

Dynamic logics of the region-based theory of discrete spaces

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Abstract

The aim of this paper is to give new kinds of modal logics suitable for reasoning about regions in discrete spaces. We call them dynamic logics of the region-based theory of discrete spaces. These modal logics are linguistic restrictions of propositional dynamic logic with the global diamond E . Their formulas are equivalent to Boolean combinations of modal formulas like $E(A \wedge \langle \alpha \rangle B)$ where A and B are Boolean terms and α is a relational term. Examining what we can say about dynamic models when we use formulas to describe them, we successively address the axiomatization/completeness issue and the decidability/complexity issue of our dynamic logics of the region-based theory of discrete spaces.

1 Introduction

Continuous and discrete region-based theories of space

One of the main subject of Qualitative Spatial Reasoning (QSR, see [5]) is the Region-based Theory of Space (RTS). The philosophical origins of RTS go back to de Laguna [6] and Whitehead [25]; the first mathematical results had been obtained by Tarski [21] and later by Grzegorzcyk [10] and Clarke [3]. Randel, Cui and Cohn [16] had introduced the Region Connection Calculus (RCC), which now is a central topic in QSR research. Recent results in RTS, concentrated mainly on the correspondence with the point-based approach, are Roeper [17], Mormann [13], Pratt and Schoop [15], Vakarelov, Dimov, Düntsch and Bennett [23].

In contrast with the classical approach to the theory of space, in which the notion of point is one of the basic primitive notions of the theory, RTS is based on the primitive notion of *region* and some operations and relations between regions. In most popular systems of RTS the operations over regions, which allow to obtain new regions from given ones, are the operations of join, meet and complement, satisfying the axioms of Boolean algebra. The relations between regions are the *contact* relation and some mereological relations like *part-of*, *overlap* and their derivatives. The standard models

of RTS are in some classes of topological spaces. If X is a given topological space, then regions in X are identified with regular open or regular closed sets. If we denote by cl and int the topological operations of closure and interior, then a subset a of X is called regular open if $a = int(cl(a))$ and dually, a is called regular closed if $a = cl(int(a))$. The Boolean operations in the set of regular closed or open sets are defined as follows:

For regular closed sets: $a \vee b = a \cup b$ – the join, $a^* = cl(X \setminus a)$ – the complement, $a \wedge b = (a^* \vee b^*)^*$ – the meet. The definitions for regular open sets are dual: $a \wedge b = a \cap b$ – the meet, $a^* = int(X \setminus a)$ – the complement, $a \vee b = (a^* \wedge b^*)^*$ – the join. Note that in both cases the regions form a Boolean algebra and the two Boolean algebras are isomorphic, which enables to consider only one of them.

The relation of contact between two regions, commonly denoted by C , is defined as follows:

For regular open regions: aCb iff $cl(a) \cap cl(b) \neq \emptyset$,

For regular closed regions: aCb iff $a \cap b \neq \emptyset$.

The part-of relation \leq is identified with the Boolean inclusion and all other standard relations between regions studied in RTS are definable by contact and part-of.

In some RTS, like for instance RCC, contact is the only primitive relation and part-of is definable by it: $a \leq b$ iff $(\forall c)(aCc \rightarrow bCc)$. In this case only some good topological spaces, as for instance regular spaces, give models for the contact relation which satisfies this equivalence.

Note that topology is a theory of such special properties of space like continuity, infinite divisibility, density etc, which are in a sense opposite to *discreteness*, considered here not in his trivial topological sense. Let us call systems of RTS, which admits such topological models *continuous* RTS. Consequently, continuous RTS exclude some real-world situations, arising from the practice of Artificial Intelligence, such as digital images and image analysis, computer vision, discrete geometry etc. Formal methods for the above practical domains are studied in a field commonly called *digital topology* (see Rosenfeld [18] for an introduction and many references). Thus a need of building a Discrete RTS arises, with non-topological models, suitable for modeling discrete situations such as existing of undivisible atomic regions, relations between such atoms like different kinds of adjacencies suitable for describing configurations of atoms, etc. To this end Smith [20] proposes mathematical structures similar to topological spaces, namely the closure spaces of Čech. Note that each topological space is a closure space but not the converse. More closed to the formal approach of most RTS is the approach by Galton [8, 9], who proposed the so called adjacency spaces.

An adjacency space is a pair (X, R) where X is a nonempty set whose elements are intuitively treated as small indivisible regions, called cells, and R is a relation between regions called adjacency relation. Originally Galton

[8] assumes R to be reflexive and symmetric relation which we will call Galton's type adjacency. In this paper however, as this has been done also in [7, 1], we will consider the more general situation, where R can be an arbitrary relation. Galton also considers multiple-adjacency spaces with more than one adjacency relation and this idea will heavily be used in this paper. A good example of adjacency space is the chess-board desk with squares as cells. If, for two squares x and y , we define to be adjacent if they have a common point then we obtain a reflexive and symmetric adjacency between squares, which is just a Galton's type adjacency. But we may specify additionally the place of y with respect to x in 9 different ways: to be equal to x , to be on the left of x , to be on the right (of x), to be on the top, to be on the bottom, to be up on the left (of x), to be up on the right, to be down on the left, and to be down on the right. In all cases we always assume that x and y have a common point. Some of these relations are converses of others, for instance the relation "to be on the left" is the converse of the relation "to be on the right" and vice versa; the relation "to be up on the right" is just a composition of the relations "to be on the right" and "to be on the top" and so on. More generally, any relational system $(X, \{R_i : i \in I\})$, where X is a non-empty set and $\{R_i : i \in I\}$ is a family of binary relations in X can be considered as a multiple-adjacency space. The case where $\{R_i : i \in I\}$ is the set of all binary relations of X will be of special importance in this paper. This set is closed with respect to all set-relational operations considered in Dynamic Logic like converse, union, composition, reflexive and transitive closure (called here shortly star $(*)$ operation) and test $?$, which assign to each subset A of X the partial identity relation $A? =_{def} \{(x, x) : x \in A\}$.

Following Galton, we define regions in an adjacency space (X, R) to be arbitrary subsets of X equipped with the standard Boolean operations between subsets. The relation C of contact between two regions a, b of X is defined as follows:

$$aCb \text{ iff } (\exists x \in a)(\exists y \in b)xRy.$$

By means of the contact and the part-of relation, which is identified with the set-inclusion, one can define all other typical region relations. If we have many adjacency relations, then for each relation R of adjacency we have a corresponding contact relation C_R .

Galton also indicates how to associate to an adjacency space (X, R) the structure of closure space in the sense of Čech and, in this way, how to incorporate the approach of Smith [20]. The set $N_x = \{y : xRy\}$ is called *immediate neighbourhood* of x . Define $cl(a) = \{x : a \cap N_x \neq \emptyset\}$ and $int(a) = \{x : N_x \subseteq a\} = X \setminus cl(X \setminus a)$. Then, (X, cl) is a closure space if R is a reflexive relation. Note that in general cl is not a topological closure, because it does not satisfy the idempotence property $cl(cl(a)) = cl(a)$. It is a topological closure iff R is a reflexive and transitive relation. This special topology, determined by R is known as Alexandroff's topology.

The aim of the paper

Galton's approach to put adjacency spaces on the base of discrete RTS have been used in the literature. In [7] Düntsch and Vakarelov extended the proximity approach from [22, 23] for continuous RTS to the case of discrete RTS, based on adjacency spaces.

Li and Ying [12] introduced the logical system GRCC – Generalized RCC, for which they prove correctness in the class of all connected adjacency spaces of Galton's type.

In [1] we propose a framework for building modal-like logics named Region-based Modal Logics of Space – RMLS, suitable for reasoning about regions in a given RTS. The approach was unified for both continuous RTS and discrete RTS based on adjacency spaces. We have used in [1] a modal-like language, which is a multimodal version of the language of an old and forgotten modal system of von Wright [26] called a new system of modal logic. Similar language have been also used by Wolter and Zakharyashev [24] for a logic of some continuous RTS, and by Li and Ying for GRCC.

The language of RMLS, we have considered in [1], consists of:

- a denumerable set of *Boolean variables* (to be interpreted by regions),
- *Boolean operations* (to be interpreted as operations on regions): \cap (Boolean meet), \cup (Boolean join), \sim (Boolean complement) and Boolean constants 0 and 1,
- *Propositional connectives* $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$.
- *Binary modal connectives* contact C , part-of \leq .

The set of *Boolean terms* is defined in a standard way: from Boolean variables and Boolean constants f, t by means of Boolean operations. *Atomic formulas* called also basic modalities, are in the form aCb and $a \leq b$ where a and b are Boolean terms. *Complex formulas* or simply *formulas* are built from atomic formulas by means of propositional connectives. By means of basic modalities one can define some other modalities: equality – $(a = b) =_{def} (a \leq b) \wedge (b \leq a)$, inequality – $(a \neq b) =_{def} \neg(a = b)$, overlap – $aOb =_{def} (a \cap b) \neq 0$, non-tangential part – $(a \ll b) =_{def} \neg(aC\tilde{b})$, etc.

