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Eliminating unorthodox derivation rules in an axiom system for iteration-free PDL with intersection

Philippe Balbiani
Institut de recherche en informatique de Toulouse, Université Paul Sabatier
118 route de Narbonne, 31062 Toulouse Cedex 4, France
balbiani@irit.fr

Abstract. We devote this paper to the completeness of an axiom system for $PDL_0^{\cap}$ — a variant of PDL which includes the program operations of composition and intersection. Most of the difficulty in the proof of the completeness theorem for $PDL_0^{\cap}$ lies in the fact that intersection of accessibility relations is not modally definable. We overcome this difficulty by considering the concepts of theory and large program. Theories are sets of formulas that contain $PDL_0^{\cap}$ and are closed under the inference rule of modus ponens. Large programs are built up from program variables and theories by means of the operations of composition and intersection, just as programs are built up from program variables and tests. Adapting these concepts to the subordination method, we can prove the completeness of our deductive system for $PDL_0^{\cap}$.

Keywords: Propositional modal logic, propositional dynamic logic, composition of programs, intersection of programs, test on a formula, deductive completeness, subordination method.

1. Introduction

Propositional dynamic logic — PDL — was first defined by Fischer and Ladner [9]. It is based on the idea of associating with each program γ of a programming language a modality $[\gamma]$, the formula $[\gamma]\phi$ being read "whenever program γ terminates, it must do so in a state satisfying formula ϕ". Syntactically, PDL is a modal logic with an algebraic structure in the set of modalities: composition $(\gamma;\delta)$ of programs γ and δ corresponding to the composition of the accessibility relations $R(\gamma)$ and $R(\delta)$, test $\phi?$ on formula ϕ corresponding to the partial identity relation in the subsets of the PDL-models in which the formula ϕ is true, union $(\gamma\cup\delta)$ of programs γ and δ

corresponding to the union of the accessibility relations $R(\gamma)$ and $R(\delta)$, iteration $\gamma^\star$ of program γ corresponding to the reflexive and transitive closure of the accessibility relation $R(\gamma)$. At the most intuitive level, an intended interpretation of these operations is as follows: executing composition $(\gamma;\delta)$ means "first do γ and then do δ", executing test $\phi?$ means "if ϕ then skip else abort", executing union $(\gamma \cup \delta)$ means "do γ or do δ nondeterministically", executing iteration $\gamma^\star$ means "repeat γ a finite number of times". There are by now a number of books [12, 13, 16] and survey papers [15, 19, 22] treating PDL and a great deal is known concerning its complexity and its proof theory: it is shown in [24] how the satisfiability problem for PDL can be solved in deterministic exponential-time whereas an elementary proof of the completeness of the deductive system formulated by Segerberg [25] is given in [18].

Additional topics related to PDL include results concerning comparative power of expression, decidability, complexity and completeness of a number of interesting variants obtained by extending or restricting PDL in various ways: deterministic programs, restricted tests, nonregular programs, programs as automata, complementation and intersection of programs, converse and infinite computations, etc. One of the most interesting extensions of PDL is PDL with intersection — $PDL^\cap$. The intersection operation $\cap$ is a program operation which allows two programs to be run in parallel. The intersection $(\gamma \cap \delta)$ of programs γ and δ corresponds to the intersection of the accessibility relations $R(\gamma)$ and $R(\delta)$: therefore the formula $[\gamma \cap \delta]\phi$ means that "every possible execution of programs γ and δ in parallel leads to a state in which formula ϕ is true". The chief feature of the intersection operation is that intersection of accessibility relations is not definable in the ordinary language of PDL, although it becomes definable in a modal language strengthened by the introduction of data constants [23]. As a result, less is known about the complexity and the proof theory of $PDL^\cap$. Decidability of the satisfiability problem for $PDL^\cap$ is proved in Danecki [8] but it is not known at present whether the upper bound of deterministic exponential-time carries over to $PDL^\cap$. The intersection operation is also investigated in Harel [14] which gives the proof that the satisfiability problem for $PDL^\cap$ is undecidable if the semantics is modified so as to refer only to deterministic programs variables. Concerning completeness, there is no known complete axiomatization for $PDL^\cap$.

Completeness results for a syntactically restricted version of $PDL^\cap$ in which programs are built up from program variables by means of the operations of composition and intersection are given in [1, 2]. The completeness proof treated in Balbiani [1] draws from the subordination method of Hughes and Cresswell [17] whereas Balbiani and Fariñas del Cerro [2] base their line of reasoning on a suitable modification of the mosaic method of Marx and Venema [21]. Seeing that both arguments consist in a step-by-step method for constructing irreflexive Kripke models, this brings us to the question of whether the proofs in [1, 2] can be extended in the presence of tests. Balbiani and Vakarelov [3] present a deductive system for a relative modal logic with composition, intersection and tests — $PDL_0^\cap$ — which explores the following idea: although intersection of accessibility relations is not definable in the ordinary language of PDL, it becomes definable in a modal language strengthened by the introduction of propositional quantifiers. Instead of using axioms to define intersection of accessibility relations in the language of PDL enlarged with propositional quantifiers, the idea developed in [3] is to add an unorthodox derivation rule for intersection. This rule has a considerable effect on the canonical model whose states are all maximal sets of the logic of interest: it makes the canonical model standard for intersection, but nevertheless it does not make it standard for composition. By means of simple constructs called "large programs", Balbiani and Vakarelov [3] overcome this limitation of the special derivation rule for intersection, and prove the completeness of their deductive system through a slight but determining modification of the canonical model construction. More recently, see [4], large programs have become fundamental in the proof of the completeness of an axiom system for $PDL^{\cap}$ using the special derivation rule for intersection. Seeing that axiomatizations using special derivation rules may not have all the nice mathematical properties that orthodox axiomatizations have, an interesting point concerns the question whether the rule for intersection can be eliminated from the deductive systems elaborated in [3, 4].

This paper deals with $PDL_0^{\cap}$. Through an axiom system, it provides a syntactical characterization of the property "formula ϕ is valid in all Kripke models". The completeness of our axiom system is proved at the end of a thorough study of theories and large programs: theories are sets of formulas that contain $PDL_0^{\cap}$ and are closed under the inference rule of modus ponens whereas large programs are built up from program variables and theories by means of the operations of composition and intersection. The concept of theory has a lot to do with the concept of maximal consistent set of formulas. Its significance derives from the importance usually assigned to the canonical model theorem. As for the concept of large program, why is it so interesting? Large programs are built up from program variables and theories just as programs are built up from program variables and tests. They generalize programs, seeing that every large program corresponds to a set of programs. The concept of theory and the concept of large program allow us to introduce the concept of large model, that is, a triple (W, R, V) where W is a nonempty set of states, R is a function from the set of all large programs into the set of all binary relations on W and V is a function from W into the set of all theories. An examination of large models reveals that every formula satisfied in a large model is satisfied in a Kripke model (proposition 6.1). The concept of theory and the concept of large program also give rise to the concept of subordination model, a weak form of the concept of large model. An examination of subordination models reveals that every formula consistent with our axiom system is satisfied in a subordination model (proposition 7.1). Finally, by a step-by-step construction, we show how to transform any subordination model into a large model satisfying the same formulas (proposition 8.1). This demonstrates that we have solved the problem concerned with the completeness of our axiom system: we have established that any formula consistent with our axiom system can be satisfied in a Kripke model.

The section-by-section ventilation of the paper is as follows. Section 2 defines the language of $PDL_0^{\cap}$ whereas section 3 introduces its Kripke models. In section 4, an axiom system is given which provides a syntactical characterization of the property "ϕ is valid in all Kripke models". To carry out the proof of its completeness, we need to learn more about theories and large programs in section 5 and section 6. In the last two sections, section 7 and section 8, we present subordination models and give a proof of the completeness of our axiom system. We assume some familiarity with modal logic. Readers wanting more details may refer, for example, to [6] or [7].

2. Syntax

This section presents the formal language of our relative modal logic.

Definition 2.1. The set of all formulas and the set of all programs are inductively defined by the following equations:

- $\phi ::= p \mid \bot \mid [\gamma]\phi$;
- $\gamma ::= \pi \mid (\gamma;\delta) \mid (\gamma \cap \delta) \mid \phi?$;

where p ranges over a countably infinite set of propositional variables and π ranges over a countably infinite set of program variables.

We will use ϕ, ψ, χ, etc, for formulas whereas we will use γ, δ, λ, etc, for programs. The intended meanings of the constructs of our relative modal logic are:

- $\bot$: "false";
- $[\gamma]\phi$: "every terminating execution of γ brings about ϕ";
- $(\gamma;\delta)$: "first do γ and then do δ";
- $(\gamma \cap \delta)$: "do γ and do δ in parallel";
- $\phi?$: "if ϕ then skip else abort".

A number of other constructs can be defined in terms of the primitive ones.

