Euclidean plane. A countable structure approximating $\mathcal{F}^2_{\mathbb{R}}$ is $\mathcal{F}^2_{\mathbb{Q}}$: its set of lines is made up of all lines in the Euclidean plane containing at least two points with rational coordinates. Clearly, both affine planes are standard. To illustrate the value of countable standard affine planes, we shall prove the following proposition:

**Proposition 2.** *Let $\mathcal{F}$ and $\mathcal{F}'$ be standard affine planes. If $\mathcal{F}$ is countable then $\mathcal{F}$ is elementary embeddable in $\mathcal{F}'$.*

*Proof.* Let $\mathcal{F} = (L, P, C)$ and $\mathcal{F}' = (L', P', C')$ be standard affine planes. Suppose that $\mathcal{F}$ is countable, we demonstrate that $\mathcal{F}$ is elementary embeddable in $\mathcal{F}'$. We need to consider an injective mapping $g$ on the partition of $L$ into the partition of $L'$. For each equivalence class $[a]$ in the partition of $L$, we also need an injective mapping $h_{[a]}$ on $[a]$ into $g([a])$. Now let $f$ be the mapping on $L$ into $L'$ defined with $f(a) = h_{[a]}(a)$. To see that $f$ is an elementary embedding of $\mathcal{F}$ into $\mathcal{F}'$, we invite the reader to show by induction on the complexity of formulas $\phi(x_1, \ldots, x_n)$ in $x_1$, …, $x_n$ and $P$, $C$ and $=$, that for all lines $a_1$, …, $a_n$ in $L$, $\mathcal{F} \models \phi(x_1, \ldots, x_n)[a_1, \ldots, a_n]$ iff $\mathcal{F}' \models \phi(x_1, \ldots, x_n)[f(a_1), \ldots, f(a_n)]$. $\dashv$

As a corollary we obtain that:

**Proposition 3.** *Any two standard affine planes are elementary equivalent.*

The *first-order theory $SAP$ of standard affine planes* has the following list of axioms: $IRREF$, $TRANS$, $UNIV$, $DENS_0$, $DENS_1$, …. There are several results about $SAP$:

**Proposition 4.** **(i)** *SAP is ω-categorical;* **(ii)** *SAP is not categorical in any uncountable power;* **(iii)** *SAP is maximal consistent;* **(iv)** *SAP is complete with respect to* $\mathcal{F}^2_{\mathbb{R}}$ *and* $\mathcal{F}^2_{\mathbb{Q}}$*;* **(v)** *SAP is decidable;* **(vi)** *the membership problem in SAP is PSPACE-complete;* **(vii)** *SAP is not axiomatizable with finitely many variables, and hence, it is not finitely axiomatizable.*

*Proof.* **(i)** Rather like the proof of proposition 2.

**(ii)** Let $\alpha$ be an uncountable power. We demonstrate that we can find standard affine planes $\mathcal{F} = (L, P, C)$ and $\mathcal{F}' = (L', P', C')$ of power $\alpha$ such that $\mathcal{F}$ and $\mathcal{F}'$ are not isomorphic. Let $S$ be a set of power $\alpha$. If $L = S \times \mathbb{N}$ and $L' = \mathbb{N} \times S$ then the affine planes $\mathcal{F} = (L, P, C)$ and $\mathcal{F}' = (L', P', C')$ of power $\alpha$ defined by:

$$P = \{((a,i),(b,j))\colon a = b \text{ and } i \neq j\} \text{ and } C = \{((a,i),(b,j))\colon a \neq b\},$$
$$P' = \{((i,a),(j,b))\colon i = j \text{ and } a \neq b\} \text{ and } C' = \{((i,a),(j,b))\colon i \neq j\},$$

are standard. The proof that $\mathcal{F}$ and $\mathcal{F}'$ are not isomorphic, which is not difficult, is left as an exercise.

**(iii)** By proposition 3.

**(iv)** Immediately follows from **(iii)**, since $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{Q}}$ are models of $SAP$.

**(v)** Simple application of item **(iii)**.

**(vi)** Let $EQ$ be the first-order theory of identity, =, and $EQ^\infty$ be the first-order theory of identity in all infinite sets. Indeed, $EQ^\infty$ coincides with the set of all true sentences in an arbitrary infinite set. According to Stockmeyer [14], the membership problem in $EQ$ is PSPACE-complete, but here we make use of the following

*Claim.* The membership problem in $EQ^\infty$ is PSPACE-complete.

Let first complete the proof of **(iii)**. Observe that $SAP$ is conservative extension of $EQ^\infty$, since any model of $EQ^\infty$ can be extended to a standard line-based affine plane, i.e. to a model $SAP$, almost in the same way as in the proof of **(ii)**. Therefore, the membership problem in $SAP$ is PSPACE-hard. Now, let we see that it is in PSPACE. Observe that the submodel $\mathcal{F} = (L, P, C)$ of $\mathcal{F}^2_{\mathbb{R}}$ restricted to nonvertical lines is standard. By proposition 3, the membership problem in $SAP$ is equivalent to the truth problem in $\mathcal{F}$. Remark that any line $a$ in $L$ univocally corresponds to a Cartesian equation of the form $v = a_1 \times u + a_2$. Obviously, for all lines $b$ in $L$, the corresponding Cartesian equation $v = b_1 \times u + b_2$ is such that $P(a, b)$ iff $a_1 = b_1$ and $a_2 \neq b_2$, $C(a, b)$ iff $a_1 \neq b_1$ and $a = b$ iff $a_1 = b_1$ and $a_2 = b_2$. It follows that the truth problem in $\mathcal{F}$ is equivalent to a particular membership problem in $EQ^\infty$. According to the above claim, this problem can be decided in PSPACE. Therefore, the membership problem in $SAP$ is PSPACE-complete.