The semantics of this language is a Kripke-like semantics over adjacency spaces (X, R) . By a valuation in (X, R) we mean any mapping v that associates a subset $v(p)$ of X to any given Boolean variable p and such that $v(f) = \emptyset$ and $v(t) = X$. By a model we understand triples of the form $M = (X, R, v)$ where v is a valuation. Valuations then are extended to arbitrary Boolean terms in a natural way. This means that Boolean terms are interpreted by regions in the adjacency space (X, R) . Formulas are evaluated in a given model $M = (X, R, v)$ as follows. Namely we define the satisfaction of a formula ϕ in a model M – in symbols $M \models \phi$, inductively as follows:

- For basic modalities we have:
- $M \models (a \leq b)$ iff $v(a) \subseteq v(b)$,

- $M \models (aCb)$ iff $(\exists x \in v(a))(\exists y \in v(b))(xRy)$ iff $v(a)Cv(b)$.

For complex formulas we have:

- $M \models \neg\phi$ iff $M \not\models \phi$, $M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$, $M \models \phi \vee \psi$ iff $M \models \phi$ or $M \models \psi$.

Note that in contrast with the standard modal logic, where a formula ϕ is evaluated at a given state $x \in X$, formulas now are evaluated globally in the model.

If Σ is a given class of adjacency spaces then $\mathcal{L}(\Sigma)$ is the set of all formulas satisfied in all models over the spaces from Σ and $\mathcal{L}(\Sigma)$ is called the logic of Σ . In [1] we axiomatize the logics of some important classes of adjacency spaces, for instance the logic \mathcal{L}_{all} of all adjacency spaces, the logic of all Galton's type adjacency spaces, the logic of all Galton's type adjacency spaces in which the relation R is connected in a graph sense. This logic is just the system GRCC considered by Li and Ying [12]. Adapting filtration from the standard modal logic we have proved decidability of a great variety of logics including GRCC. Namely we have proved that every axiomatic extension of \mathcal{L}_{all} with finitely many axioms possess finite model property and is decidable.

The aim of this paper is to extend some results from [1] for the case of multiple-adjacency spaces in which the set of adjacency relations is closed with respect to some set-relational operations, as in Dynamic Logic. The obtained logical systems are called Dynamic Logics of the Region-based theory of Discrete Spaces – DLDS. In this paper, we extend the language for RMLS described above with a denumerable set of relational variables (considered now not as relational terms but as adjacency relations) and the standard set of PDL operations over binary relations: union, converse, composition, and test. Then the set of relational terms is defined in a standard way from relational variables by means of relational operations. In analogy with PDL, the operation of test transforms each Boolean term A into a relational term $A?$. Instead of one contact modality C we have now many such modalities – \diamond_{α} , for each relational term α . The semantics of such an extension of the language of RMLS is obtained by a straightforward combination of the semantics of PDL for relational terms and the semantics of RMLS described above. We consider in this paper the following question: axiomatization of the logics corresponding to some reasonable subsystems of the above language. We also address decidability and complexity issues. Let us note that, according to the language and the semantics, we have a great analogy with PDL, when starting to develop the axiomatization problem, the analogy totally disappears. The reason is that, while in PDL the standard relational operations are modally definable by the corresponding axioms, in the case of the present language the composition is not modally definable by means of some formulas of the language. So in order to obtain completeness theorems we have to consider some additional rules of inference for composition. We do this on several steps, starting with a minimal subsystem in which every-

thing is modally definable and then extend this system first with a rule for composition.

The structure of the paper

The paper is organized as follows. In section 2, we introduce the basic language for our dynamic logic of discrete spaces. This language is based on the relational constructs of union, tests and converse. We address the axiomatization and the decidability issues of our logic. Section 3 extends these results to a logic with the relational construct of sequential composition.

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2 The basic language

The main interest for doing dynamic logic is in reasoning about relational terms. At the basic level there are relational variables. We start with three connectives: converse α^{-1} corresponding to the converse of accessibility relation R_α , union $\alpha \cup \beta$ corresponding to the union of the accessibility relations R_α and R_β and test $A?$ corresponding to the partial identity in the subsets of the dynamic models which are values of the Boolean term A .

2.1 Syntax and semantics

It is now time to meet the syntax and the semantics of the dynamic logics we will be working with. Let Φ be a countable set of Boolean variables, with typical members denoted p, q , etc. We are allowed to define Boolean terms over Φ , with typical members denoted A, B , etc, as follows:

- Every Boolean variable p is a Boolean term,
- f (“empty subset”) is a Boolean term,
- If A is a Boolean term then \tilde{A} (“complement of A ”) is a Boolean term,
- If A and B are Boolean terms then $(A \cap B)$ (“ A intersection B ”) is a Boolean term.

Let Π be a countable set of relational variables, with typical members denoted a, b , etc. We are allowed to define relational terms over Φ and Π , with typical members denoted α, β , etc, as follows:

- Every relational variable a is a relational term,

- If A is a Boolean term then $A?$ (“test A ”) is a relational term,
- If α and β are relational terms then $(\alpha \cup \beta)$ (“ α union β ”) is a relational term,
- If α is a relational term then α^{-1} is a relational term.

The formulas of the language of our dynamic logics, with typical members denoted ϕ , ψ , etc, are defined as follows:

- If α is a relational term and A and B are Boolean terms then $\diamond_\alpha(A, B)$ (“ A and B are possible with respect to α ”) is a formula,
- \perp (“false”) is a formula,
- If ϕ is a formula then $\neg\phi$ (“not ϕ ”) is a formula,
- If ϕ and ψ are formulas then $(\phi \vee \psi)$ (“ ϕ or ψ ”) is a formula.

A number of other constructs can be defined in terms of the primitive ones as follows:

- t (“full subset”) is \tilde{f} ,
- $(A \cup B)$ (“ A union B ”) is $(\widetilde{\tilde{A} \cap \tilde{B}})$,
- $\square_\alpha(A, B)$ (“ A and B are necessary with respect to α ”) is $\neg\diamond_\alpha(\tilde{A}, \tilde{B})$,
- \top (“true”) is $\neg\perp$,
- $(\phi \wedge \psi)$ (“ ϕ and ψ ”) is $\neg(\neg\phi \vee \neg\psi)$,
- $(\phi \rightarrow \psi)$ (“ ϕ only if ψ ”) is $(\neg\phi \vee \psi)$,
- $(\phi \leftrightarrow \psi)$ (“ ϕ if and only if ψ ”) is $((\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi))$.

In addition, we define:

- $(A \leq B)$ (A is part of B) is $\neg\diamond_{t?}(A, \tilde{B})$,
- $(A = B)$ (“ A is equal to B ”) is $((A \leq B) \wedge (B \leq A))$,
- $(A \neq B)$ (“ A is not equal to B ”) is $\neg(A = B)$.

We follow the standard rules for omission of the parentheses. The number of symbols in ϕ may be called the length of ϕ . We denote it by $|\phi|$. A dynamic frame is an ordered pair $\mathcal{F} = (W, R)$ where W is a nonempty set of points and R is a function assigning to each relational variable a a binary relation R_a on W . A dynamic model on \mathcal{F} is an ordered triple $\mathcal{M} = (W, R, V)$ where V is a function assigning to each Boolean variable p a subset $V(p)$ of W . In \mathcal{M} we extend V and R to the set of all Boolean terms and to the set of all relational terms by a parallel induction as follows:

- $V(f) = \emptyset$,
- $V(\tilde{A}) = W \setminus V(A)$,
- $V(A \cap B) = V(A) \cap V(B)$,
- $R_{A?} = \{(x, x) : x \in V(A)\}$,
- $R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$,
- $R_{\alpha^{-1}} = (R_\alpha)^{-1}$.

We inductively define the notion of a formula being satisfied in \mathcal{M} as follows:

- $\mathcal{M} \models \diamond_\alpha(A, B)$ iff $(\exists x \in W)(\exists y \in W)(xR_\alpha y \text{ and } x \in V(A) \text{ and } y \in V(B))$,
- $\mathcal{M} \not\models \perp$,
- $\mathcal{M} \models \neg\phi$ iff $\mathcal{M} \not\models \phi$,
- $\mathcal{M} \models \phi \vee \psi$ iff $\mathcal{M} \models \phi$ or $\mathcal{M} \models \psi$.

The defined constructs inherit their meanings from these definitions:

- $V(t) = W$,
- $V(A \cup B) = V(A) \cup V(B)$,
- $\mathcal{M} \models \square_\alpha(A, B)$ iff $(\forall x \in W)(\forall y \in W)(xR_\alpha y \Rightarrow x \in V(A) \text{ or } y \in V(B))$,
- $\mathcal{M} \models \top$,
- $\mathcal{M} \models \phi \wedge \psi$ iff $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$,
- $\mathcal{M} \models \phi \rightarrow \psi$ iff $\mathcal{M} \models \phi$ only if $\mathcal{M} \models \psi$,
- $\mathcal{M} \models \phi \leftrightarrow \psi$ iff $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \psi$,
- $\mathcal{M} \models A \leq B$ iff $V(A) \subseteq V(B)$,
- $\mathcal{M} \models A = B$ iff $V(A) = V(B)$,
- $\mathcal{M} \models A \neq B$ iff $V(A) \neq V(B)$.