Definition 2.2. The other constructs are defined as follows:

- $(\phi \to \psi)$ is defined as $[\phi?]\psi$;
- $\neg\phi$ is defined as $(\phi \to \bot)$;
- $\top$ is defined as $\neg\bot$;
- $(\phi \vee \psi)$ is defined as $(\neg\phi \to \psi)$;
- $(\phi \wedge \psi)$ is defined as $\neg(\phi \to \neg\psi)$;
- $(\phi \leftrightarrow \psi)$ is defined as $((\phi \to \psi) \wedge (\psi \to \phi))$;
- $\langle\gamma\rangle\phi$ is defined as $\neg[\gamma]\neg\phi$.

It is well worth noting that each formula or each program is a finite string of symbols, these symbols coming from a countable alphabet. It follows that there are countably many formulas and countably many programs. As usual, we follow the standard rules for omission of the parentheses. Obviously, programs are built up from program variables and tests by means of the operations ; and $\cap$. Let $\gamma(\phi_1?,\ldots,\phi_K?)$ be a program with $\phi_1?,\ldots,\phi_K?$ a sequence of some of its tests. The result of the replacement of $\phi_1?,\ldots,\phi_K?$ in their places with other tests $\psi_1?,\ldots,\psi_K?$ is another program which will be denoted $\gamma(\psi_1?,\ldots,\psi_K?)$.

3. Kripke models

Our task is now to present the possible-worlds semantics of our relative modal logic.

Definition 3.1. A Kripke model is a triple $\mathcal{M} = (W, R, V)$ where W is a nonempty set of states, R is a function from the set of all programs into the set of all binary relations on W and V is a function from W into the set of all sets of formulas.

We will use x, y, z, etc, for states. The set W of states in Kripke model $\mathcal{M} = (W, R, V)$ is to be regarded as the set of all possible states in a computation process. In Kripke model $\mathcal{M} = (W, R, V)$, the function R from the set of all programs into the set of all binary relations on W associates with each program γ the binary relation $R(\gamma)$ on W with $xR(\gamma)y$ meaning that state y can be reached from state x by performing program γ while the function V from W into the set of all sets of formulas associates with each state x the set $V(x)$ of formulas with $\phi \in V(x)$ meaning that formula ϕ is true at x.

Definition 3.2. State y can be reached from state x by performing program γ in Kripke model $\mathcal{M} = (W, R, V)$ iff $xR(\gamma)y$. Formula ϕ is true at state x in Kripke model $\mathcal{M} = (W, R, V)$ iff $\phi \in V(x)$.

We want the property "state y can be reached from state x by performing program γ" and the property "formula ϕ is true at state x" to reflect the intended meanings of the constructs of our relative modal logic. This leads naturally to the concept of standard Kripke model introduced below.

Definition 3.3. A Kripke model $\mathcal{M} = (W, R, V)$ will be defined to be standard if it satisfies the following conditions:

1. $\bot \notin V(x)$;
2. $[\gamma]\phi \in V(x)$ iff for all states y, if $xR(\gamma)y$ then $\phi \in V(y)$;
3. $R(\gamma; \delta) = \{(x, y)$: there is a state z such that $xR(\gamma)z$ and $zR(\delta)y\}$;
4. $R(\gamma \cap \delta) = R(\gamma) \cap R(\delta)$;
5. $R(\phi?) = \{(x, y)$: $x = y$ and $\phi \in V(y)\}$.

It follows immediately from definition 3.3 that

Lemma 3.1. *Let $\mathcal{M} = (W, R, V)$ be a Kripke model. If $\mathcal{M} = (W, R, V)$ is standard then:*

- $\phi \to \psi \in V(x)$ *iff if* $\phi \in V(x)$ *then* $\psi \in V(x)$*;*
- $\neg\phi \in V(x)$ *iff* $\phi \notin V(x)$*;*
- $\top \in V(x)$*;*
- $\phi \vee \psi \in V(x)$ *iff* $\phi \in V(x)$ *or* $\psi \in V(x)$*;*
- $\phi \wedge \psi \in V(x)$ *iff* $\phi \in V(x)$ *and* $\psi \in V(x)$*;*
- $\phi \leftrightarrow \psi \in V(x)$ *iff* $\phi \in V(x)$ *iff* $\psi \in V(x)$*;*
- $\langle\gamma\rangle\phi \in V(x)$ *iff there is a state* y *such that* $xR(\gamma)y$ *and* $\phi \in V(y)$*.*

For technical reasons, we use the concept of assignment of sets of formulas to states rather than the more usual concept of assignment of sets of states to propositional variables. It is easy to see that this slight change does not affect any of the results if the concept of satisfiability and the concept of validity are still defined in the usual way.

Definition 3.4. Formula ϕ is satisfied in Kripke model $\mathcal{M} = (W, R, V)$ iff there is a state x such that $\phi \in V(x)$ whereas ϕ is valid in $\mathcal{M}$ iff for all states x, $\phi \in V(x)$.

4. Axiom system

A typical feature of our relative modal logic is that it is a propositional modal logic with an algebraic structure in the set of modalities. In order to formalize this algebraic structure, the function dim from the set of all programs into the set of all positive integers and the binary relations $\equiv$ and $\preceq$ on the set of all programs will be needed.

Definition 4.1. Let dim be the function from the set of all programs into the set of all positive integers inductively defined as follows:

- $\dim(\pi) = 1$;
- $\dim(\gamma;\delta) = \dim(\gamma) + \dim(\delta)$;
- $\dim(\gamma \cap \delta) = \min\{\dim(\gamma), \dim(\delta)\}$;
- $\dim(\phi?) = 0$.

Let $\equiv$ be the binary relation on the set of all programs such that $\gamma \equiv \delta$ iff there is a positive integer k such that $\gamma \equiv_k \delta$ where $\equiv_0, \equiv_1, \ldots$ is the sequence of binary relations on the set of all programs defined inductively as follows:

- $\gamma \equiv_0 \gamma$;
- If $\gamma \equiv_k \delta$ then $\delta \equiv_{k+1} \gamma$;
- If $\gamma \equiv_k \delta$ and $\delta \equiv_l \lambda$ then $\gamma \equiv_{k+l+1} \lambda$;
- If $\gamma \equiv_k \gamma'$ and $\delta \equiv_l \delta'$ then $\gamma;\delta \equiv_{k+l+1} \gamma';\delta'$ and $\gamma \cap \delta \equiv_{k+l+1} \gamma' \cap \delta'$;
- $\gamma;(\delta;\lambda) \equiv_0 (\gamma;\delta);\lambda$.

Let $\preceq$ be the binary relation on the set of all programs such that $\gamma \preceq \delta$ iff there is a positive integer k such that $\gamma \preceq_k \delta$ where $\preceq_0, \preceq_1, \ldots$ is the sequence of binary relations on the set of all programs defined inductively as follows:

- $\gamma \preceq_0 \gamma$;
- If $\gamma \preceq_k \delta$ and $\delta \preceq_l \lambda$ then $\gamma \preceq_{k+l+1} \lambda$;
- If $\gamma \preceq_k \gamma'$ and $\delta \preceq_l \delta'$ then $\gamma;\delta \preceq_{k+l+1} \gamma';\delta'$ and $\gamma \cap \delta \preceq_{k+l+1} \gamma' \cap \delta'$;
- If $\gamma \equiv \delta$ then $\gamma \preceq_0 \delta$;
- $\gamma \cap \delta \preceq_0 \gamma$ and $\gamma \cap \delta \preceq_0 \delta$;
- If $\gamma \preceq_k \delta$ and $\gamma \preceq_l \lambda$ then $\gamma \preceq_{k+l+1} \delta \cap \lambda$;
- If $\dim(\gamma) = 0$ then $\gamma;\delta \preceq_0 \delta$;
- If $\dim(\delta) = 0$ then $\gamma;\delta \preceq_0 \gamma$;
- $\gamma \preceq_0 \gamma;\top?$;
- $\delta \preceq_0 \top?;\delta$;
- $\langle\gamma \cap \top?\rangle\top? \preceq_0 \gamma$;
- $(\phi \wedge \psi)? \preceq_0 \phi?;\psi?$.

It is a simple matter to check that dim is computable in linear time. Benanav, Kapur and Narendran [5] show that $\equiv$ is computable in linear time whereas the only complexity result known about $\preceq$ is that it is recursively enumerable. By definition 4.1, we infer immediately that

Lemma 4.1.
- *$\equiv$ is a congruence on the set of all programs such that:*
  - $\gamma; (\delta; \lambda) \equiv (\gamma; \delta); \lambda$*;*
- *$\preceq$ is an ordering on the set of all programs containing $\equiv$ and such that:*
  - $\gamma \cap \delta \preceq \gamma$ *and* $\gamma \cap \delta \preceq \delta$*;*
  - *If* $\gamma \preceq \delta$ *and* $\gamma \preceq \lambda$ *then* $\gamma \preceq \delta \cap \lambda$*;*
  - *If* $\dim(\gamma) = 0$ *then* $\gamma; \delta \preceq \delta$*;*
  - *If* $\dim(\delta) = 0$ *then* $\gamma; \delta \preceq \gamma$*;*
  - $\gamma \preceq \gamma; \top?$*;*
  - $\delta \preceq \top?; \delta$*;*
  - $\langle \gamma \cap \top? \rangle \top? \preceq \gamma$*;*
  - $(\phi \wedge \psi)? \preceq \phi?; \psi?$*.*

The next lemma shows a close connection between the function dim and the binary relations $\equiv$ and $\preceq$.