*Proof of the claim.* Let $m$ be a positive integer. Then for any first-order sentence $\varphi$ containing only identity and not more than $m$ individual variables, the following equivalence holds:

$$\mathcal{A} \models \varphi \quad \text{iff} \quad \mathcal{B} \models \varphi$$

for arbitrary structures $\mathcal{A}$ and $\mathcal{B}$ having at least $m$ elements. Indeed, this can be quite easily checked and we invite the reader to do this exercise, for example, considering $m$-pebble game between $\mathcal{A}$ and $\mathcal{B}$. So,

$$\varphi \in EQ^{\infty} \quad \text{iff} \quad \sigma_m \vee \varphi \in EQ,$$

where $\sigma_m$ says "there are not more than $m$ elements", i.e. $\sigma_m$ can be the sentence $\exists x_1 \ldots \exists x_m \forall y(y = x_1 \vee \cdots \vee y = x_m)$. Obviously PSPACE-completeness of $EQ$, [14], and the size of $\sigma_m$ give that the membership problem in $EQ^{\infty}$ is in PSPACE. For the PSPACE-hardness we shall imitate the validity in a substructure with given size. Suppose that the sentence $\varphi$ has exactly $m$ variables. Without loss of generality we can assume $2 \leq m$ and choose different individual variables $x_1, \ldots, x_m$ not occurring in $\varphi$. Let $k$ be arbitrary integer such that $0 < k \leq m$. Define $\varphi_{(k)}$ to be the formula obtained from $\varphi$ after replacing each subformula of the type $\exists x\psi$ by $\exists x((x = x_1 \vee \cdots \vee x = x_k) \wedge \psi)$ and each subformula of the type $\forall x\psi$ by $\forall x((x = x_1 \vee \cdots \vee x = x_k) \rightarrow \psi)$. Finally, denote by $\varphi^{(k)}$ the sentence $\exists x_1 \ldots \exists x_k (\bigwedge_{1 \leq i < j \leq k}(x_i \neq x_j) \wedge \varphi_{(k)})$ if $k \geq 2$, and by $\varphi^{(1)}$ the sentence $\exists x_1 \varphi_{(1)}$. The reader is invited to verify that for any structure $\mathcal{A}$ holds: $\mathcal{A} \models \varphi^{(k)}$ iff $\varphi$ is true in the structures with exactly $k$ elements and $\mathcal{A}$ has at least $k$ elements. Now, since

$$\varphi \in EQ \quad \text{iff} \quad \varphi^{(1)} \wedge \varphi^{(2)} \wedge \cdots \wedge \varphi^{(m)} \in EQ^{\infty},$$

and the last sentence has size polynomial in the size of $\varphi$, again by Stockmeyer's theorem it follows the PSPACE-hardness of $EQ^{\infty}$. This completes the proof of the claim.

**(vii)** For the sake of contradiction suppose that the theory $SAP$ is axiomatized with the set $\Gamma$ of sentences in which are used individual variables only among $x_1, \ldots, x_k$. Let us consider the line-based affine plane $\mathcal{F} = (L, P, C)$, where

$L = \{(a, i) : 0 \leq a < k \text{ and } 0 \leq i < k\}$,
$P = ((a, i), (b, j)) : a = b \text{ and } i \neq j\}$,
$C = ((a, i), (b, j)) : a \neq b\}$.

Let $n$ be an arbitrary nonnegative integer. Consider the $k$-pebble game with $n$ moves over $\mathcal{F}$ and $\mathcal{F}^2_{\mathbb{Q}}$. It is easy to find a winning strategy for Duplicator (the second player) in this game. Therefore $\mathcal{F} \models \Gamma$ if and only if $\mathcal{F}^2_{\mathbb{Q}} \models \Gamma$. Since $\Gamma$ axiomatize $SAP$, we conclude that $\mathcal{F}$ is a model of $SAP$. Clearly $\mathcal{F} \not\models DENS_{k-1}$, which is a desired contradiction. $\dashv$

### 2.2 Convergence alone

The first-order language of the discussion above is inextricably tied up with the properties of $P$, $C$ and $=$. Of course, there is nothing against investigating the pure $C$-fragment only; the pure $P$-fragment being equivalent to the full language. All the more so since:

**Proposition 5.** *Equality and parallelism are not first-order definable with $C$ in any standard affine plane.*

*Proof.* Let $\mathcal{F} = (L, P, C)$ be a standard affine plane. We demonstrate that equality and parallelism are not first-order definable with $C$ in $\mathcal{F}$. Seeing that $\mathcal{F} \models \forall x \forall y (P(x, y) \leftrightarrow (x \neq y \wedge \neg C(x, y))$, it is sufficient to show that equality is not first-order definable with $C$ in $\mathcal{F}$. Assume that there is a formula $\phi(x, y)$ in $x$, $y$ and $C$, such that $\mathcal{F} \models \forall x \forall y (x = y \leftrightarrow \phi(x, y))$. Let $a$ be a line in $L$. We need to consider a surjective mapping $g$ on $[a]$ into itself. Now let $f$ be the mapping on $L$ into itself defined as follows:

$$f(b) = \begin{cases} b & b \notin [a], \\ g(b) & b \in [a]. \end{cases}$$

As a simple exercise we invite the reader to show by induction on the complexity of formulas $\psi(x_1, \ldots, x_n)$ in $x_1$, ..., $x_n$ and $C$, that for all lines $a_1$, ..., $a_n$ in $L$, $\mathcal{F} \models \psi(x_1, \ldots, x_n)[a_1, \ldots, a_n]$ iff $\mathcal{F} \models \psi(x_1, \ldots, x_n)[f(a_1), \ldots, f(a_n)]$. Hence, for all lines $a, b$ in $L$, $\mathcal{F} \models \phi(x, y)[a, b]$ iff $\mathcal{F} \models \phi(x, y)[f(a), f(b)]$. It follows that for all lines $a, b$ in $L$, $\mathcal{F} \models x = y[a, b]$ iff $\mathcal{F} \models x = y[f(a), f(b)]$. If $g$ is not injective then we can find lines $a$ and $b$ in $L$ such that $\mathcal{F} \not\models x = y[a, b]$ and $\mathcal{F} \models x = y[f(a), f(b)]$, a contradiction. $\dashv$

This observation leads us to consider a line-based first-order theory based solely on $C$, i.e. $C$ will be the unique predicate symbol considered from now on in this section. A *weak line-based affine plane* is a relational structure of the form $\mathcal{F} = (L, C)$ where $L$ is a nonempty set of *lines* and $C$ is a binary relation on $L$. We shall say that a weak affine plane $\mathcal{F} = (L, C)$ is *standard* iff it satisfies the following sentences:

$IRREF^-$ $\forall x \neg C(x, x)$
$TRANS^-$ $\forall x \forall y \forall z (\neg C(x, y) \wedge C(y, z) \rightarrow C(x, z))$
$DENS_n^-$ $\forall x \forall y_1 \ldots \forall y_n (C(x, y_1) \wedge \ldots \wedge C(x, y_n) \rightarrow \exists z (C(x, z) \wedge C(z, y_1) \wedge \ldots \wedge C(z, y_n)))$

Let $\mathcal{F} = (L, C)$ be a standard weak affine plane. It is easy to see that the sentences $\forall x \forall y (C(x, y) \rightarrow C(y, x))$ and $\forall x \forall y \exists z (C(x, z) \wedge C(z, y))$ are true in $\mathcal{F}$. Hence $\neg C$ is an equivalence relation on $L$ such that the partition of $L$ modulo $\neg C$ consists of infinitely many equivalence classes. Examples of standard weak affine planes are $\mathcal{F}_{\mathbb{R}}^{2-}$ and $\mathcal{F}_{\mathbb{Q}}^{2-}$, the pure $C$-fragments of $\mathcal{F}_{\mathbb{R}}^2$ and $\mathcal{F}_{\mathbb{Q}}^2$. To study standard weak affine planes more precisely, the concept of quotient will be of use to us. Let $\mathcal{F} = (L, C)$ be a standard weak affine plane. The *quotient of $\mathcal{F}$ modulo* $\neg C$ is the standard weak affine plane $\mathcal{F}/[\cdot] = (L/[\cdot], C/[\cdot])$ defined as follows:

$L/[\cdot] = \{[a] \colon a \in L\}$,
$C/[\cdot] = \{([a], [b]) \colon C(a, b)\}$.

Observe that for all lines $a$, $b$ in $L$, $C/[\cdot]([a], [b])$ iff $[a] \neq [b]$. A straightforward consequence of our definition is that:

**Proposition 6.** *Let $\mathcal{F}$ and $\mathcal{F}'$ be standard weak affine planes. If $\mathcal{F}$ is countable then $\mathcal{F}/[\cdot]$ is elementary embeddable in $\mathcal{F}'$.*

*Proof.* Let $\mathcal{F} = (L, C)$ and $\mathcal{F}' = (L', C')$ be standard weak affine planes. Suppose that $\mathcal{F}$ is countable, we demonstrate that $\mathcal{F}/[\cdot]$ is elementary embeddable in $\mathcal{F}'$. We need to consider an injective mapping $g$ on the partition of $L$ into the partition of $L'$. Now let $f$ be a mapping on the partition of $L$ into $L'$ such that:

For all lines $a$ in $L$, $f([a])$ belongs to $g([a])$.

To see that $f$ is an elementary embedding of $\mathcal{F}/[\cdot]$ into $\mathcal{F}'$, we invite the reader to show by induction on the complexity of formulas $\phi(x_1, \ldots, x_n)$ in $x_1, \ldots, x_n$ and $C$, that for all lines $a_1, \ldots, a_n$ in $L$, $\mathcal{F}/[\cdot] \models \phi(x_1, \ldots, x_n)[[a_1], \ldots, [a_n]]$ iff $\mathcal{F}' \models \phi(x_1, \ldots, x_n)[f([a_1]), \ldots, f([a_n])]$. $\dashv$

As a corollary we obtain that:

**Proposition 7.** *Any two standard weak affine planes are elementary equivalent.*

The *first-order theory $SAP^-$ of standard weak affine planes* has the following list of axioms: $IRREF^-$, $TRANS^-$, $DENS_0^-$, $DENS_1^-$, .... There are several results about $SAP^-$:

**Proposition 8.** **(i)** *$SAP^-$ is not $\omega$-categorical;* **(ii)** *$SAP^-$ is not categorical in any uncountable power;* **(iii)** *$SAP^-$ is maximal consistent;* **(iv)** *$SAP^-$ is complete with respect to $\mathcal{F}_{\mathbb{R}}^{2-}$ and $\mathcal{F}_{\mathbb{Q}}^{2-}$;* **(v)** *$SAP^-$ is decidable;* **(vi)** *the membership problem in $SAP^-$ is PSPACE-complete;* **(vii)** *$SAP^-$ is not axiomatizable with finitely many variables, and hence, it is not finitely axiomatizable.*

*Proof.* **(i)** The proof that $\mathcal{F}_{\mathbb{Q}}^{2-}$ and $\mathcal{F}_{\mathbb{Q}}^{2-}/[\cdot]$ are not isomorphic, which is not difficult, is left as an exercise.

**(ii)** Similar to the proof of item **(i)**.

**(iii)** By proposition 7.

**(iv)** Immediately follows from **(iii)**, since $\mathcal{F}_{\mathbb{R}}^{2-}$ and $\mathcal{F}_{\mathbb{Q}}^{2-}$ are models of $SAP^-$.

**(v)** Simple application of item **(iii)**.

**(vi)** Rather like the proof of item **(vi)** in proposition 4.

**(vii)** Similar to the proof of item **(vii)** in proposition 4. $\dashv$

The reader may easily verify that every theorem of $SAP^-$ is also a theorem of $SAP$. We will now prove the converse result, that is to say:

**Proposition 9.** *$SAP$ is a conservative extension of $SAP^-$.*

*Proof.* By item **(iv)** in proposition 4 and item **(iv)** in proposition 8. $\dashv$

## 3 Modal logic

It is now time to meet the modal languages we will be working with, generalizing the modal languages introduced by Balbiani and Goranko [2]. What we would like to do in this section is study a modal logic of lines in a space of dimension 2. We assume some familiarity with modal logic. Readers wanting more details may refer, for example, to [4] or to [5]. Our line-based modal logic is based on the idea of associating with parallelism and convergence the modal connectives $[P]$ and $[C]$, with the formulas $[P]\phi$ and $[C]\phi$ being read "for all parallel lines, $\phi$" and "for all convergent lines, $\phi$". The *formulas* are given by the rule:

$$\phi ::= p \mid \neg\phi \mid \phi \vee \psi \mid [P]\phi \mid [C]\phi,$$

where $p$ ranges over a countable set of *propositional variables*. Let the *size of* $\phi$, denoted $\mid \phi \mid$, be the number of symbols occurring in $\phi$. We adopt the standard definitions for the remaining Boolean operations and for the diamond modality. The concept of *subformula* is standard, the expression $Sf(\phi)$ denoting the set of all subformulas of formula $\phi$.