If $\mathcal{F} = (W, R)$ is a dynamic frame then we say that formula ϕ is valid in \mathcal{F} , in symbols $\mathcal{F} \models \phi$, iff for all dynamic models $\mathcal{M} = (W, R, V)$ on \mathcal{F} , $\mathcal{M} \models \phi$ whereas we say that formula ϕ is satisfiable in \mathcal{F} iff there is a dynamic model $\mathcal{M} = (W, R, V)$ on \mathcal{F} such that $\mathcal{M} \models \phi$. If \mathcal{C} is a class of dynamic frames then we say that formula ϕ is valid in \mathcal{C} , in symbols $\mathcal{C} \models \phi$, iff for all dynamic frames \mathcal{F} in \mathcal{C} , $\mathcal{F} \models \phi$ whereas we say that formula ϕ is satisfiable in \mathcal{C} iff there is a dynamic frame \mathcal{F} in \mathcal{C} such that ϕ is satisfiable in \mathcal{F} . Let:

- \mathcal{C}_{all} be the class of all dynamic frames,
- \mathcal{C}_{fin} be the class of all finite dynamic frames,
- For all positive integers n , \mathcal{C}_n be the class of all finite dynamic frames containing at most n points.

2.2 Axiomatization and completeness

We aim to give a complete Hilbert-style deductive system L_{all} for validity in \mathcal{C}_{all} .

2.2.1 A deductive system

L_{all} is an extension of the deductive system for nondegenerate Boolean algebra:

- $A \cup B = B \cup A$,
- $A \cap B = B \cap A$,
- $A \cup (B \cap C) = (A \cup B) \cap C$,
- $A \cap (B \cup C) = (A \cap B) \cup C$,
- $A \cup (A \cap B) = A$,
- $A \cap (A \cup B) = A$,
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
- $\tilde{\tilde{A}} = A$,
- $f = A \cap \tilde{A}$,
- $t \neq f$.

Its proper axioms are all instances of the following schemata:

$$S_1 \quad A = B \rightarrow (\diamond_\alpha(A, C) \leftrightarrow \diamond_\alpha(B, C)),$$

$$S_2 \quad A = B \rightarrow (\diamond_\alpha(C, A) \leftrightarrow \diamond_\alpha(C, B)),$$

$$S_3 \quad \neg \diamond_\alpha(f, A),$$

$$S_4 \quad \neg \diamond_\alpha(A, f),$$

$$S_5 \quad \diamond_\alpha(A, C) \rightarrow \diamond_\alpha(A \cap B, C) \vee \diamond_\alpha(A \cap \tilde{B}, C),$$

$$S_6 \quad \diamond_\alpha(C, A) \rightarrow \diamond_\alpha(C, A \cap B) \vee \diamond_\alpha(C, A \cap \widetilde{B}),$$

$$S_7 \quad \diamond_\alpha(A \cap B, C) \rightarrow \diamond_\alpha(A, C) \wedge \diamond_\alpha(B, C),$$

$$S_8 \quad \diamond_\alpha(C, A \cap B) \rightarrow \diamond_\alpha(C, A) \wedge \diamond_\alpha(C, B),$$

$$S_9 \quad \diamond_{C?}(A, B) \leftrightarrow A \cap B \cap C \neq f,$$

$$S_{10} \quad \diamond_{\alpha \cup \beta}(A, B) \leftrightarrow \diamond_\alpha(A, B) \vee \diamond_\beta(A, B),$$

$$S_{11} \quad \diamond_{\alpha^{-1}}(A, B) \leftrightarrow \diamond_\alpha(B, A).$$

Let Γ be a set of formulas. A formula ϕ is called derivable from Γ in L_{all} , denoted $\Gamma \vdash_{L_{all}} \phi$, iff it can be obtained from the above axioms, from the tautologies of classical propositional logic and formulas from Γ by applying the following inference rule of modus ponens:

MP Given ϕ and $\phi \rightarrow \psi$, infer ψ .

An alternative formulation of derivability is:

- $\Gamma \vdash_{L_{all}} \phi$ iff ϕ belongs to the least set of formulas that contains Γ , all axioms and is closed under modus ponens.

A formula ϕ is called provable in L_{all} , denoted $\vdash_{L_{all}} \phi$, iff ϕ is derivable from \emptyset in L_{all} .

2.2.2 Soundness

The following theorem can be easily proved.

Theorem 1 *Let ϕ be a formula and Γ be a set of formulas such that $\Gamma \vdash_{L_{all}} \phi$. If all formulas from Γ are satisfied in a dynamic model \mathcal{M} , then $\mathcal{M} \models \phi$. Hence, if $\vdash_{L_{all}} \phi$ then $\mathcal{C}_{all} \models \phi$.*

Proof: Obviously, dynamic models satisfy the conditions which are needed to validate the axioms of L_{all} and *MP* is correct with respect to satisfiability in a dynamic model, i.e. if $\mathcal{M} \models \psi_1$ and $\mathcal{M} \models \psi_1 \rightarrow \psi_2$ then $\mathcal{M} \models \psi_2$. As a result, every formula derivable from Γ in L_{all} is satisfied in \mathcal{M} . Hence, every formula provable in L_{all} is valid in \mathcal{C}_{all} . \dashv

2.2.3 Theories

A construction called canonical model will enable us to prove the completeness of derivability in L_{all} , i.e. the strong completeness theorem: for any set of formulas Γ and any formula ϕ , if $\Gamma \not\vdash_{L_{all}} \phi$ then there is a dynamic model which satisfy all formulas from $\Gamma \cup \{\neg\phi\}$. The idea behind the canonical model construction is that every model $\mathcal{M} = (W, V, R)$ is associated with a set of formulas, namely $th(\mathcal{M}) = \{\phi: \mathcal{M} \models \phi\}$, that contains L_{all} and that

is closed under *MP*. By a theory we mean any set of formulas that satisfies these two conditions. So, for any set of formulas Γ the set $\{\phi : \Gamma \vdash_{L_{all}} \phi\}$ is a theory. On the other side, if S is a theory then for every formula ϕ , $\phi \in S$ iff $S \vdash_{L_{all}} \phi$. Hence, $\{\phi : \Gamma \vdash_{L_{all}} \phi\}$ is the least theory containing Γ . Obviously the least theory is L_{all} and the greatest theory is the set of all formulas. The latter theory is called trivial theory. A set of formulas Γ is called consistent if \perp is not derivable from Γ in L_{all} . Hence, a theory S is consistent iff \perp is not in S . The reader may easily verify that for all theories S , the following conditions are equivalent:

- S is not trivial,
- S is consistent,
- For all formulas ϕ , either $\phi \notin S$ or $\neg\phi \notin S$.

If S is a theory and ϕ is a formula then define $S + \phi = \{\psi : \phi \rightarrow \psi \in S\}$. The reader may easily verify the deduction theorem for derivability in L_{all} , i.e. that $S + \phi$ is the least theory containing $S \cup \{\phi\}$. The theory $S + \phi$ is inconsistent iff $\neg\phi \in S$. A set of formulas $\Gamma \cup \{\neg\phi\}$ is consistent iff $\Gamma \not\vdash_{L_{all}} \phi$. A theory S is called maximal if it is consistent and for all formulas ϕ , $\phi \in S$ or $\neg\phi \in S$. To carry out the completeness proof of L_{all} , we need to learn a little more about maximal theories. The lemma below is the Lindenbaum's lemma for L_{all} .

Lemma 1 *Let S_0 be a consistent theory. Then there is a maximal theory S such that $S_0 \subseteq S$.*

Proof: Let us arrange all the formulas in an infinite sequence ϕ_0, ϕ_1, \dots . We define the theory S as the union of a chain S_0, S_1, \dots of consistent theories as follows:

- For all positive integers n , if $S_n + \phi_n$ is consistent then $S_{n+1} = S_n + \phi_n$ else $S_{n+1} = S_n + \neg\phi_n$.