Lemma 4.2. *Let γ, δ be programs.*

1. *If* $\gamma \equiv \delta$ *then* $\dim(\gamma) = \dim(\delta)$*;*
2. *If* $\gamma \preceq \delta$ *then* $\dim(\gamma) \leq \dim(\delta)$*.*

Proof:
(1): The proof runs by induction on the least positive integer k such that $\gamma \equiv_k \delta$.
(2): The proof runs by induction on the least positive integer k such that $\gamma \preceq_k \delta$. □

To study more deeply the relationship between tests and $\preceq$, let us consider the function f from the set of all programs into itself defined as follows:

- $f(\pi) = \pi$;
- $f(\gamma; \delta) = f(\gamma); \top?; (\delta)$;
- $f(\gamma \cap \delta) = f(\gamma) \cap f(\delta)$;
- $f(\phi?) = \phi?$.

Not surprisingly, we have

Lemma 4.3. *Let γ be a program.*

1. $f(\gamma) \preceq \gamma$*;*
2. $\gamma \preceq f(\gamma)$*.*

Proof:
(1): By induction on the construction of a program.
(2): By induction on the construction of a program. □

For our purpose, the crucial properties of standard Kripke models are detailed in the following lemma.

Lemma 4.4. *Let $\mathcal{M} = (W, R, V)$ be a Kripke model. If $\mathcal{M} = (W, R, V)$ is standard then:*

1. *$xR(\gamma; \delta)y$ iff there is a state z such that $xR(\gamma)z$ and $zR(\delta)y$;*
2. *If $\gamma \equiv \delta$ then $xR(\gamma)y$ iff $xR(\delta)y$;*
3. *If $\dim(\gamma) = 0$ then if $xR(\gamma)y$ then $x = y$;*
4. *If $\gamma \preceq \delta$ then if $xR(\gamma)y$ then $xR(\delta)y$;*
5. *If $xR(\gamma(\phi?))y$ then $xR(\gamma((\phi \wedge \psi)?))y$ or $xR(\gamma((\phi \wedge \neg\psi)?))y$;*
6. *If $xR(\gamma((\phi \vee \psi)?))y$ then $xR(\gamma(\phi?))y$ or $xR(\gamma(\psi?))y$;*
7. *$xR(\overline{(\gamma; (\psi \wedge [(\delta; \phi?; \gamma) \cap \psi?]\bot)?; \delta) \cap \phi?})y$.*

Proof:
(1): The proof is by an easy verification.
(2): The proof runs by induction on the least positive integer k such that $\gamma \equiv_k \delta$.
(3): By induction on the construction of a program.
(4): The proof runs by induction on the least positive integer k such that $\gamma \preceq_k \delta$.
(5): By induction on the construction of a program.
(6): By induction on the construction of a program.
(7): The proof is by an easy verification. □

This motivates the following axiom system.

Definition 4.2. Let $PDL_0^{\cap}$ be the smallest normal logic that contains the axioms given below:

A_1 $\langle\gamma; \delta\rangle\phi \leftrightarrow \langle\gamma\rangle\langle\delta\rangle\phi$;
A_2 If $\gamma \equiv \delta$ then $\langle\gamma\rangle\phi \leftrightarrow \langle\delta\rangle\phi$;
A_3 If $\dim(\gamma) = 0$ then $\langle\gamma\rangle\phi \rightarrow \phi$;
A_4 If $\gamma \preceq \delta$ then $\langle\gamma\rangle\phi \rightarrow \langle\delta\rangle\phi$;
A_5 $\langle\gamma(\phi?)\rangle\chi \rightarrow \langle\gamma((\phi \wedge \psi)?)\rangle\chi \vee \langle\gamma((\phi \wedge \neg\psi)?)\rangle\chi$;
A_6 $\langle\gamma((\phi \vee \psi)?)\rangle\chi \rightarrow \langle\gamma(\phi?)\rangle\chi \vee \langle\gamma(\psi?)\rangle\chi$;
A_7 $[(\gamma; (\psi \wedge [(\delta; \phi?; \gamma) \cap \psi?]\bot)?; \delta) \cap \phi?]\bot$.

A formula ϕ is called provable in $PDL_0^{\cap}$, in symbols $\vdash_{PDL_0^{\cap}} \phi$, if ϕ belongs to $PDL_0^{\cap}$.

In other words, ϕ is provable in $PDL_0^{\cap}$ if ϕ can be derived from the axioms of $PDL_0^{\cap}$ by applying the inference rules of $PDL_0^{\cap}$, i.e. modus ponens "given $\phi \rightarrow \psi$ and ϕ, prove ψ" and generalization "given ϕ, prove $[\gamma]\phi$". It is well worth noting that the use of the binary relation $\preceq$ on the set of all programs in the proof theory of $PDL_0^{\cap}$ has a lot to do with the use of the binary relation $\subseteq$ on the set of all Boolean expressions in the proof theory of BML, the Boolean modal logic brought in by Gargov, Passy and Tinchev [11] and furthered by Gargov and Passy [10]. It should be noticed that the axiomatization of $PDL_0^{\cap}$ given in definition 4.2 is based on a recursively enumerable set of axioms. It is a well-known fact that our axiom system can be turned into an equivalent recursive set of axioms by means of Craig's trick, the question whether an equivalent finite set of axioms exists or not remaining unsettled. There is also the related question of how to eliminate the use of $\preceq$ in our axiomatization. The truth of the matter is that there is no easy answer to the issue at stake here. So we will not consider this question before section 8. Using lemma 4.4, the reader may easily verify that standard Kripke models satisfy the conditions which are needed to verify the axioms of $PDL_0^\cap$. As a result it is straightforward to prove by induction on the assumed derivation of ϕ from the axioms of $PDL_0^\cap$ that if ϕ is provable then ϕ is valid in all standard Kripke models. Hence

Proposition 4.1. *Let ϕ be a formula. If $\vdash_{PDL_0^\cap} \phi$ then ϕ is valid in all standard Kripke models.*

It follows that $PDL_0^\cap$ is sound with respect to the class of all standard Kripke models. The completeness of $PDL_0^\cap$ is more difficult to establish than its soundness and we defer proving that $PDL_0^\cap$ is complete with respect to the class of all standard Kripke models till section 8. To end this section, we present some useful results.

Lemma 4.5. *Let ϕ be a formula and $\gamma(\phi?)$ be a program with $\phi?$ one of its tests.*

1. *For all formulas ψ, if $\vdash_{PDL_0^\cap} \neg\phi$ then $\vdash_{PDL_0^\cap} [\gamma(\phi?)]\psi$;*
2. *For all formulas ψ, χ, $\vdash_{PDL_0^\cap} [\gamma((\phi \wedge \neg\psi)?)]\chi \to ([\gamma(\psi?)]\chi \to [\gamma(\phi?)]\chi)$;*
3. *For all formulas ψ, χ, if $\vdash_{PDL_0^\cap} \phi \to \psi$ then $\vdash_{PDL_0^\cap} [\gamma(\psi?)]\chi \to [\gamma(\phi?)]\chi$.*

Proof:
(1): By induction on the construction of a program.
(2): We demonstrate for all formulas ψ, χ, $\vdash_{PDL_0^\cap} [\gamma((\phi\wedge\neg\psi)?)]\chi \to ([\gamma(\psi?)]\chi \to [\gamma(\phi?)]\chi)$. Let ψ, χ be formulas. The proof that $(\phi\wedge\psi)? \preceq \psi?$, which is not difficult (using lemma 4.1), is left as an exercise. It follows that $\gamma((\phi\wedge\psi)?) \preceq \gamma(\psi?)$. Consequently $\vdash_{PDL_0^\cap} [\gamma(\psi?)]\chi \to [\gamma((\phi\wedge\psi)?)]\chi$ (using axiom A_4). Hence $\vdash_{PDL_0^\cap} [\gamma((\phi \wedge \neg\psi)?)]\chi \to ([\gamma(\psi?)]\chi \to [\gamma(\phi?)]\chi)$ (using axiom A_5).
(3): We demonstrate for all formulas ψ, χ, if $\vdash_{PDL_0^\cap} \phi \to \psi$ then $\vdash_{PDL_0^\cap} [\gamma(\psi?)]\chi \to [\gamma(\phi?)]\chi$. Let ψ, χ be formulas such that $\vdash_{PDL_0^\cap} \phi \to \psi$. It follows that $\vdash_{PDL_0^\cap} \neg(\phi \wedge \neg\psi)$. By item (1), $\vdash_{PDL_0^\cap} [\gamma((\phi \wedge \neg\psi)?)]\chi$. Therefore $\vdash_{PDL_0^\cap} [\gamma(\psi?)]\chi \to [\gamma(\phi?)]\chi$ (using item (2)). □