### 3.1 Parallelism and convergence

A *Kripke model* is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F} = (L, P, C)$ is an affine plane and $V$ is a *valuation on* $\mathcal{F}$, i.e. a function assigning to each line $a$ in $L$ a set $V(a)$ of propositional variables. If $\mathcal{M} = (L, P, C, V)$ is a Kripke model and $a$ is a line in $L$ then the relation "$\phi$ is *true in* $\mathcal{M}$ *at* $a$", denoted $\mathcal{M}, a \models \phi$, is defined inductively on the complexity of formulas $\phi$ as usual. In particular:

$\mathcal{M}, a \models [P]\phi$ iff for all $b \in L$ with $P(a, b)$, we have $\mathcal{M}, b \models \phi$,
$\mathcal{M}, a \models [C]\phi$ iff for all $b \in L$ with $C(a, b)$, we have $\mathcal{M}, b \models \phi$.

Formula $\phi$ is *true in Kripke model* $\mathcal{M}$, in symbols $\mathcal{M} \models \phi$, iff $\mathcal{M}, a \models \phi$ for all $a \in L$. $\phi$ is said to be *valid in affine plane* $\mathcal{F} = (L, P, C)$, in symbols $\mathcal{F} \models \phi$, iff $\mathcal{M} \models \phi$ for all models $\mathcal{M} = (L, P, C, V)$ based on $\mathcal{F}$. The following formulas are valid in all standard affine planes:

$\phi \rightarrow [P]\langle P\rangle\phi$,
$\phi \rightarrow [C]\langle C\rangle\phi$,
$\phi \wedge [P]\phi \rightarrow [P][P]\phi$,
$[C]\phi \rightarrow [P][C]\phi$,
$\phi \wedge [P]\phi \wedge [C]\phi \rightarrow [C][C]\phi$,
$\langle P\rangle\phi_1 \wedge \ldots \wedge \langle P\rangle\phi_n \rightarrow \langle P\rangle(\langle P\rangle\phi_1 \wedge \ldots \wedge \langle P\rangle\phi_n)$,
$\langle C\rangle\phi_1 \wedge \ldots \wedge \langle C\rangle\phi_n \rightarrow \langle C\rangle(\langle C\rangle\phi_1 \wedge \ldots \wedge \langle C\rangle\phi_n)$.

Let $ML(SAP)$ be the smallest normal modal logic, in the language just described, that contains the above formulas as proper axioms. It is a simple exercise in modal logic to check that if $\phi$ is a theorem of $ML(SAP)$ then $\phi$ is valid in every standard affine plane. Now we come to prove the converse proposition: if $\phi$ is valid in every standard affine plane then $\phi$ is a theorem of $ML(SAP)$. Let

$\mathcal{F} = (L, P, C)$ be a generated subframe of the canonical frame for $ML(SAP)$. Seeing that the proper axioms of $ML(SAP)$ are all Sahlqvist formulas, it is easy to get information about the structure of $\mathcal{F}$:

$$\forall x \forall y (P(x, y) \to P(y, x)),$$
$$\forall x \forall y (C(x, y) \to C(y, x)),$$
$$\forall x \forall y \forall z (P(x, y) \wedge P(y, z) \to x = z \vee P(x, z)),$$
$$\forall x \forall y \forall z (P(x, y) \wedge C(y, z) \to C(x, z)),$$
$$\forall x \forall y \forall z (C(x, y) \wedge C(y, z) \to x = z \vee P(x, z) \vee C(x, z)),$$
$$\forall x \forall y_1 \ldots \forall y_n (P(x, y_1) \wedge \ldots \wedge P(x, y_n) \to \exists z (P(x, z) \wedge P(z, y_1) \wedge \ldots \wedge P(z, y_n))),$$
$$\forall x \forall y_1 \ldots \forall y_n (C(x, y_1) \wedge \ldots \wedge C(x, y_n) \to \exists z (C(x, z) \wedge C(z, y_1) \wedge \ldots \wedge C(z, y_n))).$$

This motivates the following definition. A rooted affine plane $\mathcal{F} = (L, P, C)$ is said to be *prenormal* iff it satisfies the above first-order conditions. Two simple observations. First, the Sahlqvist formula $[C]\phi \to [C][P]\phi$ corresponds to the first-order condition $\forall x \forall y \forall z (C(x, y) \wedge P(y, z) \to C(x, z))$ which is true in all prenormal affine planes. Hence it is a theorem of $ML(SAP)$. Second, a prenormal affine plane $\mathcal{F} = (L, P, C)$ where $P$ and $C$ are irreflexive relations on $L$ is standard. As an immediate consequence of the Sahlqvist completeness theorem, we obtain that if $\phi$ is valid in every prenormal affine plane then $\phi$ is a theorem of $ML(SAP)$. We will now show that:

**Proposition 10.** *Every prenormal affine plane is a bounded morphic image of a standard affine plane.*

*Proof.* Our first claim is that every prenormal affine plane is a bounded morphic image of a prenormal affine plane $\mathcal{F}' = (L', P', C')$ where $P'$ is an irreflexive relation on $L'$. Our second claim is that every prenormal affine plane $\mathcal{F} = (L, P, C)$ where $P$ is an irreflexive relation on $L$ is a bounded morphic image of a standard affine plane. To prove our first claim, consider a prenormal affine plane $\mathcal{F} = (L, P, C)$. The affine plane $\mathcal{F}' = (L', P', C')$ where:

- $L' = \{(a, 0)\colon a \in L\} \cup \{(a, i)\colon a \in L,\ i \geq 1 \text{ and } P(a, a)\}$,
- For all $a, b$ in $L$ and for all $i, j \geq 0$, $P'((a, i), (b, j))$ iff $P(a, b)$ and either $a \neq b$ or $i \neq j$,
- For all $a, b$ in $L$ and for all $i, j \geq 0$, $C'((a, i), (b, j))$ iff $C(a, b)$,

is obviously prenormal. What is more, its relation $P'$ is irreflexive on $L'$. Now, let $f$ be the mapping from $L'$ to $L$ defined as follows for all $a$ in $L$ and for all $i \geq 0$, $f((a, i)) = a$. We claim that $f$ is a bounded morphism from $\mathcal{F}'$ to $\mathcal{F}$, as the reader is asked to show. To prove our second claim, consider a prenormal affine plane $\mathcal{F} = (L, P, C)$ where $P$ is an irreflexive relation on $L$. The affine plane $\mathcal{F}' = (L', P', C')$ where:

- $L' = \{(a, 0)\colon a \in L\} \cup \{(a, i)\colon a \in L,\ i \geq 1 \text{ and } C(a, a)\}$,
- For all $a, b$ in $L$ and for all $i, j \geq 0$, $P'((a, i), (b, j))$ iff $P(a, b)$ and $i = j$,
- For all $a, b$ in $L$ and for all $i, j \geq 0$, $C'((a, i), (b, j))$ iff $C(a, b)$, either $a \neq b$ or $i \neq j$ and either $\neg P(a, b)$ or $i \neq j$,

is obviously prenormal. What is more, its relations $P'$ and $C'$ are irreflexive on $L'$. Now, let $f$ be the mapping from $L'$ to $L$ defined as follows for all $a$ in $L$ and for all $i \geq 0$, $f((a,i)) = a$. We claim that $f$ is a bounded morphism from $\mathcal{F}'$ to $\mathcal{F}$, as the reader is asked to show. $\dashv$

Hence prenormal affine planes and standard affine planes validate the same formulas. These considerations prove that:

**Proposition 11.** *The following conditions are equivalent:* **(i)** *$\phi$ is a theorem of $ML(SAP)$;* **(ii)** *$\phi$ is valid in every standard affine plane;* **(iii)** *$\phi$ is valid in every prenormal affine plane.*

By proposition 11, $ML(SAP)$ is sound and complete with respect to the class of all standard affine planes, a first-order definable class of affine planes. Hence, $ML(SAP)$ is also sound and complete with respect to the class of all countable standard affine planes. By proposition 3, we obtain that any two standard affine planes are elementary equivalent. We will now show that any two standard affine planes are modally equivalent. In order to prepare for the proof, let us consider a formula $\phi$ and a countable model $\mathcal{M} = (L, P, C, V)$. Restricting our discussion to the propositional variables actually occurring in $\phi$, let $\mathcal{V}$ be the set of all sets of sets of propositional variables. Remark that $Card(\mathcal{V}) \leq 2^{2^{|\phi|}}$. Define the functions $\gamma$ and $\delta$ from the partition of $L$ into $\mathcal{V}$ as follows:

$$\gamma([a]) = \{V(b) \colon \neg C(a,b) \text{ and } V(b) \neq V(c) \text{ for each line } c \text{ such that } P(b,c)\},$$
$$\delta([a]) = \{V(b) \colon \neg C(a,b) \text{ and } V(b) = V(c) \text{ for some line } c \text{ such that } P(b,c)\}.$$

For our purpose, the crucial properties of $\gamma$ and $\delta$ are the following:

- We can find $\Xi$ in $\mathcal{V}$ such that $\gamma([\omega]) \cup \delta([\omega]) = \Xi$ for countably many equivalence classes $[\omega]$ in the partition of $L$,
- For all equivalence classes $[a]$ in the partition of $L$, $\delta([a]) \neq \emptyset$.

**Proposition 12.** *Any two standard affine planes are modally equivalent.*

*Proof.* Let $\mathcal{F} = (L, P, C)$ and $\mathcal{F}' = (L', P', C')$ be standard affine planes. We demonstrate that $\mathcal{F}$ and $\mathcal{F}'$ are modally equivalent. Without loss of generality, we may assume that $\mathcal{F}$ is countable. If $\mathcal{F}$ and $\mathcal{F}'$ are not modally equivalent then there are two cases: either there is a formula $\phi$ such that $\mathcal{F} \models \phi$ and $\mathcal{F}' \not\models \phi$ or there is a formula $\phi$ such that $\mathcal{F} \not\models \phi$ and $\mathcal{F}' \models \phi$. In the first case, there is a formula $\phi$ such that $\mathcal{F} \models \phi$ and $\mathcal{F}' \not\models \phi$. Hence, $\phi$ is valid in every countable standard affine plane and $\phi$ is not a theorem of $ML(SAP)$, a contradiction. In the second case, there is a formula $\phi$ such that $\mathcal{F} \not\models \phi$ and $\mathcal{F}' \models \phi$. We restrict our discussion to the set of all propositional variables actually occurring in $\phi$. Let $\mathcal{V}$ be the set of all sets of sets of propositional variables. Since $\mathcal{F} \not\models \phi$, then there is a model $\mathcal{M} = (\mathcal{F}, V)$ based on $\mathcal{F}$ such that $\mathcal{M} \not\models \phi$. In order to contradict $\mathcal{F}' \models \phi$, we need to define a valuation $V'$ on $\mathcal{F}'$ such that $(\mathcal{F}', V') \not\models \phi$. By proposition 2, there is an elementary embedding $f$ of $\mathcal{F}$ into $\mathcal{F}'$. Let $a'$ be a line in $L'$. If $a'$ belongs to $f(L)$ then there is a line $a$ in $L$ such that $f(a) = a'$