Note that for all positive integers n , if S_n is consistent then S_{n+1} is consistent. For suppose not. Then there is a positive integer n such that S_n is consistent and S_{n+1} is inconsistent. Hence $S_n + \phi_n$ is inconsistent and $S_n + \neg\phi_n$ is inconsistent. Consequently $\neg\phi_n \in S_n$ and $\neg\neg\phi_n \in S_n$ contradicting the consistency of S_n . Since S_0 is consistent, then for all positive integers n , S_n is consistent. Hence S is a maximal theory such that $S_0 \subseteq S$. \dashv

Let S be a maximal theory. A set x of Boolean terms is said to be S -consistent iff for all Boolean terms A_1, \dots, A_i in x , $A_1 \cap \dots \cap A_i \neq f \in S$. Remark that for all sets x of Boolean terms, if x is S -consistent then x is consistent, i.e. for all Boolean terms A_1, \dots, A_i in x , $A_1 \cap \dots \cap A_i = f$ is not a Boolean tautology. A set x of Boolean terms is called S -maximal if

it is S -consistent and for all Boolean terms A , $A \in x$ or $\widetilde{A} \in x$. The reader may easily verify that if x is an S -maximal set then

- if $A, B \in x$ then $A \cap B \in x$,
- if $A \in x$ then for any Boolean term B , $A \cup B \in x$.

Let S be a maximal theory and α be a relational term. A pair (x, y) of sets of Boolean terms is said to be S -compatible with respect to α iff for all Boolean terms A_1, \dots, A_i in x and for all Boolean terms B_1, \dots, B_j in y , $\diamond_\alpha(A_1 \cap \dots \cap A_i, B_1 \cap \dots \cap B_j) \in S$. Remark that for all sets x and y of Boolean terms, if the pair (x, y) is S -compatible with respect to α then both x and y are S -consistent.

Lemma 2 *Let S be a maximal theory and α be a relational term. If z and t are sets of Boolean terms such that the pair (z, t) is S -compatible with respect to α then there are S -maximal sets x and y of Boolean terms such that $z \subseteq x$, $t \subseteq y$ and the pair (x, y) is S -compatible with respect to α .*

Proof: Let A_0, A_1, \dots be an enumeration of the set of all Boolean terms. We define the sets x and y of Boolean terms as the unions of the chains x_0, x_1, \dots and y_0, y_1, \dots of sets of Boolean terms as follows:

- $x_0 = z$ and $y_0 = t$,
- For all positive integers n , if the pair $(x_n \cup \{A_n\}, y_n \cup \{A_n\})$ is S -compatible then $x_{n+1} = x_n \cup \{A_n\}$ and $y_{n+1} = y_n \cup \{A_n\}$ else if the pair $(x_n \cup \{A_n\}, y_n \cup \{\widetilde{A}_n\})$ is S -compatible then $x_{n+1} = x_n \cup \{A_n\}$ and $y_{n+1} = y_n \cup \{\widetilde{A}_n\}$ else if the pair $(x_n \cup \{\widetilde{A}_n\}, y_n \cup \{A_n\})$ is S -compatible then $x_{n+1} = x_n \cup \{\widetilde{A}_n\}$ and $y_{n+1} = y_n \cup \{A_n\}$ else $x_{n+1} = x_n \cup \{\widetilde{A}_n\}$ and $y_{n+1} = y_n \cup \{\widetilde{A}_n\}$.

First, remark that the pair (x_0, y_0) is S -compatible with respect to α . Second, note that for all positive integers n , if the pair (x_n, y_n) is S -compatible with respect to α then the pair (x_{n+1}, y_{n+1}) is also S -compatible with respect to α . For suppose not. Then there is a positive integer n such that the pair (x_n, y_n) is S -compatible with respect to α and the pair (x_{n+1}, y_{n+1}) is S -incompatible with respect to α . Since the pair (x_{n+1}, y_{n+1}) is S -incompatible with respect to α , then $\diamond_\alpha(B_1 \cap \dots \cap B_i \cap A_n, C_1 \cap \dots \cap C_j \cap A_n) \notin S$ for some Boolean terms B_1, \dots, B_i in x_n and for some Boolean terms C_1, \dots, C_j in y_n , $\diamond_\alpha(B'_1 \cap \dots \cap B'_{i'} \cap A_n, C'_1 \cap \dots \cap C'_{j'} \cap \widetilde{A}_n) \notin S$ for some Boolean terms $B'_1, \dots, B'_{i'}$ in x_n and for some Boolean terms $C'_1, \dots, C'_{j'}$ in y_n , $\diamond_\alpha(B''_1 \cap \dots \cap B''_{i''} \cap \widetilde{A}_n, C''_1 \cap \dots \cap C''_{j''} \cap A_n) \notin S$ for some Boolean terms $B''_1, \dots, B''_{i''}$ in x_n and for some Boolean terms $C''_1, \dots, C''_{j''}$ in y_n and $\diamond_\alpha(B'''_1 \cap \dots \cap B'''_{i'''} \cap \widetilde{A}_n, C'''_1 \cap \dots \cap C'''_{j'''} \cap \widetilde{A}_n) \notin S$ for some Boolean terms $B'''_1, \dots, B'''_{i'''}$ in x_n and for some Boolean terms $C'''_1, \dots, C'''_{j'''}$ in y_n .

Consequently $\diamond_\alpha(B_1 \cap \dots \cap B_i \cap B'_1 \cap \dots \cap B'_i \cap B''_1 \cap \dots \cap B''_i \cap B'''_1 \cap \dots \cap B'''_i, C_1 \cap \dots \cap C_j \cap C'_1 \cap \dots \cap C'_j \cap C''_1 \cap \dots \cap C''_j \cap C'''_1 \cap \dots \cap C'''_j) \notin S$ contradicting the S -compatibility of the pair (x_n, y_n) with respect to α . It follows that for all positive integers n , the pair (x_n, y_n) is S -compatible with respect to α . Thus x and y are S -maximal sets of Boolean terms such that $z \subseteq x, t \subseteq y$ and the pair (x, y) is S -compatible with respect to α . \dashv

From lemma ??, we immediately infer the following lemma.

Lemma 3 *Let S be a maximal theory and z be an S -consistent set of Boolean terms. Then there is an S -maximal set x of Boolean terms such that $z \subseteq x$.*

Proof: Let $A_1, \dots, A_i \in z$ and $B_1, \dots, B_j \in z$. Since z is S -consistent, $A_1 \cap \dots \cap A_i \cap B_1 \cap \dots \cap B_j \neq f$ is in S . Therefore, $\diamond_{t?}(A_1 \cap \dots \cap A_i, B_1 \cap \dots \cap B_j)$ is in S . So, the pair (z, z) is S -compatible with respect to $t?$ and lemma 2 can be applied. \dashv

2.2.4 Completeness

We are now ready to prove the completeness result for L_{all} .

Theorem 2 *For any consistent set of formulas there is a dynamic model satisfying all of them. Hence, for any formula ϕ if $\mathcal{C}_{all} \models \phi$, then $\vdash_{L_{all}} \phi$.*

Proof: Let Γ be a consistent set of formulas. By lemma 1 there is a maximal theory S extending the consistent theory $\{\phi : \Gamma \vdash_{L_{all}} \phi\}$. Now we shall define a dynamic model and we shall prove that the formulas from S are satisfied in them.

Let $\mathcal{F} = (W, R)$ be the structure defined as follows:

- W is the set of all S -maximal sets of Boolean terms,
- For all relational variables a , R_a is the binary relation on W such that for all x and y in W , $xR_a y$ iff $(\forall A \in x)(\forall B \in y)(\diamond_a(A, B) \in S)$, i.e. R_a is the set of all pairs (x, y) of S -maximal sets of Boolean terms such that (x, y) is S -compatible with respect to a .

Since $t \neq f$ is in S , then by lemma 2, W is a nonempty set. Hence \mathcal{F} is a dynamic frame. We shall call it the canonical frame of L_{all} defined by S . Let $\mathcal{M} = (W, R, V)$ be the dynamic model on \mathcal{F} defined by:

- For all Boolean variables p , $V(p) = \{x \in W : p \in x\}$.

We shall call it the canonical model of L_{all} defined by S .

Now it is time to show that for all Boolean terms A, B , for all $x, y \in W$ and for all relational terms α :

1. $V(A) = \{x \in W : A \in x\}$,

2. $xR_\alpha y$ iff $\forall C \forall D (C \in x \text{ and } D \in y \Rightarrow \diamond_\alpha(C, D) \in S)$, i.e. $xR_\alpha y$ iff the pair (x, y) is S -compatible with respect to α ,
3. If $\mathcal{M} \models \diamond_\alpha(A, B)$ then $\diamond_\alpha(A, B) \in S$,
4. If $\diamond_\alpha(A, B) \in S$ then $\mathcal{M} \models \diamond_\alpha(A, B)$.

(1) The reader is invited to prove the claim by induction on A .

(2) This claim follows by induction on α . The base case, where α is a relational variable, holds by the definition of R . So let us turn to the cases $A?$, $\alpha \cup \beta$, α^{-1} .

Case $A?$. Suppose that $xR_{A?}y$. Then $x = y$ and y is in $V(A)$. If (x, y) is not S -compatible with respect to $A?$ then there is a Boolean term $C \in x$ and there is a Boolean term $D \in y$ such that $\diamond_{A?}(C, D) \notin S$. Since $x = y$ and $y \in V(A)$, then $C \cap D \cap A \in y$. Therefore $C \cap D \cap A \neq f \in S$ contradicting $\diamond_{A?}(C, D) \notin S$.