5. Theories

This section is about using theories to define the canonical model for $PDL_0^\cap$. The study of canonical models has been central to modal logic since the work of Lemmon and Scott [20]. Canonical models are useful for proving semantically driven completeness results for Sahlqvist axioms, the most commonly encountered axioms in the ordinary modal language. Every Sahlqvist axiom is first-order definable and is canonical for the first-order property it defines. In the ordinary language of PDL, $\langle\gamma;\delta\rangle\phi \leftrightarrow \langle\gamma\rangle\langle\delta\rangle\phi$ is a Sahlqvist axiom which defines the first-order property corresponding to the composition of accessibility relations. Seeing that the first-order property corresponding to the intersection of accessibility relations is not definable in the language of our relative modal logic, the canonical model for $PDL_0^\cap$ will not be of use to us for proving that $PDL_0^\cap$ is complete with respect to the class of all standard Kripke models. Nonetheless, its definition as well as the systematic study of theories pave the way for the completeness result proved in section 8. This section has three main parts. The first introduces the concept of theory and presents some useful results. The second part is essentially a detailed exposition of three lemmas: the Lindenbaum's lemma, the diamond lemma and the composition lemma. The final part defines the canonical model for $PDL_0^\cap$ and gives a brief account of its properties.

5.1. Theories: definition and useful results

Referring to proposition 4.1, it is well worth noting that for all standard Kripke models $\mathcal{M} = (W, R, V)$ and for all states x, $V(x)$ is a set of formulas that contains $PDL_0^\cap$ and is closed under the inference rule of modus ponens. This motivates the following definition.

Definition 5.1. A theory is any set of formulas that contains $PDL_0^\cap$ and is closed under the inference rule of modus ponens.

We will use S, T, U, etc, for theories. Let us be clear that the set of all theories is a partially ordered set with respect to set inclusion. The least element is $PDL_0^\cap$ and the greatest element is the set of all formulas. Not surprisingly, we have

Lemma 5.1. *Let S be a theory. The following conditions are equivalent:*

1. *S is the set of all formulas;*
2. *There is a formula ϕ such that $\phi \in S$ and $\neg\phi \in S$;*
3. *$\bot \in S$.*

Referring to lemma 5.1, we define what it means for a theory to be consistent.

Definition 5.2. A theory S will be defined to be consistent if for all formulas ϕ, $\phi \notin S$ or $\neg\phi \notin S$.

By duality, we define what it means for a theory to be maximal.

Definition 5.3. A theory S will be defined to be maximal if for all formulas ϕ, $\phi \in S$ or $\neg\phi \in S$.

The reader may easily check (using lemma 5.1) that there is only one inconsistent theory: the set of all formulas. We now establish a lemma which shows that in respect of the Boolean operations, a maximal consistent theory looks like a state at which the true formulas are the formulas in the theory. Its proof can be found in most elementary logic texts.

Lemma 5.2. *Let S be a maximal consistent theory.*

- *$\phi \rightarrow \psi \in S$ iff if $\phi \in S$ then $\psi \in S$;*
- *$\neg\phi \in S$ iff $\phi \notin S$;*
- *$\top \in S$;*
- *$\phi \vee \psi \in S$ iff $\phi \in S$ or $\psi \in S$;*
- *$\phi \wedge \psi \in S$ iff $\phi \in S$ and $\psi \in S$;*
- *$\phi \leftrightarrow \psi \in S$ iff $\phi \in S$ iff $\psi \in S$.*

In order to define the canonical model for $PDL_0^\cap$, we need yet another definition.

Definition 5.4. If γ is a program and S is a theory then let $[\gamma]S = \{\phi\colon [\gamma]\phi \in S\}$.

One can easily establish the following results.

Lemma 5.3. *Let* ϕ *be a formula and* S *be a theory.*

1. $S \subseteq [\phi?]S$;
2. $\phi \in [\phi?]S$.

Lemma 5.4. *Let* γ *be a program and* S *be a theory. Then* $[\gamma]S$ *is a theory.*

Lemma 5.5. *Let* ϕ *be a formula and* S *be a theory. If* $\phi \notin S$ *then for all formulas* ψ, $\phi \notin [\psi?]S$ *or* $\phi \notin [\neg\psi?]S$.

Lemma 5.6. *Let* γ *be a program and* S, T *be theories. If* T *is maximal and* $[\gamma]S \subseteq T$ *then for all formulas* ϕ, $[\gamma][\phi?]S \subseteq T$ *or* $[\gamma][\neg\phi?]S \subseteq T$.

5.2. Three lemmas

Three lemmas support the technique of the canonical model for $PDL_0^\cap$. There is the Lindenbaum's lemma, the diamond lemma and the composition lemma.

Lemma 5.7. (Lindenbaum's lemma) *Let* ϕ *be a formula and* S *be a theory. If* $\phi \notin S$ *then there is a maximal theory* T *such that* $S \subseteq T$ *and* $\phi \notin T$.

Proof:
Suppose $\phi \notin S$, we demonstrate there is a maximal theory T such that $S \subseteq T$ and $\phi \notin T$. Let us suppose that the set of all formulas is arranged in some determinate order ψ_0, ψ_1, ... Now we define an infinite sequence $T_0, T_1, \ldots$ of theories such that for all positive integers i, the following conditions are satisfied:

$C_1(i)$ For all positive integers j, if $i > j$ then $\psi_j \in T_i$ or $\neg\psi_j \in T_i$;
$C_2(i)$ $\phi \notin T_i$.

Let $T_0 = S$. Note that the conditions $C_1(0)$ and $C_2(0)$ are satisfied. Let i be a positive integer. Given T_i such that the conditions $C_1(i)$ and $C_2(i)$ are satisfied, we let T_{i+1} be $[\psi_i]T_i$ if $\phi \notin [\psi_i]T_i$. If $\phi \in [\psi_i]T_i$ then by lemma 5.5, $\phi \notin [\neg\psi_i]T_i$ and we let T_{i+1} be $[\neg\psi_i]T_i$. By lemma 5.3, it follows that the conditions $C_1(i+1)$ and $C_2(i+1)$ are satisfied. Now we put $T = T_0 \cup T_1 \cup \ldots$. □

Lemma 5.8. (Diamond lemma) *Let* γ *be a program,* ϕ *be a formula and* S *be a theory. If* $[\gamma]\phi \notin S$ *then there is a maximal theory* T *such that* $[f(\gamma)]S \subseteq T$ *and* $\phi \notin T$.

Proof:
Suppose $[\gamma]\phi \notin S$, we demonstrate there is a maximal theory T such that $[f(\gamma)]S \subseteq T$ and $\phi \notin T$. Since $[\gamma]\phi \notin S$, it follows that $[f(\gamma)]\phi \notin S$ (using item 2 of lemma 4.3 and axiom A_4). Thus $\phi \notin [f(\gamma)]S$. By lemma 5.7, there is a maximal theory T such that $[f(\gamma)]S \subseteq T$ and $\phi \notin T$. □

Lemma 5.9. (Composition lemma) *Let γ, δ be programs and S, T be theories. If T is maximal, $\bot \notin T$ and $[\gamma;\delta]S \subseteq T$ then there is a maximal theory U such that $\bot \notin U$, $[\gamma]S \subseteq U$ and $[\delta]U \subseteq T$.*

Proof:
Suppose T is maximal, $\bot \notin T$ and $[\gamma;\delta]S \subseteq T$, we demonstrate there is a maximal theory U such that $\bot \notin U$, $[\gamma]S \subseteq U$ and $[\delta]U \subseteq T$. Let us suppose that the set of all formulas is arranged in some determinate order $\phi_0, \phi_1, \ldots$. Now we define an infinite sequence $U_0, U_1, \ldots$ of theories such that for all positive integers i, the following conditions are satisfied:

$C_1(i)$ For all positive integers j, if $i > j$ then $\phi_j \in U_i$ or $\neg\phi_j \in U_i$;

$C_2(i)$ $[\gamma]S \subseteq U_i$;

$C_3(i)$ $[\delta]U_i \subseteq T$.