and define $V'(a') = V(a)$. Otherwise, there are two cases: either $a'$ is parallel with $f(a)$ for some line $a$ in $L$ or $a'$ is convergent with $f(a)$ for each line $a$ in $L$. In the first case, $a'$ is parallel with $f(a)$ for some line $a$ in $L$. Reminding that $\delta([a]) \neq \emptyset$, select a set $\lambda$ of propositional variables in $\delta([a])$ and define $V'(a') = \lambda$. In the second case, $a'$ is convergent with $f(a)$ for each line $a$ in $L$. Seeing that we can find $\Xi$ in $\mathcal{V}$ such that $\gamma([\omega]) \cup \delta([\omega]) = \Xi$ for countably many equivalence classes $[\omega]$ in the partition of $L$, select an equivalence class $[\omega]$ such that $\gamma([\omega]) \cup \delta([\omega]) = \Xi$. First, remark that we can find sets $\theta_1, \ldots, \theta_m$ of propositional variables such that $\gamma([\omega]) = \{\theta_1, \ldots, \theta_m\}$. Fix lines $a'_1, \ldots, a'_m$ in $[a']$ and define $V'(a'_1) = \theta_1, \ldots, V'(a'_m) = \theta_m$. Second, notice that we can find sets $\lambda_1, \ldots, \lambda_n$ of propositional variables such that $\delta([\omega]) = \{\lambda_1, \ldots, \lambda_n\}$. Let $\{B'_1, \ldots, B'_n\}$ be a partition of $[a'] \setminus \{a'_1, \ldots, a'_m\}$ such that $B'_1, \ldots, B'_n$ are infinite subsets of $L'$ and define $V'(a'_1) = \lambda_1$ for each line $a'_1$ in $B'_1, \ldots, V'(a'_n) = \lambda_n$ for each line $a'_n$ in $B'_n$. As a simple exercise we invite the reader to show by induction on the complexity of formulas $\psi$ in $Sf(\phi)$ that for all lines $a$ in $L$, $\mathcal{M}, a \models \psi$ iff $(\mathcal{F}', V'), f(a) \models \psi$. Since $\mathcal{M} \not\models \phi$, then $(\mathcal{F}', V') \not\models \phi$. ⊣

An important related result is that:

**Proposition 13.** *The following conditions are equivalent:* **(i)** *$\phi$ is a theorem of $ML(SAP)$;* **(ii)** *$\mathcal{F}^2_{\mathbb{R}} \models \phi$;* **(iii)** *$\mathcal{F}^2_{\mathbb{Q}} \models \phi$.*

*Proof.* **(i)** implies **(ii)**: By proposition 11.
**(ii)** implies **(iii)**: By proposition 12.
**(iii)** implies **(i)**: By proposition 11 and proposition 12. ⊣

Our next result deals with the relationship between $ML(SAP)$ and finite prenormal affine planes.

**Proposition 14.** *$ML(SAP)$ has the polysize frame property with respect to the set of all finite prenormal affine planes.*

*Proof.* The fundamental construction underlying our proof is that of selective filtration. Take a formula $\phi$ such that $\phi$ is not a theorem of $ML(SAP)$. Hence, there is a countable standard affine plane $\mathcal{F} = (L, P, C)$ such that $\mathcal{F} \not\models \phi$. It follows that there is a model $\mathcal{M} = (\mathcal{F}, V)$ based on $\mathcal{F}$ such that $\mathcal{M} \not\models \phi$. We start our selective filtration of $\mathcal{M}$ through $Sf(\phi)$ by selecting a line $a$ in $L$ such that $\mathcal{M}, a \not\models \phi$. Reminding that $\delta([a]) \neq \emptyset$, choose a set $\lambda$ of propositional variables in $\delta([a])$ and select a new line $a^\star$ in $L$ such that $a^\star$ belongs to $[a]$ and $V(a^\star) = \lambda$. Seeing that we can find $\Xi$ in $\mathcal{V}$ such that $\gamma([\omega]) \cup \delta([\omega]) = \Xi$ for countably many equivalence classes $[\omega]$ in the partition of $L$, select a new line $\omega$ in $L$ such that $[a] \cap [\omega] = \emptyset$ and $\gamma([\omega]) \cup \delta([\omega]) = \Xi$. Recalling that $\delta([\omega]) \neq \emptyset$, choose a set $\lambda$ of propositional variables in $\delta([\omega])$ and select a new line $\omega^\star$ in $L$ such that $\omega^\star$ belongs to $[\omega]$ and $V(\omega^\star) = \lambda$. Now we define an infinite sequence $L_0, L_1, \ldots$ of subsets of $L$ such that for all positive integers $i$, the following conditions are satisfied:

$C_1(i)$ For all positive integers $j$, if $i > j$ then for all lines $b$ in $L_j$ and for all formulas $[P]\psi$ in $Sf(\phi)$, if $\mathcal{M}, b \not\models [P]\psi$ then we can find a line $c$ in $L_i$ such that $P(b, c)$ and $\mathcal{M}, c \not\models \psi$,

$C_2(i)$ For all positive integers $j$, if $i > j$ then for all lines $b$ in $L_j$ and for all formulas $[C]\psi$ in $Sf(\phi)$, if $\mathcal{M}, b \not\models [C]\psi$ then we can find a line $c$ in $L_i$ such that $C(b, c)$ and $\mathcal{M}, c \not\models \psi$,

$C_3(i)$ $a \in L_i$.

Let $L_0 = \{a, a^\star, \omega, \omega^\star\}$. Note that the conditions $C_1(0)$, $C_2(0)$ and $C_3(0)$ are satisfied. Let $i$ be a positive integer. Given $L_i$ such that the conditions $C_1(i)$, $C_2(i)$ and $C_3(i)$ are satisfied, we let $L_{i+1}$ be the subset of $L$ defined by the following algorithm:

```
begin
L_{i+1} := L_i;
for all lines b in L_i and for all formulas [P]ψ in Sf(φ) do
    if M, b ⊭ [P]ψ and there is no line c in L_{i+1} such that P(b, c) and
    M, c ⊭ ψ then
        begin
        select a line c in L such that P(b, c) and M, c ⊭ ψ;
        L_{i+1} := L_{i+1} ∪ {c}
        end;
for all lines b in L_i and for all formulas [C]ψ in Sf(φ) do
    if M, b ⊭ [C]ψ and there is no line c in L_{i+1} such that C(b, c) and
    M, c ⊭ ψ then
        begin
        select a line c in L such that C(b, c) and M, c ⊭ ψ;
        L_{i+1} := L_{i+1} ∪ {c};
        if there is no line d in L_{i+1} such that P(c, d) then
            begin
            choose a set λ of propositional variables in δ([c]);
            select a line d* in L such that P(c, d*) and V(d*) = λ;
            L_{i+1} := L_{i+1} ∪ {d*}
            end
        end
end.
```

It follows immediately from the definition of the algorithm that the conditions $C_1(i+1)$, $C_2(i+1)$ and $C_3(i+1)$ are satisfied. The affine plane $\mathcal{F}' = (L', P', C')$ where:

$L' = L_0 \cup L_1 \cup \ldots$,
$P'(b, c)$ iff either $P(b, c)$ or $b$ and $c$ are one and the same starred line,
$C'(b, c)$ iff either $C(b, c)$ or $b$ belongs to $[\omega]$ or $c$ belongs to $[\omega]$,

is obviously prenormal. Define $n_1$ to be the number of $[P]$-boxed formulas in $Sf(\phi)$ and $n_2$ to be the number of $[C]$-boxed formulas in $Sf(\phi)$. We claim that $Card(L') \leq (2 \times n_1 + n_2 + 2) \times (2 \times n_2 + 2)$, as the reader is asked to show.