Suppose that (x, y) is S -compatible with respect to $A?$ and that not $xR_{A?}y$. It follows that $x \neq y$ or y is not in $V(A)$. If $x \neq y$ then there is a Boolean term $C' \in x$ and $C' \notin y$ or there is a Boolean term $C'' \in x$ and $C'' \notin y$. In the former case thus $\widetilde{C'} \in y$ and $\diamond_{A?}(C', \widetilde{C'}) \in S$. Consequently $C' \cap \widetilde{C'} \cap C \neq f \in S$ contradicting the consistency of S . In the latter case thus $\widetilde{C''} \in x$ and $\diamond_{A?}(\widetilde{C''}, C'') \in S$. Consequently $\widetilde{C''} \cap C'' \cap A \neq f \in S$ contradicting the consistency of S . Hence $x = y$. It follows that $y \notin V(A)$. Moreover $\bar{A} \in x$ and $\bar{A} \in y$. Therefore $\bar{A} \cap \bar{A} \cap A \neq f \in S$ contradicting the consistency of S .

Case $\alpha \cup \beta$. Suppose that $xR_{\alpha \cup \beta}y$. It follows that $xR_\alpha y$ or $xR_\beta y$. Hence the pair (x, y) is S -compatible with respect to α or the pair (x, y) is S -compatible with respect to β . Let C be a Boolean term in x and D be a Boolean term in y . If the pair (x, y) is S -compatible with respect to α then $\diamond_\alpha(C, D) \in S$. If the pair (x, y) is S -compatible with respect to β then $\diamond_\beta(C, D) \in S$. In both cases $\diamond_{\alpha \cup \beta}(C, D) \in S$. Therefore the pair (x, y) is S -compatible with respect to $\alpha \cup \beta$.

Suppose that (x, y) is S -compatible with respect to $\alpha \cup \beta$ and that not $xR_{\alpha \cup \beta}y$. Then not $xR_\alpha y$ and not $xR_\beta y$. Hence $\diamond_\alpha(C', D') \notin S$ for some Boolean term $C' \in x$ and for some Boolean term $D' \in y$ and $\diamond_\beta(C'', D'') \notin S$ for some Boolean term $C'' \in x$ and for some Boolean term $D'' \in y$. It follows that $\diamond_{\alpha \cup \beta}(C' \cap C'', D' \cap D'') \in S$. Hence $\diamond_\alpha(C' \cap C'', D' \cap D'') \in S$ or $\diamond_\beta(C' \cap C'', D' \cap D'') \in S$. Therefore $\diamond_\alpha(C', D') \in S$ or $\diamond_\beta(C'', D'') \in S$ contradicting $\diamond_\alpha(C', D') \notin S$ and $\diamond_\beta(C'', D'') \notin S$.

Case α^{-1} . Suppose that $xR_{\alpha^{-1}}y$. Then $yR_\alpha x$ and hence the pair (y, x) is S -compatible with respect to α . Therefore, if $C \in x$ and $D \in y$, then $\diamond_\alpha(D, C) \in S$, which implies $\diamond_{\alpha^{-1}}(C, D) \in S$.

Suppose that (x, y) is S -compatible with respect to α^{-1} . Let C and D be Boolean terms, $C \in x$ and $D \in y$. Then $\diamond_{\alpha^{-1}}(C, D) \in S$ and hence

$\diamond_\alpha(D, C) \in S$. So (y, x) is S -compatible with respect to α and by induction hypothese $yR_\alpha x$. Therefore $xR_{\alpha^{-1}}y$.

This finishes the proof by induction on α that the pair (x, y) is S -compatible with respect to α iff $xR_\alpha y$.

(3) This claim follows immediately by (1) and (2). Let α be a relational term and A, B be Boolean terms. If $\mathcal{M} \models \diamond_\alpha(A, B)$, then there are $x, y \in W$ such that $x \in V(A)$, $y \in V(B)$ and $xR_\alpha y$. Hence $A \in x$, $B \in y$ and (x, y) is S -compatible with respect to α . So, $\diamond_\alpha(A, B) \in S$.

(4) If $\diamond_\alpha(A, B)$ is in S then the pair $(\{A\}, \{B\})$ is S -compatible with respect to α . Hence by lemma 2, there are S -maximal sets x_0 and y_0 of Boolean terms such that $A \in x_0$, $B \in y_0$ and the pair (x_0, y_0) is S -compatible with respect to α . Then by (1) and (2), $x_0 \in V(A)$, $y_0 \in V(B)$ and $x_0R_\alpha y_0$. Therefore $\mathcal{M} \models \diamond_\alpha(A, B)$.

Now it is quite easy to prove by induction on ψ , using the maximality of S , that for all formulas ψ , $\mathcal{M} \models \psi$ iff $\psi \in S$. Since $\Gamma \subseteq S$, we conclude that all formulas from Γ are satisfied in \mathcal{M} .

Suppose that $\mathcal{C}_{all} \models \phi$. Then $\vdash_{L_{all}} \phi$, because, otherwise, the set $\{\neg\phi\}$ is consistent and therefore there is a dynamic model satisfying $\neg\phi$. \dashv

2.3 Decidability and complexity

Our intention is to prove that \mathcal{C}_{all} -satisfiability is NP -complete.

2.3.1 Lower bound

In the first place we prove that NP is a lower bound.

Theorem 3 \mathcal{C}_{all} -satisfiability is NP -hard.

Proof: Let A be a Boolean term. The reader may easily verify that $A \neq f$ is satisfiable in \mathcal{C}_{all} iff $A = f$ is not a Boolean tautology. From the fact that Boolean consistency is NP -hard it follows that \mathcal{C}_{all} -satisfiability is NP -hard. \dashv

2.3.2 Upper bound

In the second place we prove that NP is an upper bound.

Theorem 4 \mathcal{C}_{all} -satisfiability is in NP .

Proof: We can show that \mathcal{C}_{all} -satisfiability is in NP by appealing to the following result:

Lemma 4 Let ϕ be a formula. If $\mathcal{C}_{2 \times |\phi|} \models \phi$ then $\mathcal{C}_{all} \models \phi$.

Proof: Suppose $\mathcal{C}_{2 \times |\phi|} \models \phi$, we demonstrate $\mathcal{C}_{all} \models \phi$. If $\mathcal{C}_{all} \not\models \phi$ then there is a dynamic frame $\mathcal{F} = (W, R)$ such that $\mathcal{F} \not\models \phi$. Hence there is a dynamic model $\mathcal{M} = (W, R, V)$ on \mathcal{F} such that $\mathcal{M} \not\models \phi$. Let:

- $\phi_1 \wedge \dots \wedge \phi_n$,

be a conjunctive normal form of ϕ where for all i in $\{1, \dots, n\}$, ϕ_i is a formula of the form $\diamond_{\alpha_{(1)}^i}(E_{(1)}^i, F_{(1)}^i) \vee \dots \vee \diamond_{\alpha_{(n_i^\diamond)}^i}(E_{(n_i^\diamond)}^i, F_{(n_i^\diamond)}^i) \vee \neg \diamond_{\beta_{(1)}^i}(G_{(1)}^i, H_{(1)}^i) \vee \dots \vee \neg \diamond_{\beta_{(n_i^\neg)}^i}(G_{(n_i^\neg)}^i, H_{(n_i^\neg)}^i)$. Since $\mathcal{M} \not\models \phi$, then there is i in $\{1, \dots, n\}$ such that $\mathcal{M} \not\models \phi_i$. Thus $\mathcal{M} \models \neg \diamond_{\alpha_{(k)}^i}(E_{(k)}^i, F_{(k)}^i)$, for all $k = 1, \dots, n_i^\diamond$, and $\mathcal{M} \models \diamond_{\beta_{(\ell)}^i}(G_{(\ell)}^i, H_{(\ell)}^i)$, for all $\ell = 1, \dots, n_i^\neg$. It follows that W contains points $y_{(1)}, \dots, y_{(n_i^\neg)}$ and points $z_{(1)}, \dots, z_{(n_i^\diamond)}$ such that for all j in $\{1, \dots, n_i^\diamond\}$:

- $y_{(j)} R_{\beta_{(j)}^i} z_{(j)}$, $y_{(j)} \in V(G_{(j)}^i)$ and $z_{(j)} \in V(H_{(j)}^i)$.

Let $W' = \{y_{(1)}, \dots, y_{(n_i^\neg)}, z_{(1)}, \dots, z_{(n_i^\diamond)}\}$ and R' and V' be the restrictions of R and V to W' , respectively. Let $\mathcal{F}' = (W', R')$ and $\mathcal{M}' = (W', R', V')$. Obviously W' contains at most $2 \times |\phi|$ points and for all $x, y \in W'$:

- $x \in V(A)$ iff $x \in V'(A)$, for all Boolean terms A ,
- $x R_\alpha y$ iff $x R'_\alpha y$, for all relational terms α .