Let $U_0 = [\gamma]S$. Note that the conditions $C_1(0)$, $C_2(0)$ and $C_3(0)$ are satisfied. Let i be a positive integer. Given U_i such that the conditions $C_1(i)$, $C_2(i)$ and $C_3(i)$ are satisfied, we let U_{i+1} be $[\phi_i?]U_i$ if $[\delta][\phi_i?]U_i \subseteq T$. If $[\delta][\phi_i?]U_i \nsubseteq T$ then by lemma 5.6, $[\delta][\neg\phi_i?]U_i \subseteq T$ and we let U_{i+1} be $[\neg\phi_i?]U_i$. By lemma 5.3, it follows that the conditions $C_1(i+1)$, $C_2(i+1)$ and $C_3(i+1)$ are satisfied. Now we put $U = U_0 \cup U_1 \cup \ldots$. □

In fact, the proofs of these lemmas, especially the proof of the Lindenbaum's lemma, can be found in most elementary logic texts. In section 6.2, we will provide strong versions of the diamond lemma and the composition lemma.

5.3. The canonical model for $PDL_0^{\cap}$

We are now in a position to be able to construct the canonical model for $PDL_0^{\cap}$.

Definition 5.5. The canonical model for $PDL_0^{\cap}$ is the triple $\mathcal{M}_c = (W_c, R_c, V_c)$ where W_c is the set of all maximal consistent theories, R_c is the function from the set of all programs into the set of all binary relations on W_c such that $R_c(\gamma) = \{(S,T)\colon [\gamma]S \subseteq T\}$ and V_c is the function from W_c into the set of all sets of formulas such that $V_c(S) = S$.

The trouble is that the canonical model for $PDL_0^{\cap}$ is a Kripke model subject to the conditions of definition 3.3 but the condition (4).

Lemma 5.10. *1. $\bot \notin V_c(S)$;*

2. $[\gamma]\phi \in V_c(S)$ iff for all maximal consistent theories T, if $SR_c(\gamma)T$ then $\phi \in V_c(T)$;

3. $R_c(\gamma;\delta) = \{(S,T)$: there is a maximal consistent theory U such that $SR_c(\gamma)U$ and $UR_c(\delta)T\}$;

4. $R_c(\gamma \cap \delta) \subseteq R_c(\gamma) \cap R_c(\delta)$;

5. $R_c(\phi?) = \{(S,T)$: $S = T$ and $\phi \in V_c(T)\}$.

Proof:
(1): By lemma 5.1.
(2): By lemma 5.8.
(3): By lemma 5.9.
(4): The proof is immediate.
(5): The proof is by an easy verification.

□

The reason that the canonical model for $PDL_0^\cap$ does not satisfy the condition (4) of definition 3.3 is that intersection of accessibility relations is not definable in the language of our relative modal logic. That is why the technique using the canonical model for $PDL_0^\cap$ does not seem to be a great help to us for proving that $PDL_0^\cap$ characterizes the set of all formulas that are valid in all standard Kripke models. In this respect, the concept of large program will be of use to us.

6. Large programs

The main idea of this section can be traced back to Passy and Tinchev [23]: add symbols intended to name states in Kripke models. We choose a system of such symbols: for all theories S, we have the symbols $S?$ which we call large test. But the crucial notion in this section is the new concept of large program brought in by Balbiani and Vakarelov [3, 4]. Large programs are built up from program variables and large tests by means of the operation ; and $\cap$. They generalize programs, seeing that every large program corresponds to a set of programs. The section has three main parts. The first introduces large programs and is essentially a detailed exposition of their properties. In the second part of the section we develop techniques for proving strong versions of the diamond lemma and the composition lemma. In the last part, the concept of large program allows us to introduce the concept of large model. Large models generalize Kripke models, seeing that every large model is a triple (W, R, V) where W is a nonempty set of states, R is a function from the set of all large programs into the set of all binary relations on W and V is a function from W into the set of all theories.

6.1. Large programs: definition and useful results

Now, for the definition of large programs.

Definition 6.1. The set of all large programs is defined inductively as follows:

- If π is a program variable then π is a large program;
- If Γ, Δ are large programs then $(\Gamma; \Delta)$ is a large program;
- If Γ, Δ are large programs then $(\Gamma \cap \Delta)$ is a large program;
- If S is a theory then $S?$ is a large program.

Large programs of the form $S?$ will also be called large tests.

We will use Γ, Δ, Λ, etc, for large programs. Let us be clear that each large program is a finite string of symbols, these symbols coming from an uncountable alphabet. It follows that there are uncountably many large programs. For convenience, we omit the parentheses in accordance with the standard rules. It is essential to realize that large programs are built up from program variables and large tests by means of the operations ; and $\cap$. Let $\Gamma(S_1?, \ldots, S_K?)$ be a large program with $S_1?, \ldots, S_K?$ a sequence of some of its large tests. The result of the replacement of $S_1?, \ldots, S_K?$ in their places with other large tests $T_1?, \ldots, T_K?$ is another large program which will be denoted $\Gamma(T_1?, \ldots, T_K?)$. A large program $\Gamma(S_1?, \ldots, S_K?)$ with $S_1?, \ldots, S_K?$ the sequence of all its large tests will be defined to be maximal if the theories $S_1, \ldots, S_K$ are maximal. It appears that every large program, whether it is maximal or not, can be associated with a set of programs by means of the function ker from the set of all large programs into the set of all sets of programs.

Definition 6.2. Let ker be the function from the set of all large programs into the set of all sets of programs such that:

- $\ker(\pi) = \{\pi\}$;
- $\ker(\Gamma; \Delta) = \{\gamma; \delta \colon \gamma \in \ker(\Gamma) \text{ and } \delta \in \ker(\Delta)\}$;
- $\ker(\Gamma \cap \Delta) = \{\gamma \cap \delta \colon \gamma \in \ker(\Gamma) \text{ and } \delta \in \ker(\Delta)\}$;
- $\ker(S?) = \{\phi? \colon \phi \in S\}$.

Let $\Gamma(S_1?, \ldots, S_K?)$ be a large program with $S_1?, \ldots, S_K?$ the sequence of all its large tests. The result of the replacement of $S_1?, \ldots, S_K?$ in their places with tests $\phi_1?, \ldots, \phi_K?$ is a program which will be denoted $\Gamma(\phi_1?, \ldots, \phi_K?)$. Let us be clear that if $\phi_1 \in S_1$, …, $\phi_K \in S_K$ then $\Gamma(\phi_1?, \ldots, \phi_K?) \in \ker(\Gamma(S_1?, \ldots, S_K?))$. Reciprocally, if $\gamma(\phi_1?, \ldots, \phi_K?)$ is a program with $\phi_1?, \ldots, \phi_K?$ the sequence of all its tests then the result of the replacement of $\phi_1?, \ldots, \phi_K?$ in their places with large tests $S_1?, \ldots, S_K?$ is a large program which will be denoted $\gamma(S_1?, \ldots, S_K?)$. It is essential to realize that if $\phi_1 \in S_1$, …, $\phi_K \in S_K$ then $\gamma(\phi_1?, \ldots, \phi_K?) \in \ker(\gamma(S_1?, \ldots, S_K?))$. Our intuitive understanding of large programs tells us that any large program Γ is, roughly speaking, equivalent to the infinite intersection $\gamma_0 \cap \gamma_1 \cap \ldots$ where γ_0, γ_1, … is an arrangement of $\ker(\Gamma)$ in some determinate order. Since large programs correspond to sets of programs, the function dim and the binary relations $\equiv$ and $\preceq$ can be defined as follows.

Definition 6.3. Let dim be the function from the set of all large programs into the set of all positive integers such that:

- $\dim(\pi) = 1$;
- $\dim(\Gamma; \Delta) = \dim(\Gamma) + \dim(\Delta)$;
- $\dim(\Gamma \cap \Delta) = \min\{\dim(\Gamma), \dim(\Delta)\}$;
- $\dim(S?) = 0$.

Let $\equiv$ be the binary relation on the set of all large programs such that $\Gamma \equiv \Delta$ iff for all programs γ, if $\gamma \in \ker(\Gamma)$ then there is a program δ such that $\delta \in \ker(\Delta)$ and $\gamma \equiv \delta$ and for all programs δ, if $\delta \in \ker(\Delta)$ then there is a program γ such that $\gamma \in \ker(\Gamma)$ and $\gamma \equiv \delta$. Let $\preceq$ be the binary relation on the set of all large programs such that $\Gamma \preceq \Delta$ iff for all programs δ, if $\delta \in \ker(\Delta)$ then there is a program γ such that $\gamma \in \ker(\Gamma)$ and $\gamma \preceq \delta$.

Not surprisingly, we have

Lemma 6.1. *Let* Γ *be a large program and* γ *be a program. If* $\gamma \in \ker(\Gamma)$ *then* $\dim(\gamma) = \dim(\Gamma)$.

Proof:
The proof runs by induction on the construction of a large program. □

To obtain results for large programs similar to the results obtained for programs in lemma 4.1, lemma 4.2 and lemma 5.4, the following definition is required.

Definition 6.4. If Γ is a large program and S is a theory then let $[\Gamma]S = \{\phi$: there is a program γ such that $\gamma \in \ker(\Gamma)$ and $[\gamma]\phi \in S\}$.