To complete the proof we show by induction on the complexity of formulas $\psi$ in $Sf(\phi)$ that for all lines $b$ in $L'$, $\mathcal{M}, b \models \psi$ iff $(\mathcal{F}', V'), b \models \psi$ where $V'$ is the restriction of $V$ to $L'$. The base case follows from the definition of $V'$. We leave the Boolean cases to the reader. It remains to deal with the modalities. The right to left direction is more or less immediate from the definition of $L'$. For the left to right direction, a more delicate approach is needed.

Consider a formula $[P]\psi$ in $Sf(\phi)$. Let $b$ be a line in $L'$. Suppose $\mathcal{M}, b \models [P]\psi$, we demonstrate $(\mathcal{F}', V'), b \models [P]\psi$. Let $c$ be a line in $L'$ such that $P'(b, c)$. Hence, either $P(b, c)$ or $b$ and $c$ are one and the same starred line. In the first case, $P(b, c)$. Therefore, $\mathcal{M}, c \models \psi$ and, by induction hypothesis, $(\mathcal{F}', V'), c \models \psi$. In the second case, $b$ and $c$ are one and the same starred line. Therefore, $\mathcal{M}, c \models \psi$ and, by induction hypothesis, $(\mathcal{F}', V'), c \models \psi$.

Consider a formula $[C]\psi$ in $Sf(\phi)$. Let $b$ be a line in $L'$. Suppose $\mathcal{M}, b \models [C]\psi$, we demonstrate $(\mathcal{F}', V'), b \models [C]\psi$. Let $c$ be a line in $L'$ such that $C'(b, c)$. Hence, either $C(b, c)$ or $b$ belongs to $[\omega]$ or $c$ belongs to $[\omega]$. In the first case, $C(b, c)$. Therefore, $\mathcal{M}, c \models \psi$ and, by induction hypothesis, $(\mathcal{F}', V'), c \models \psi$. In the second case, $b$ belongs to $[\omega]$. Therefore, $\mathcal{M}, c \models \psi$ and, by induction hypothesis, $(\mathcal{F}', V'), c \models \psi$. In the third case, $c$ belongs to $[\omega]$. Therefore, $\mathcal{M}, c \models \psi$ and, by induction hypothesis, $(\mathcal{F}', V'), c \models \psi$. $\dashv$

An important related corollary is that:

**Proposition 15.** *The membership problem in $ML(SAP)$ is NP-complete.*

### 3.2 Convergence alone

The remainder of this section is devoted to studying the pure $[C]$-fragment of our line-based modal logic; the pure $[P]$-fragment being studied by Balbiani and Goranko [2]. A *Kripke model* is now a pair $\mathcal{M} = (\mathcal{F}, V)$ with $\mathcal{F}$ a weak affine plane and $V$ a valuation on $\mathcal{F}$. The notion of a formula $\phi$ being true in a Kripke model $\mathcal{M} = (L, C, V)$, where $\mathcal{F}$ is a weak affine plane, at a line $a$ in $L$, notation $\mathcal{M}, a \models \phi$, is defined inductively as for the full language. It is a simple matter to check that the following formulas are valid in all standard weak affine planes:

$\phi \to [C]\langle C\rangle\phi$,
$[C]\phi \to [C][C](\phi \vee [C]\phi)$,
$[C][C]\phi \to [C][C][C]\phi$,
$\langle C\rangle\phi_1 \wedge \ldots \wedge \langle C\rangle\phi_n \to \langle C\rangle(\langle C\rangle\phi_1 \wedge \ldots \wedge \langle C\rangle\phi_n)$.

Let $ML(SAP^{-})$ be the smallest normal modal logic, with $[C]$, that contains the above formulas as proper axioms. It is a simple exercise in modal logic to check that if $\phi$ is a theorem of $ML(SAP^{-})$ then $\phi$ is valid in every standard weak affine plane. Now we prove the converse proposition: if $\phi$ is valid in every standard weak affine plane then $\phi$ is a theorem of $ML(SAP^{-})$. Let $\mathcal{F} = (L, C)$ be a generated subframe of the canonical frame for $ML(SAP^{-})$. The proper axioms of $ML(SAP^{-})$ are all Sahlqvist formulas. Hence, $\mathcal{F}$ satisfies the following conditions:

$\forall x \forall y (C(x,y) \rightarrow C(y,x))$,
$\forall x \forall y \forall z \forall t (C(x,y) \wedge C(y,z) \rightarrow (\neg C(x,z) \wedge C(z,t) \rightarrow C(x,t)))$,
$\forall x \forall y \forall z \forall t (C(x,y) \wedge C(y,z) \wedge C(z,t) \rightarrow \exists u (C(x,u) \wedge C(u,t)))$,
$\forall x \forall y_1 \ldots \forall y_n (C(x,y_1) \wedge \ldots \wedge C(x,y_n) \rightarrow \exists z (C(x,z) \wedge C(z,y_1) \wedge \ldots \wedge C(z,y_n)))$.