The reader is asked to show that $\mathcal{M}' \models \neg \diamond_{\alpha_{(k)}^i}(E_{(k)}^i, F_{(k)}^i)$, for all $k = 1, \dots, n_i^\diamond$, and $\mathcal{M}' \models \diamond_{\beta_{(\ell)}^i}(G_{(\ell)}^i, H_{(\ell)}^i)$, for all $\ell = 1, \dots, n_i^\neg$. Therefore $\mathcal{M}' \not\models \phi_i$. Thus $\mathcal{M}' \not\models \phi$. Hence $\mathcal{F}' \not\models \phi$ contradicting $\mathcal{C}_{2 \times |\phi|} \models \phi$. \dashv

So it remains to prove the existence of an algorithm in NP that solves \mathcal{C}_{all} -satisfiability. Given a formula ϕ , we non-deterministically choose a frame $\mathcal{F} = (W, R)$ whose size is linear in the size of ϕ and a model $\mathcal{M} = (W, R, V)$ on \mathcal{F} . Because \mathcal{M} is polysize in $|\phi|$, we can check in time polynomial in $|\phi|$ whether $\mathcal{M} \models \phi$. \dashv

3 Adding composition

This section introduces the relational operation of sequential composition: for two relational α and β , we form the composition $\alpha; \beta$.

3.1 Syntax and semantics

Suppose again that we have fixed a countable set Φ of Boolean variables and a countable set Π of relational variables. Let us now consider the smallest set of relational terms over Φ and Π containing the relational variables, tests over Φ and all the relationals terms constructed over them using the constructors $^{-1}$ (“converse”), \cup (“union”) and $;$ (“sequential composition”). A dynamic model $\mathcal{M} = (W, R, V)$ on a dynamic frame $\mathcal{F} = (W, R)$ is now such that for all relational terms α and β ,

$$R_{\alpha; \beta} = \{(x, y) : (\exists z \in W)(x R_\alpha z \text{ and } z R_\beta y)\}.$$

3.2 Axiomatization and completeness

We attack the problem of axiomatization of validity in \mathcal{C}_{all} by a method using an additional inference rule for composition. Let us briefly mention the reason to do this. In classical multimodal language we can modally define by a single schema the condition on accessibility relations corresponding to the composition of two relations. But the well known schema has nested modalities! It is not difficult to see that no set of modal formulas with modal depth not greater than 1 can define the composition. Since in the considered language no diamond is in the scope of other one, it is not surprising that we have proved that in the language with the set of relational terms a , b , $a;b$ no set of formulas can define by validity in the frame the semantical condition for the composition. However, in the case when the relational terms are exactly a , b and $a;b$ we can axiomatize validity in the class of all frames, where $R_{a;b} = \{(x, y) : (\exists z \in W)(xR_a z \text{ and } zR_b y)\}$, using only the axiom schema S_{12} (see below). Unfortunately, we cannot use this proof in the present dynamic context. This is the reason why we use below the inference rule R . As a result the strong completeness theorem is proved.

3.2.1 A deductive system

Let L_{all}^i be the extension of the deductive system L_{all} which proper axioms are all instances of the following schema:

$$S_{12} \quad \diamond_{\alpha;\beta}(A, B) \rightarrow \diamond_{\alpha}(A, C) \vee \diamond_{\beta}(\tilde{C}, B).$$

Let Γ be a set of formulas. A formula ϕ is called derivable from Γ in L_{all}^i , denoted $\Gamma \vdash_{L_{all}^i} \phi$, iff it can be derived from the above axioms and from the tautologies of classical propositional logic by applying modus ponens and the following inference rule of composition:

$$R: \text{ Given } \phi \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, B) \text{ for some Boolean variable } p \text{ not occurring in } \phi \rightarrow \diamond_{\alpha;\beta}(A, B), \text{ infer } \phi \rightarrow \diamond_{\alpha;\beta}(A, B).$$

A formula ϕ is called provable in L_{all}^i , denoted $\vdash_{L_{all}^i} \phi$, if it is derivable from the empty set in L_{all}^i .

As an illustration of R 's usefulness let us demonstrate that the formula $\diamond_{\alpha;(\beta;\gamma)}(A, B) \rightarrow \diamond_{(\alpha;\beta);\gamma}(A, B)$ is provable in L_{all}^i .

1. $\diamond_{\alpha;(\beta;\gamma)}(A, B) \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta;\gamma}(\tilde{p}, B)$, where p is a variable not occurring in $\diamond_{\alpha;(\beta;\gamma)}(A, B)$, is an instance of axiom S_{12} ;
2. $\diamond_{\beta;\gamma}(\tilde{p}, B) \rightarrow \diamond_{\beta}(\tilde{p}, q) \vee \diamond_{\gamma}(\tilde{q}, B)$, where q is a variable not occurring in $\diamond_{\alpha;(\beta;\gamma)}(A, B) \wedge \neg \diamond_{\gamma}(\tilde{q}, B)$ and different from p , is an instance of axiom S_{12} ;
3. $\diamond_{\alpha;(\beta;\gamma)}(A, B) \wedge \neg \diamond_{\gamma}(\tilde{q}, B) \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, q)$, Boolean consequence of line 1 and line 2;

4. $\diamond_{\alpha;(\beta;\gamma)}(A, B) \wedge \neg \diamond_{\gamma}(\tilde{q}, B) \rightarrow \diamond_{\alpha;\beta}(A, q)$, an application of R_i to line 3;
5. $\diamond_{\alpha;(\beta;\gamma)}(A, B) \rightarrow \diamond_{\alpha;\beta}(A, q) \vee \diamond_{\gamma}(\tilde{q}, B)$, Boolean consequence of line 4;
6. $\diamond_{\alpha;(\beta;\gamma)}(A, B) \rightarrow \diamond_{(\alpha;\beta);\gamma}(A, B)$, an application of R_i to line 5.

3.2.2 Soundness

The rule R_i preserves the property of validity in dynamic frames, as shows the next lemma.

Lemma 5 *Let $\mathcal{F} = (W, R)$ be a dynamic frame. Let ϕ be a formula, α and β be relational terms and A and B be Boolean terms. If $\mathcal{F} \models \phi \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, B)$ for some Boolean variable p not occurring in $\phi \rightarrow \diamond_{\alpha;\beta}(A, B)$, then $\mathcal{F} \models \phi \rightarrow \diamond_{\alpha;\beta}(A, B)$.*

Proof: Let $\mathcal{F} \models \phi \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, B)$ for some Boolean variable p not occurring in $\phi \rightarrow \diamond_{\alpha;\beta}(A, B)$, we demonstrate $\mathcal{F} \models \phi \rightarrow \diamond_{\alpha;\beta}(A, B)$. For the sake of contradiction, suppose $\mathcal{F} \not\models \phi \rightarrow \diamond_{\alpha;\beta}(A, B)$. Then there is a dynamic model $\mathcal{M} = (W, R, V)$ on \mathcal{F} such that $\mathcal{M} \not\models \phi \rightarrow \diamond_{\alpha;\beta}(A, B)$. Consequently $\mathcal{M} \models \phi$ and $\mathcal{M} \not\models \diamond_{\alpha;\beta}(A, B)$. Let V' be the function assigning to each Boolean variable q a subset $V'(q)$ of W such that:

$$V'(q) = \begin{cases} V(q), & \text{if } q = p, \\ W \setminus \{y : (\exists x \in V(A))(xR_{\alpha}y)\}, & \text{if } q \neq p. \end{cases}$$

Let \mathcal{M}' be the dynamic model on \mathcal{F} defined by V' . Let V' and R' be the extensions of V' and R to the set of all Boolean terms and the set of all relational terms, respectively. Since p does not occur in $\phi \rightarrow \diamond_{\alpha;\beta}(A, B)$, then $V'(A) = V(A)$, $V'(B) = V(B)$, $R'_{\alpha} = R_{\alpha}$, $R'_{\beta} = R_{\beta}$, $R'_{\alpha;\beta} = R_{\alpha;\beta}$, $\mathcal{M}' \models \phi$ and $\mathcal{M}' \not\models \diamond_{\alpha;\beta}(A, B)$. Since $\mathcal{F} \models \phi \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, B)$, then $\mathcal{M}' \models \phi \rightarrow \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, B)$. Therefore $\mathcal{M}' \models \diamond_{\alpha}(A, p) \vee \diamond_{\beta}(\tilde{p}, B)$. If $\mathcal{M}' \models \diamond_{\alpha}(A, p)$ then there is x in $V'(A)$ and there is y in $V'(p)$ such that $xR'_{\alpha}y$. Thus $x \in V(A)$, $y \in W \setminus \{y : (\exists x \in V(A))(xR_{\alpha}y)\}$ and $xR_{\alpha}y$, which is impossible. If $\mathcal{M}' \models \diamond_{\beta}(\tilde{p}, B)$ then there is x in $V'(\tilde{p})$ and there is y in $V'(B)$ such that $xR'_{\beta}y$. Since $x \in V'(\tilde{p}) = W \setminus V'(p) = \{y : (\exists x \in V(A))(xR_{\alpha}y)\}$, then there is $z \in V(A)$ such that $zR_{\alpha}x$. Consequently $z \in V'(A)$, $y \in V'(B)$ and $zR'_{\alpha;\beta}y$ contradicting $\mathcal{M}' \not\models \diamond_{\alpha;\beta}(A, B)$. \dashv

The following theorem can be easily proved.