To continue, another technical lemma is necessary.

Lemma 6.2. *Let* Γ *be a large program and* γ', γ'' *be programs. If* $\gamma' \in \ker(\Gamma)$ *and* $\gamma'' \in \ker(\Gamma)$ *then there is a program* γ *such that* $\gamma \in \ker(\Gamma)$, $\gamma \preceq \gamma'$ *and* $\gamma \preceq \gamma''$.

Proof:
Let us assume for the sake of the argument that large program Γ contains exactly one occurrence of a large test, say, $S?$. Suppose $\gamma' \in \ker(\Gamma(S?))$ and $\gamma'' \in \ker(\Gamma(S?))$, we demonstrate there is a program γ such that $\gamma \in \ker(\Gamma(S?))$, $\gamma \preceq \gamma'$ and $\gamma \preceq \gamma''$. Since $\gamma' \in \ker(\Gamma(S?))$ and $\gamma'' \in \ker(\Gamma(S?))$, then there are formulas ϕ', ϕ'' such that $\phi' \in S$, $\gamma' = \Gamma(\phi'?)$, $\phi'' \in S$ and $\gamma'' = \Gamma(\phi''?)$. Since $\phi' \to (\phi'' \to \phi' \wedge \phi'') \in S$, then $\phi' \wedge \phi'' \in S$. Now we put $\gamma = \Gamma((\phi' \wedge \phi'')?)$. Thus $\gamma \in \ker(\Gamma(S?))$. The proof that $(\phi' \wedge \phi'')? \preceq \phi'?$ and $(\phi' \wedge \phi'')? \preceq \phi''?$, which is not difficult (using lemma 4.1), is left as an exercise. Therefore $\gamma \preceq \gamma'$ and $\gamma \preceq \gamma''$. □

Now we have all means whereby we may prove the counterparts of lemma 4.1, lemma 4.2 and lemma 5.4.

Lemma 6.3. *1.* $\equiv$ *is a congruence on the set of all large programs such that:*

- $\Gamma; (\Delta; \Lambda) \equiv (\Gamma; \Delta); \Lambda$.

2. $\preceq$ *is an ordering on the set of all large programs containing* $\equiv$ *and such that:*

  (a) $\Gamma \cap \Delta \preceq \Gamma$ *and* $\Gamma \cap \Delta \preceq \Delta$*;*

  (b) If $\Gamma \preceq \Delta$ *and* $\Gamma \preceq \Lambda$ *then* $\Gamma \preceq \Delta \cap \Lambda$*;*

  (c) If $\dim(\Gamma) = 0$ *then* $\Gamma; \Delta \preceq \Delta$*;*

  (d) If $\dim(\Delta) = 0$ *then* $\Gamma; \Delta \preceq \Gamma$*;*

  (e) If S *is maximal,* $\bot \notin S$ *and* $[\Gamma \cap S?]S \subseteq S$ *then* $S? \preceq \Gamma$*;*

  (f) $S? \preceq S?; S?$.

Proof:
This lemma is the counterpart of lemma 4.1.
(1) The proof is by an easy verification.
(2) The proof is by an easy verification.
(2a) The proof is by an easy verification.
(2b) Suppose $\Gamma \preceq \Delta$ and $\Gamma \preceq \Lambda$, we demonstrate $\Gamma \preceq \Delta \cap \Lambda$. Let δ, λ be programs such that $\delta \in \ker(\Delta)$ and $\lambda \in \ker(\Lambda)$. Since $\Gamma \preceq \Delta$ and $\Gamma \preceq \Lambda$, therefore there are programs γ', γ'' such that $\gamma' \in \ker(\Gamma)$, $\gamma' \preceq \delta$, $\gamma'' \in \ker(\Gamma)$ and $\gamma'' \preceq \lambda$. By lemma 6.2, there is a program γ such that $\gamma \in \ker(\Gamma)$, $\gamma \preceq \gamma'$ and $\gamma \preceq \gamma''$. Since $\gamma' \preceq \delta$ and $\gamma'' \preceq \lambda$, then $\gamma \preceq \delta$ and $\gamma \preceq \lambda$. Hence $\gamma \preceq \delta \cap \lambda$. The result follows: $\Gamma \preceq \Delta \cap \Lambda$.
(2c) Suppose $\dim(\Gamma) = 0$, we demonstrate $\Gamma; \Delta \preceq \Delta$. Let δ be a program such that $\delta \in \ker(\Delta)$. Let γ be a program such that $\gamma \in \ker(\Gamma)$. By lemma 6.1 and with the fact that $\dim(\Gamma) = 0$, it follows that $\dim(\gamma) = 0$. Hence $\gamma; \delta \preceq \delta$. The result follows: $\Gamma; \Delta \preceq \Delta$.
(2d) The proof is similar to the proof of (2c).
(2e) Suppose S is maximal, $\bot \not\in S$ and $[\Gamma \cap S?]S \subseteq S$, we demonstrate $S? \preceq \Gamma$. Let γ be a program such that $\gamma \in \ker(\Gamma)$. Since $\bot \not\in S$ and $[\Gamma \cap S?]S \subseteq S$, consequently $\bot \not\in [\Gamma \cap S?]S$. Since $\top \in S$, $\gamma \in \ker(\Gamma)$ and $\bot \not\in [\Gamma \cap S?]S$, then $[\gamma \cap \top?]\bot \not\in S$. Since S is maximal, it follows that $\neg[\gamma \cap \top?]\bot \in S$. Hence $\langle\gamma \cap \top?\rangle\top \in S$. The result follows: $S? \preceq \Gamma$.
(2f) We demonstrate $S? \preceq S?; S?$. Let ϕ, ψ be formulas such that $\phi \in S$ and $\psi \in S$. Since $\phi \to (\psi \to \phi \wedge \psi) \in S$, then $\phi \wedge \psi \in S$. The result follows: $S? \preceq S?; S?$. □

Lemma 6.4. *Let* Γ, Δ *be large programs.*

1. *If* $\Gamma \equiv \Delta$ *then* $\dim(\Gamma) = \dim(\Delta)$*;*
2. *If* $\Gamma \preceq \Delta$ *then* $\dim(\Gamma) \leq \dim(\Delta)$.

Proof:
This lemma corresponds to lemma 4.2.
(1) By item (1) of lemma 4.2 and lemma 6.1.
(2) By item (2) of lemma 4.2 and lemma 6.1. □

Lemma 6.5. *Let* Γ *be a large program and* S *be a theory. Then* $[\Gamma]S$ *is a theory.*

Proof:
This lemma is the matching piece to lemma 5.4.
We demonstrate $[\Gamma]S$ contains $PDL_0^{\cap}$. Suppose $\vdash_{PDL_0^{\cap}} \phi$. Let $\gamma \in \ker(\Gamma)$. Since $\vdash_{PDL_0^{\cap}} \phi$, consequently $\vdash_{PDL_0^{\cap}} [\gamma]\phi$. Hence $[\gamma]\phi \in S$. Since $\gamma \in \ker(\Gamma)$, hence $\phi \in [\Gamma]S$.
We demonstrate $[\Gamma]S$ is closed under the inference rule of modus ponens. Suppose $\phi \to \psi \in [\Gamma]S$ and $\phi \in [\Gamma]S$. It follows that there are programs γ', γ'' such that $\gamma' \in \ker(\Gamma)$, $[\gamma'](\phi \to \psi) \in S$, $\gamma'' \in \ker(\Gamma)$ and $[\gamma'']\phi \in S$. By lemma 6.2, hence there is a program γ such that $\gamma \in \ker(\Gamma)$, $\gamma \preceq \gamma'$ and $\gamma \preceq \gamma''$. It follows that $\vdash_{PDL_0^{\cap}} [\gamma'](\phi \to \psi) \to [\gamma](\phi \to \psi)$ and $\vdash_{PDL_0^{\cap}} [\gamma'']\phi \to [\gamma]\phi$ (using axiom A_4). Hence $[\gamma'](\phi \to \psi) \to [\gamma](\phi \to \psi) \in S$ and $[\gamma'']\phi \to [\gamma]\phi \in S$. Since $[\gamma'](\phi \to \psi) \in S$ and $[\gamma'']\phi \in S$, consequently $[\gamma](\phi \to \psi) \in S$ and $[\gamma]\phi \in S$. Since $[\gamma](\phi \to \psi) \to ([\gamma]\phi \to [\gamma]\psi) \in S$, then $[\gamma]\psi \in S$. Since $\gamma \in \ker(\Gamma)$, thus $\psi \in [\Gamma]S$. □

To end this section, we present some useful results. The reader may notice that the proofs of lemma 6.10, lemma 6.11 and lemma 6.13 are the only places in the paper where axiom A_6 and axiom A_7 are used.