Let $\mathcal{F}$ be a rooted weak affine plane. We shall say that $\mathcal{F}$ is *prenormal* iff it satisfies the conditions above. Remark that a prenormal weak affine plane $\mathcal{F} = (L, C)$ where $C$ is an irreflexive relation on $L$ is standard. Unsurprisingly, if $\phi$ is valid in every prenormal weak affine plane then $\phi$ is a theorem of $ML(SAP^-)$. We now make the following claim:

**Proposition 16.** *Every prenormal weak affine plane is a bounded morphic image of a standard weak affine plane.*

*Proof.* Consider a prenormal weak affine plane $\mathcal{F} = (L, C)$. The weak affine plane $\mathcal{F}' = (L', C')$ where:

- $L' = \{(a, 0) \colon a \in L\} \cup \{(a, i) \colon a \in L,\ i \geq 1 \text{ and } C(a, a)\}$,
- For all $a, b$ in $L$ and for all $i, j \geq 0$, $C'((a, i), (b, j))$ iff $C(a, b)$ and either $a \neq b$ or $i \neq j$,

is obviously prenormal. What is more, its relation $C'$ is irreflexive on $L'$. Hence, it is standard. Now, let $f$ be the mapping from $L'$ to $L$ defined as follows for all $a$ in $L$ and for all $i \geq 0$, $f((a, i)) = a$. We claim that $f$ is a bounded morphism from $\mathcal{F}'$ to $\mathcal{F}$, as the reader is asked to show. ⊣

Hence prenormal weak affine planes and standard weak affine planes validate the same formulas. These considerations prove that:

**Proposition 17.** *The following conditions are equivalent:* **(i)** *$\phi$ is a theorem of $ML(SAP^-)$;* **(ii)** *$\phi$ is valid in every standard weak affine plane;* **(iii)** *$\phi$ is valid in every prenormal weak affine plane.*

By proposition 17, $ML(SAP^-)$ is sound and complete with respect to the class of all standard weak affine planes, a first-order definable class of weak affine planes. Hence, $ML(SAP^-)$ is also sound and complete with respect to the class of all countable standard weak affine planes. The reader may easily verify that every theorem of $ML(SAP^-)$ is also a theorem of $ML(SAP)$. We will now prove the converse result, that is to say:

**Proposition 18.** *$ML(SAP)$ is a conservative extension of $ML(SAP^-)$.*

*Proof.* Let $\phi$ be a formula in the language with $[C]$. Suppose that $\phi$ is a theorem of $ML(SAP)$, we demonstrate that $\phi$ is a theorem of $ML(SAP^-)$. If $\phi$ is not a theorem of $ML(SAP^-)$ then there is a countable standard weak affine plane $\mathcal{F} = (L, C)$ such that $\mathcal{F} \not\models \phi$. We need to consider a bijective mapping $g$ on the partition of $\mathcal{F}^2_{\mathbb{Q}}$'s set of lines into the partition of $L$. For each equivalence class $[a]$ in the partition of $\mathcal{F}^2_{\mathbb{Q}}$'s set of lines, we also need a surjective mapping $h_{[a]}$ on $[a]$ into $g([a])$. Now let $f$ be the mapping on $\mathcal{F}^2_{\mathbb{Q}}$'s set of lines into $L$ defined as follows:

For all lines $a$ in $\mathcal{F}^2_{\mathbb{Q}}$'s set of lines, $f(a) = h_{[a]}(a)$.

The reader is asked to show that $f$ is a bounded morphism from $\mathcal{F}^2_{\mathbb{Q}}$ to $\mathcal{F}$. Consequently $\mathcal{F}^2_{\mathbb{Q}} \not\models \phi$. Thus $\phi$ is not a theorem of $ML(SAP)$, a contradiction. ⊣

By item **(i)** in proposition 8, we obtain that we can find two countable standard weak affine planes that are not isomorphic. However, we still have not shown that:

**Proposition 19.** *We can find two countable standard weak affine planes that are not modally equivalent.*

*Proof.* The proof that for all propositional variables $p$, $\mathcal{F}^{2-}_{\mathbb{Q}} \not\models p \wedge [C]p \rightarrow \langle C\rangle[C]p$ and $\mathcal{F}^{2-}_{\mathbb{Q}}/[\cdot] \models p \wedge [C]p \rightarrow \langle C\rangle[C]p$, which is not difficult, is left as an exercise. ⊣

It is nevertheless true that:

**Proposition 20.** *The following conditions are equivalent:* **(i)** *$\phi$ is a theorem of $ML(SAP^-)$;* **(ii)** *$\mathcal{F}^{2-}_{\mathbb{R}} \models \phi$;* **(iii)** *$\mathcal{F}^{2-}_{\mathbb{Q}} \models \phi$.*

*Proof.* **(i)** implies **(ii)**: By proposition 17.

**(ii)** implies **(iii)**: Suppose that $\mathcal{F}^{2-}_{\mathbb{R}} \models \phi$, we demonstrate that $\mathcal{F}^{2-}_{\mathbb{Q}} \models \phi$. If $\mathcal{F}^{2-}_{\mathbb{Q}} \not\models \phi$ then $\phi$ is not a theorem of $ML(SAP^-)$. Therefore $\phi$ is not a theorem of $ML(SAP)$. It follows that $\mathcal{F}^2_{\mathbb{R}} \not\models \phi$. Hence $\mathcal{F}^{2-}_{\mathbb{R}} \not\models \phi$, a contradiction.

**(iii)** implies **(i)**: Suppose that $\mathcal{F}^{2-}_{\mathbb{Q}} \models \phi$, we demonstrate that $\phi$ is a theorem of $ML(SAP^-)$. If $\phi$ is not a theorem of $ML(SAP^-)$ then $\phi$ is not a theorem of $ML(SAP)$. It follows that $\mathcal{F}^2_{\mathbb{Q}} \not\models \phi$. Hence $\mathcal{F}^{2-}_{\mathbb{Q}} \not\models \phi$, a contradiction. ⊣

Rather like the proof of proposition 14 one can prove that:

**Proposition 21.** *$ML(SAP^-)$ has the polysize frame property with respect to the set of all finite prenormal weak affine planes.*

One interesting corollary is that:

**Proposition 22.** *The membership problem in $ML(SAP^-)$ is NP-complete.*

## 4 Conclusion

We now naturally ask the question: what is the first-order theory of lines in space geometry and what is the corresponding modal logic? For a start, note that two lines in space geometry may have the following mutual relations: they are parallel if they lie in the same plane and never meet, they are convergent if they lie in the same plane and have exactly one common point and they are separated if they are not coplanar. These relations bring a new array of questions. Which mutual relations can first-order define the two others in the Euclidean space? Is the real line-based affine space an elementary extension of the set of all rational lines? With respect to parallelism, convergence and separation, what is the first-order theory of the real line-based affine space? Same question with only one of these mutual relations. These first-order theories are decidable since they can be embedded in elementary algebra; little seems to be known as regards their complete axiomatizations or their complexity. A systematic exploration of the properties of a first-order theory based on the relations of parallelism, convergence and separation in space affine geometry and a thorough examination of the modal logic it gives rise to require further studies.

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