Theorem 5 *Let Γ be a set of formulas valid in a dynamic frame \mathcal{F} . Let ϕ be a formula. If $\Gamma \vdash_{L_{all}^i} \phi$ then $\mathcal{F} \models \phi$. Hence, if $\vdash_{L_{all}^i} \phi$ then $\mathcal{C}_{all} \models \phi$.*

Proof: Obviously, dynamic frames satisfy the conditions which are needed to validate the axioms of L_{all}^i and MP is correct with respect to validity in a dynamic frame. For the correctness of R_i , see lemma 5 above. As a result, every formula derivable from Γ in L_{all}^i is valid in the frames validating all formulas in Γ . Hence, every formula provable in L_{all}^i is valid in \mathcal{C}_{all} . \dashv

3.2.3 Theories

To carry out the completeness proof of L_{all}^i , we need to prove a truth lemma which tells us that for all formulas ψ , ψ is satisfied in the canonical model defined by some maximal theory S iff ψ is in S . For the proof of the truth lemma, we will need theories of a special form: closed theories. By a closed theory we mean any set of formulas that contains L_{all}^i and that is closed under MP and the infinite version of the rule R^i :

R_ω^i Given $\phi \rightarrow \diamond_\alpha(A, p) \vee \diamond_\beta(\tilde{p}, B)$ for any Boolean variable p , infer $\phi \rightarrow \diamond_{\alpha;\beta}(A, B)$.

The reader may easily verify that if S is a closed theory and ϕ is a formula then $S + \phi$ is the least closed theory containing $S \cup \{\phi\}$. The notions of consistency and maximality are the same as those defined in the section devoted to the proof of the completeness theorem of L_{all} . To carry out the completeness proof of L_{all}^i , we need to learn a little more about maximal closed theories. The lemma below is the Lindenbaum's lemma for L_{all}^i .

Lemma 6 *Let S_0 be a consistent closed theory. Then there is a maximal closed theory S such that $S_0 \subseteq S$.*

Proof: Let ϕ_0, ϕ_1, \dots be an enumeration of the set of all formulas. We define the closed theory S as the union of a chain S_0, S_1, \dots of closed theories as follows:

- For all positive integers n , if $S_n + \phi_n$ is consistent then $S_{n+1} = S_n + \phi_n$ else we consider two cases:
 1. If ϕ_n is not in a form of a conclusion of the rule R_ω^i then we let $S_{n+1} = S_n + \neg\phi_n$,
 2. If ϕ_n is in a form of a conclusion of the rule R_ω^i , namely $\phi_n = \psi \rightarrow \diamond_{\alpha;\beta}(A, B)$, then there is a Boolean variable p such that $S_n + \psi \wedge \neg(\diamond_\alpha(A, p) \vee \diamond_\beta(\tilde{p}, B))$ is consistent and we let $S_{n+1} = S_n + \psi \wedge \neg(\diamond_\alpha(A, p) \vee \diamond_\beta(\tilde{p}, B))$.

Note that for all positive integers n , if S_n is consistent then S_{n+1} is consistent. For suppose not. Then there is a positive integer n such that S_n is consistent and S_{n+1} is inconsistent. It follows that ϕ_n is not in a form of a conclusion of the rule R_ω^i . Hence $S_n + \phi_n$ is inconsistent and $S_n + \neg\phi_n$ is inconsistent. Consequently ϕ_n is in S_n and $\neg\phi_n$ is in S_n contradicting the consistency of S_n . Since S_0 is consistent, then for all positive integers n , S_n is consistent. Hence S is a maximal closed theory such that $S_0 \subseteq S$. \dashv

In the sequel we make use of the following lemma.

Lemma 7 *Let Γ be a set of formulas and ϕ be a formula. If $\Gamma \not\vdash_{L_{all}^i} \phi$ then there is a maximal closed theory S such that $\Gamma \subseteq S$ and $\neg\phi \in S$.*

Proof: The set $S_0 = \{\psi : \Gamma \vdash_{L_{all}^i} \psi\}$ is a theory. Since $\Gamma \not\vdash_{L_{all}^i} \phi$, then S_0 is consistent. Let us show that S_0 is closed. Suppose that, for any Boolean variable p , the formula $\psi \rightarrow \diamond_\alpha(A, p) \vee \diamond_\beta(\tilde{p}, B)$ is in S_0 . Since there are infinitely many Boolean variables, $\psi \rightarrow \diamond_\alpha(A, p) \vee \diamond_\beta(\tilde{p}, B) \in S_0$ for some p not occurring in $\psi \rightarrow \diamond_{\alpha;\beta}(A, B)$. Thus $\psi \rightarrow \diamond_{\alpha;\beta}(A, B) \in S_0$, by the rule R_i . Hence, S_0 is consistent and closed. Now, since $\phi \notin S_0$, the theory $S_0 + \neg\phi$ is consistent and closed. Hence, it can be extended to a maximal consistent theory, by lemma 6. \dashv

The results in lemma 2 and lemma 3 can be readily extended to L_{all}^i . Everything goes through with only minor changes when the notion of theory is replaced by closed theory.

Lemma 8 *Let S be a maximal closed theory and α be a relational terms. If z and t are sets of Boolean terms such that the pair (z, t) is S -compatible with respect to α then there are S -maximal sets x and y of Boolean terms such that $z \subseteq x$, $t \subseteq y$ and the pair (x, y) is S -compatible with respect to α .*

Lemma 9 *Let S be a maximal closed theory and z be an S -consistent set of Boolean terms. Then there is an S -maximal set x of Boolean terms such that $z \subseteq x$.*

3.2.4 Completeness

With these basic results established, we are ready to demonstrate that:

Theorem 6 *Let ϕ be a formula and Γ be a set of formulas. If $\Gamma \not\vdash_{L_{all}^i} \phi$ then there is a dynamic model \mathcal{M} satisfying all formulas in $\Gamma \cup \{\neg\phi\}$. Hence, if $\mathcal{C}_{all} \models \phi$ then $\vdash_{L_{all}^i} \phi$.*

Proof: Suppose $\Gamma \not\vdash_{L_{all}^i} \phi$. Then by lemma 7 there is a maximal closed theory S such that $\Gamma \cup \{\neg\phi\} \subseteq S$. Let $\mathcal{F} = (W, R)$ be the dynamic frame defined by S , i.e.:

- W is the set of all S -maximal sets of Boolean terms,
- For all x, y in W , $xR_a y$ iff $(\forall A \in x)(\forall B \in y)(\diamond_a(A, B) \in S)$, i.e. R_a is the set of all pairs (x, y) of S -maximal sets of Boolean terms such that (x, y) is S -compatible with respect to a .

Seeing that $t \neq f$ is in S , then, to demonstrate the nonemptiness of W , the reader may use lemma 9. Hence \mathcal{F} is a dynamic frame. We shall call it the canonical frame of L_{all}^i defined by S . Let $\mathcal{M} = (W, R, V)$ be the dynamic model on \mathcal{F} defined by:

- For all Boolean variables p , $V(p) = \{x \in W : p \in x\}$.

We shall call it the canonical model of L_{all}^i defined by S .

Now, the proof almost literally follows the proof of the completeness theorem of L_{all} : the induction step in the proof of (2) has new case for the composition; and the reference in (3) to lemma 2 have to be replaced by a reference to lemma 8. So, we demonstrate $xR_{\alpha;\beta}y$ iff the pair (x, y) is S -compatible with respect to $\alpha; \beta$.

Suppose $xR_{\alpha;\beta}y$. Then there is an S -maximal set z of Boolean terms such that $xR_{\alpha}z$ and $zR_{\beta}y$. If the pair (x, y) is not S -compatible with respect to $\alpha; \beta$ then there are Boolean terms C and D such that $C \in x$, $D \in y$ and $\diamond_{\alpha;\beta}(C, D) \notin S$. Consequently $\diamond_{\alpha}(C, p) \notin S$ and $\diamond_{\beta}(\tilde{p}, D) \notin S$ for some Boolean variable p . If $p \in z$ then the pair (x, z) is not S -compatible with respect to α contradicting $xR_{\alpha}z$. If $\tilde{p} \in z$ then the pair (z, y) is not S -compatible with respect to β contradicting $zR_{\beta}y$.