Lemma 6.6. *Let S, T be theories. $S? \preceq T?$ iff $T \subseteq S$.*

Proof:
We only prove the "only if" condition, the "if" condition being left to the reader. Suppose $S? \preceq T?$, we demonstrate $T \subseteq S$. If $T \not\subseteq S$ then there is a formula ϕ such that $\phi \in T$ and $\phi \notin S$. Since $S? \preceq T?$, therefore there is a formula ψ such that $\psi \in S$ and $\psi? \preceq \phi?$. It follows that $\vdash_{PDL_0^\cap} \psi \rightarrow \phi$ (using axiom A_4). Thus $\psi \rightarrow \phi \in S$. Since $\psi \in S$, then $\phi \in S$, a contradiction. □

Lemma 6.7. *Let Γ, Δ be large programs and S, T be theories.*

1. *$S? \preceq \Gamma; T?$ iff $S? \preceq \Gamma$ and $T \subseteq S$;*
2. *$S? \preceq T?; \Delta$ iff $T \subseteq S$ and $S? \preceq \Delta$.*

Proof:
(1) We only prove the "only if" condition, the "if" condition being left to the reader. Suppose $S? \preceq \Gamma; T?$, we demonstrate $S? \preceq \Gamma$ and $T \subseteq S$. Since $\dim(T?) = 0$, thus $\Gamma; T? \preceq \Gamma$. Since $S? \preceq \Gamma; T?$, therefore $S? \preceq \Gamma$. If $T \not\subseteq S$ then there is a formula ϕ such that $\phi \in T$ and $\phi \notin S$. Let γ be a program such that $\gamma \in \ker(\Gamma)$. Since $\phi \in T$, therefore $\gamma; \phi? \in \ker(\Gamma; T?)$. Since $S? \preceq \Gamma; T?$, therefore there is a formula ψ such that $\psi \in S$ and $\psi? \preceq \gamma; \phi?$. It follows that $\vdash_{PDL_0^\cap} \psi \rightarrow \phi$ (using axiom A_1 and axiom A_4). Hence, $\psi \rightarrow \phi \in S$. Since $\psi \in S$, then $\phi \in S$, a contradiction.
(2) We only prove the "only if" condition, the "if" condition being left to the reader. Suppose $S? \preceq T?; \Delta$, we demonstrate $T \subseteq S$ and $S? \preceq \Delta$. If $T \not\subseteq S$ then there is a formula ϕ such that $\phi \in T$ and $\phi \notin S$. Let δ be a program such that $\delta \in \ker(\Delta)$. Since $\phi \in T$, therefore $\phi?; \delta \in \ker(T?; \Delta)$. Since $S? \preceq T?; \Delta$, therefore there is a formula ψ such that $\psi \in S$ and $\psi? \preceq \phi?; \delta$. It follows that $\vdash_{PDL_0^\cap} \psi \rightarrow \phi$ (using axiom A_1 and axiom A_4). Hence, $\psi \rightarrow \phi \in S$. Since $\psi \in S$, then $\phi \in S$, a contradiction. Since $\dim(T?) = 0$, thus $T?; \Delta \preceq \Delta$. Since $S? \preceq T?; \Delta$, therefore $S? \preceq \Delta$. □

Lemma 6.8. *Let Γ, Δ, Λ be large programs and S, T be theories.*

1. *If $[\Gamma; S?]PDL_0^\cap \subseteq S$ then $\Gamma; S? \preceq \Delta; T?$ iff $\Gamma; S? \preceq \Delta$ and $T \subseteq S$;*
2. *If $\bot \notin [\Delta]S$ and S is maximal then $S?; \Delta \preceq T?; \Lambda$ iff $T \subseteq S$ and $S?; \Delta \preceq \Lambda$.*

Proof:
(1) We only prove the "only if" condition, the "if" condition being left to the reader. Suppose $[\Gamma; S?]PDL_0^\cap \subseteq S$ and $\Gamma; S? \preceq \Delta; T?$, we demonstrate $\Gamma; S? \preceq \Delta$ and $T \subseteq S$. Since $\dim(T?) = 0$, hence $\Delta; T? \preceq \Delta$. Since $\Gamma; S? \preceq \Delta; T?$, therefore $\Gamma; S? \preceq \Delta$. If $T \not\subseteq S$ then there is a formula ϕ such that $\phi \in T$ and $\phi \notin S$. Let δ be a program such that $\delta \in \ker(\Delta)$. Since $\phi \in T$, it follows that $\delta; \phi? \in \ker(\Delta; T?)$. Since $\Gamma; S? \preceq \Delta; T?$, therefore there is a program γ and there is a formula ψ such that $\gamma \in \ker(\Gamma)$, $\psi \in S$ and $\gamma; \psi? \preceq \delta; \phi?$. It follows that $\vdash_{PDL_0^\cap} [\gamma; \psi?]\phi$ (using axiom A_1

and axiom A_4). Since $\gamma \in \ker(\Gamma)$ and $\psi \in S$, hence, $\gamma;\psi? \in \ker(\Gamma;S?)$. Since $\vdash_{PDL_0^\cap} [\gamma;\psi?]\phi$, therefore $\phi \in [\Gamma;S?]PDL_0^\cap$. Since $[\Gamma;S?]PDL_0^\cap \subseteq S$, then $\phi \in S$, a contradiction.
(2) We only prove the "only if" condition, the "if" condition being left to the reader. Suppose $\bot \notin [\Delta]S$, S is maximal and $S?;\Delta \preceq T?;\Lambda$, we demonstrate $T \subseteq S$ and $S?;\Delta \preceq \Lambda$. If $T \not\subseteq S$ then there is a formula ϕ such that $\phi \in T$ and $\phi \notin S$. Let λ be a program such that $\lambda \in \ker(\Lambda)$. Since $\phi \in T$, it follows that $\phi?;\lambda \in \ker(T?;\Lambda)$. Since $S?;\Delta \preceq T?;\Lambda$, therefore there is a formula ψ and there is a program δ such that $\psi \in S$, $\delta \in \ker(\Delta)$ and $\psi?;\delta \preceq \phi?;\lambda$. Since $\bot \notin [\Delta]S$, hence $[\delta]\bot \notin S$. Since S is maximal, consequently $\langle\delta\rangle\top \in S$. Since $\psi?;\delta \preceq \phi?;\lambda$, then $\vdash_{PDL_0^\cap} \psi \to (\langle\delta\rangle\top \to \phi)$ (using axiom A_1 and axiom A_4). Thus $\psi \to (\langle\delta\rangle\top \to \phi) \in S$. Since $\psi \in S$ and $\langle\delta\rangle\top \in S$, consequently $\phi \in S$, a contradiction. Since $\dim(T?) = 0$, then $T?;\Lambda \preceq \Lambda$. Since $S?;\Delta \preceq T?;\Lambda$, it follows that $S?;\Delta \preceq \Lambda$. □

Lemma 6.9. *Let* Γ *be a large program and* S, T *be theories.*

1. *If* S *is maximal,* $\bot \notin T$, $\dim(\Gamma) = 0$ *and* $[\Gamma]S \subseteq T$ *then* $S = T$*;*
2. *If* S *is maximal,* $\bot \notin T$, $\dim(\Gamma) = 0$ *and* $[\Gamma]S \subseteq T$ *then* $T? \preceq \Gamma$.

Proof:
(1): Suppose S is maximal, $\bot \notin T$, $\dim(\Gamma) = 0$ and $[\Gamma]S \subseteq T$, we demonstrate $S = T$. If $S \neq T$ then there is a formula ϕ such that $\phi \in S$ and $\phi \notin T$ or $\phi \notin S$ and $\phi \in T$. Let γ be a program such that $\gamma \in \ker(\Gamma)$. By lemma 6.1 and the fact that $\dim(\Gamma) = 0$, it follows that $\dim(\gamma) = 0$. Thus $\vdash_{PDL_0^\cap} \phi \to [\gamma]\phi$ (using axiom A_3) and $\vdash_{PDL_0^\cap} \neg\phi \to [\gamma]\neg\phi$ (using axiom A_3). Consequently $\phi \to [\gamma]\phi \in S$ and $\neg\phi \to [\gamma]\neg\phi \in S$. In the first case, $\phi \in S$ and $\phi \notin T$. Since $\phi \to [\gamma]\phi \in S$, therefore $[\gamma]\phi \in S$. Since $\gamma \in \ker(\Gamma)$, then $\phi \in [\Gamma]S$. Since $[\Gamma]S \subseteq T$, then $\phi \in T$, a contradiction. In the second case, $\phi \notin S$ and $\phi \in T$. Since S is maximal, thus $\neg\phi \in S$. Since $\neg\phi \to [\gamma]\neg\phi \in S$, consequently $[\gamma]\neg\phi \in S$. Since $\gamma \in \ker(\Gamma)$, hence $\neg\phi \in [\Gamma]S$. Since $[\Gamma]S \subseteq T$, then $\neg\phi \in T$. Since $\phi \to (\neg\phi \to \bot) \in T$, then $\bot \in T$, a contradiction.
(2): Suppose S is maximal, $\bot \notin T$, $\dim(\Gamma) = 0$ and $[\Gamma]S \subseteq T$, we demonstrate $T? \preceq \Gamma$. Let γ be a program such that $\gamma \in \ker(\Gamma)$. By lemma 6.1 and the fact that $\dim(\Gamma) = 0$, thus $\dim(\gamma) = 0$. Since $\dim(\top?) = 0$, consequently $\gamma;\top? \preceq \gamma \cap \top?$. It follows that $\vdash_{PDL_0^\cap} \neg[\gamma]\bot \to \langle\gamma \cap \top?\rangle\top$ (using axiom A_1 and axiom A_4). Hence $\neg[\gamma]\bot \to \langle\gamma \cap \top?\rangle\top \in S$. Since $\bot \notin T$ and $[\Gamma]S \subseteq T$, then $\bot \notin [\Gamma]S$. Since $\gamma \in \ker(\Gamma)$, consequently $[\gamma]\bot \notin S$. Since S is maximal, it follows that $\neg[\gamma]\bot \in S$. Since $\neg[\gamma]\bot \to \langle\gamma \cap \top?\rangle\top \in S$, therefore $\langle\gamma \cap \top?\rangle\top \in S$. By item (2), $S = T$. Since $\langle\gamma \cap \top?\rangle\top \in S$, then $\langle\gamma \cap \top?\rangle\top \in T$. The result follows from the fact that $\langle\gamma \cap \top?\rangle\top \preceq \gamma$. □