Suppose that (x, y) is S -compatible with respect to $\alpha; \beta$. We have to prove that $xR_{\alpha;\beta}y$. Let $z = \{C : (\exists A \in x)(\diamond_{\alpha}(A, \tilde{C}) \notin S)\} \cup \{D : (\exists B \in y)(\diamond_{\beta}(\tilde{D}, B) \notin S)\}$. Suppose that z is not S -consistent. Then there are Boolean terms $C_1, \dots, C_m, D_1, \dots, D_n$ in z such that $C_1 \cap \dots \cap C_m \cap D_1 \cap \dots \cap D_n \neq f \notin S$. Therefore for some Boolean terms A_1, \dots, A_m in x , $\diamond_{\alpha}(A_1, \tilde{C}_1) \notin S, \dots, \diamond_{\alpha}(A_m, \tilde{C}_m) \notin S$ and for some Boolean terms B_1, \dots, B_n in y , $\diamond_{\beta}(\tilde{D}_1, B_1) \notin S, \dots, \diamond_{\beta}(\tilde{D}_n, B_n) \notin S$. Since A_1, \dots, A_m are in x and B_1, \dots, B_n are in y , then $\diamond_{\alpha;\beta}(A_1 \cap \dots \cap A_m, B_1 \cap \dots \cap B_n) \in S$. Consequently $\diamond_{\alpha}(A_1 \cap \dots \cap A_m, C_1 \cap \dots \cap C_m) \in S$ or $\diamond_{\beta}(C_1 \cap \dots \cap C_m, B_1 \cap \dots \cap B_n) \in S$. If $\diamond_{\alpha}(A_1 \cap \dots \cap A_m, C_1 \cap \dots \cap C_m) \in S$ then $\diamond_{\alpha}(A_1, \tilde{C}_1) \in S$ or \dots or $\diamond_{\alpha}(A_m, \tilde{C}_m) \in S$ contradicting $\diamond_{\alpha}(A_1, \tilde{C}_1) \notin S, \dots, \diamond_{\alpha}(A_m, \tilde{C}_m) \notin S$. Hence $\diamond_{\beta}(C_1 \cap \dots \cap C_m, B_1 \cap \dots \cap B_n) \in S$. Since $C_1 \cap \dots \cap C_m \cap D_1 \cap \dots \cap D_n \neq f \notin S$, then $\diamond_{\beta}(D_1 \cap \dots \cap D_n, B_1 \cap \dots \cap B_n) \in S$. Consequently $\diamond_{\beta}(\tilde{D}_1, B_1) \in S$ or \dots or $\diamond_{\beta}(\tilde{D}_n, B_n) \in S$ contradicting $\diamond_{\beta}(\tilde{D}_1, B_1) \notin S, \dots, \diamond_{\beta}(\tilde{D}_n, B_n) \notin S$. From all this it follows that z is an S -consistent set of Boolean terms. By lemma 9 there is an S -maximal set z' of Boolean terms such that $z \subseteq z'$. By definition of z , the pair (x, z') is S -compatible with respect to α and the pair (z', y) is S -compatible with respect to β . Thus $xR_{\alpha}z'$ and $z'R_{\beta}y$. Thus, $xR_{\alpha;\beta}y$. This finishes the new case $\alpha; \beta$ in the proof of (2).

Now, as in the proof of the completeness theorem for L_{all} , by induction on formulas ψ , it is easy to prove $\mathcal{M} \models \psi$ iff $\psi \in S$. Since $\Gamma \cup \neg\phi \subseteq S$, then all formulas from Γ are satisfied in \mathcal{M} and $\mathcal{M} \not\models \phi$. The case when $\Gamma = \emptyset$ corresponds to the completeness of L_{all}^i with respect to \mathcal{C}_{all} . \dashv

3.3 Decidability and complexity

For the remainder of this section we return to the decidability/complexity issue.

3.3.1 Lower bound

By utilising again the fact that Boolean consistency is *NP*-hard it is possible to prove that:

Theorem 7 \mathcal{C}_{all} -satisfiability is *NP*-hard.

3.3.2 Upper bound

Let ϕ be a formula such that $\mathcal{C}_{(|\phi|+2)\times|\phi|}^n \models \phi$. If $\mathcal{C}_{all} \not\models \phi$ then there is a dynamic frame $\mathcal{F} = (W, R)$ such that $\mathcal{F} \not\models \phi$. Hence there is a dynamic model $\mathcal{M} = (W, R, V)$ on \mathcal{F} such that $\mathcal{M} \not\models \phi$. Let:

- $\phi_1 \wedge \dots \wedge \phi_n$,

be a conjunctive normal form of ϕ where for all $i \in \{1, \dots, n\}$, ϕ_i is a formula of the form $\diamond_{\alpha_{(1)}}^i (E_{(1)}^i, F_{(1)}^i) \vee \dots \vee \diamond_{\alpha_{(n_i^\diamond)}}^i (E_{(n_i^\diamond)}^i, F_{(n_i^\diamond)}^i) \vee \neg \diamond_{\beta_{(1)}}^i (G_{(1)}^i, H_{(1)}^i) \vee \dots \vee \neg \diamond_{\beta_{(n_i^\diamond)}}^i (G_{(n_i^\diamond)}^i, H_{(n_i^\diamond)}^i)$. Since $\mathcal{M} \not\models \phi$, then there is $i \in \{1, \dots, n\}$ such that $\mathcal{M} \not\models \phi_i$. Thus for all k , $1 \leq k \leq n_i^\diamond$, $\mathcal{M} \models \neg \diamond_{\alpha_{(k)}}^i (E_{(k)}^i, F_{(k)}^i)$. and for all ℓ , $1 \leq \ell \leq n_i^{\neg \diamond}$, $\mathcal{M} \models \diamond_{\beta_{(\ell)}}^i (G_{(\ell)}^i, H_{(\ell)}^i)$. It follows that W contains points $y_{(1)}, \dots, y_{(n_i^{\neg \diamond})}$ and points $z_{(1)}, \dots, z_{(n_i^\diamond)}$ such that for all j in $\{1, \dots, n_i^{\neg \diamond}\}$:

- $y_{(j)} R_{\beta_{(j)}^i} z_{(j)}$, $y_{(j)} \in V(G_{(j)}^i)$ and $z_{(j)} \in V(H_{(j)}^i)$.

For all relational terms α , let w_α be a function assigning to each pair (x, y) of points a set $w_\alpha(x, y)$ of points such that:

- $w_a(x, y) = \emptyset$,
- $w_{C?}(x, y) = \emptyset$,
- If not $xR_{\alpha \cup \beta} y$ then $w_{\alpha \cup \beta}(x, y) = \emptyset$ else either $xR_\alpha y$ and $w_{\alpha \cup \beta}(x, y) = w_\alpha(x, y)$ or $xR_\beta y$ and $w_{\alpha \cup \beta}(x, y) = w_\beta(x, y)$,
- If not $xR_{\alpha; \beta} y$ then $w_{\alpha; \beta}(x, y) = \emptyset$ else $w_{\alpha; \beta}(x, y) = w_\alpha(x, z) \cup \{z\} \cup w_\beta(z, y)$ where z is a point such that $xR_\alpha z$ and $zR_\beta y$.

The reader may easily verify that for all relational terms α occurring in ϕ and for all pairs (x, y) of points in W , $w_\alpha(x, y)$ contains at most $|\phi|$ points. Let $\mathcal{F}' = (W', R')$ and $\mathcal{M}' = (W', R', V')$ be the restrictions of \mathcal{F} and \mathcal{M} to $\{y_{(1)}, \dots, y_{(n_i^{\neg \diamond})}, z_{(1)}, \dots, z_{(n_i^\diamond)}\} \cup \bigcup \{w_{\beta_{(j)}^i}(y_{(j)}, z_{(j)}) : j \in \{1, \dots, n_i^{\neg \diamond}\}\}$. Obviously W' contains at most $(|\phi| + 2) \times |\phi|$ points. The reader is asked to show that $\mathcal{M}' \models \neg \diamond_{\alpha_{(k)}}^i (E_{(k)}^i, F_{(k)}^i)$ for all k , $1 \leq k \leq n_i^\diamond$, and $\mathcal{M}' \models \diamond_{\beta_{(\ell)}^i} (G_{(\ell)}^i, H_{(\ell)}^i)$ for all ℓ , $1 \leq \ell \leq n_i^{\neg \diamond}$. Therefore $\mathcal{M}' \not\models \phi_i$. Thus $\mathcal{M}' \not\models \phi$. Consequently $\mathcal{F}' \not\models \phi$ contradicting $\mathcal{C}_{(|\phi|+2)\times|\phi|}^n \models \phi$. This line of reasoning proves the following lemma.

Lemma 10 *Let ϕ be a formula. If $\mathcal{C}_{(|\phi|+2)\times|\phi|} \models \phi$ then $\mathcal{C}_{all} \models \phi$.*

As in section 2, lemma 10 implies that \mathcal{C}_{all} -satisfiability is in *NP*.

Theorem 8 *\mathcal{C}_{all} -satisfiability is in *NP*.*

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