Lemma 6.10. *Let* S, T *be theories and* Γ, Δ *be large programs. If* $\bot \in [(\Delta;S?;\Gamma) \cap T?]T$ *then* $\bot \in [(\Gamma;T?;\Delta) \cap S?]S$.

Proof:
Suppose $\bot \in [(\Delta;S?;\Gamma) \cap T?]T$, we demonstrate $\bot \in [(\Gamma;T?;\Delta) \cap S?]S$. Since $\bot \in [(\Delta;S?;\Gamma) \cap T?]T$, it follows that there is a program δ, there is a formula ϕ, there is a program γ and there is a formula ψ such that $\delta \in \ker(\Delta)$, $\phi \in S$, $\gamma \in \ker(\Gamma)$, $\psi \in T$ and $[(\delta;\phi?;\gamma) \cap \psi?]\bot \in T$. Therefore $\psi \wedge [(\delta;\phi?;\gamma) \cap \psi?]\bot \in T$. Remark that $\vdash_{PDL_0^\cap} [(\gamma;(\psi \wedge [(\delta;\phi?;\gamma) \cap \psi?]\bot)?;\delta) \cap \phi?]\bot$ (axiom A_7). Hence $[(\gamma;(\psi \wedge [(\delta;\phi?;\gamma) \cap \psi?]\bot)?;\delta) \cap \phi?]\bot \in S$. Since $\gamma \in \ker(\Gamma)$, $\psi \wedge [(\delta;\phi?;\gamma) \cap \psi?]\bot \in T$, $\delta \in \ker(\Delta)$ and $\phi \in S$, consequently $\bot \in [(\Gamma;T?;\Delta) \cap S?]S$. □

Lemma 6.11. *Let* S, T *be theories and* Γ, Δ *be large programs. If* $\bot \notin S$*,* T *is maximal and* $[(\Gamma; T?; \Delta) \cap S?]S \subseteq S$ *then* $[(\Delta; S?; \Gamma) \cap T?]T \subseteq T$*.*

Proof:
Suppose $\bot \notin S$, T is maximal and $[(\Gamma; T?; \Delta) \cap S?]S \subseteq S$, we demonstrate $[(\Delta; S?; \Gamma) \cap T?]T \subseteq T$. If $[(\Delta; S?; \Gamma) \cap T?]T \not\subseteq T$ then there is a formula ϕ such that $\phi \in [(\Delta; S?; \Gamma) \cap T?]T$ and $\phi \notin T$. Hence there is a program δ, there is a formula ψ, there is a program γ and there is a formula χ such that $\delta \in \ker(\Delta)$, $\psi \in S$, $\gamma \in \ker(\Gamma)$, $\chi \in T$ and $[(\delta; \psi?; \gamma) \cap \chi?]\phi \in T$. Suppose that $[(\delta; \psi?; \gamma) \cap \chi?]\bot \notin T$. Since T is maximal, then $\langle(\delta; \psi?; \gamma) \cap \chi?\rangle\top \in T$. Since $\vdash_{PDL_0^{\cap}} [(\delta; \psi?; \gamma) \cap \chi?]\phi \to (\langle(\delta; \psi?; \gamma) \cap \chi?\rangle\top \to \phi)$ (using axiom A_4), then $[(\delta; \psi?; \gamma) \cap \chi?]\phi \to (\langle(\delta; \psi?; \gamma) \cap \chi?\rangle\top \to \phi) \in T$. Since $[(\delta; \psi?; \gamma) \cap \chi?]\phi \in T$ and $\langle(\delta; \psi?; \gamma) \cap \chi?\rangle\top \in T$, then $\phi \in T$, a contradiction. Hence, $[(\delta; \psi?; \gamma) \cap \chi?]\bot \in T$. It follows that $\chi \wedge [(\delta; \psi?; \gamma) \cap \chi?]\bot \in T$. Since $\vdash_{PDL_0^{\cap}} [(\gamma; (\chi \wedge [(\delta; \psi?; \gamma) \cap \chi?]\bot)?; \delta) \cap \psi?]\bot$ (axiom A_7), thus $[(\gamma; (\chi \wedge [(\delta; \psi?; \gamma) \cap \chi?]\bot)?; \delta) \cap \psi?]\bot \in S$. Since $\gamma \in \ker(\Gamma)$, $\chi \wedge [(\delta; \psi?; \gamma) \cap \chi?]\bot \in T$, $\delta \in \ker(\Delta)$ and $\psi \in S$, consequently $\bot \in [(\Gamma; T?; \Delta) \cap S?]S$. Since $[(\Gamma; T?; \Delta) \cap S?]S \subseteq S$, then $\bot \in S$, a contradiction. $\square$

Lemma 6.12. *Let* Γ, Δ *be large programs and* S, T, U *be theories. If* $\bot \notin T$*,* U *is maximal and* $[\Gamma; U?; \Delta]S \subseteq T$ *then* $\bot \notin U$*,* $[\Gamma]S \subseteq U$ *and* $[\Delta]U \subseteq T$*.*

Proof:
Suppose $\bot \notin T$, U is maximal and $[\Gamma; U?; \Delta]S \subseteq T$, we demonstrate $\bot \notin U$, $[\Gamma]S \subseteq U$ and $[\Delta]U \subseteq T$. If $\bot \in U$ then let γ, δ be programs such that $\gamma \in \ker(\Gamma)$ and $\delta \in \ker(\Delta)$. By item (1) of lemma 4.5 and the fact that $\vdash_{PDL_0^{\cap}} \neg\bot$, hence $\vdash_{PDL_0^{\cap}} [\gamma; \bot?; \delta]\bot$. Thus $[\gamma; \bot?; \delta]\bot \in S$. Since $\gamma \in \ker(\Gamma)$, $\delta \in \ker(\Delta)$ and $\bot \in U$, then $\bot \in [\Gamma; U?; \Delta]S$. Since $[\Gamma; U?; \Delta]S \subseteq T$, it follows that $\bot \in T$, a contradiction. If $[\Gamma]S \not\subseteq U$ then there is a formula ϕ such that $\phi \in [\Gamma]S$ and $\phi \notin U$. It follows that there is a program γ such that $\gamma \in \ker(\Gamma)$ and $[\gamma]\phi \in S$. Since U is maximal and $\phi \notin U$, then $\neg\phi \in U$. Let δ be a program such that $\delta \in \ker(\Delta)$. Since $\vdash_{PDL_0^{\cap}} [\gamma]\phi \to [\gamma; \neg\phi?; \delta]\bot$ (using axiom A_1), it follows that $[\gamma]\phi \to [\gamma; \neg\phi?; \delta]\bot \in S$. Since $[\gamma]\phi \in S$, consequently $[\gamma; \neg\phi?; \delta]\bot \in S$. Since $\gamma \in \ker(\Gamma)$, $\delta \in \ker(\Delta)$ and $\neg\phi \in U$, it follows that $\bot \in [\Gamma; U?; \Delta]S$. Since $[\Gamma; U?; \Delta]S \subseteq T$, thus $\bot \in T$, a contradiction. If $[\Delta]U \not\subseteq T$ then there is a formula ϕ such that $\phi \in [\Delta]U$ and $\phi \notin T$. Hence there is a program δ such that $\delta \in \ker(\Delta)$ and $[\delta]\phi \in U$. Let γ be a program such that $\gamma \in \ker(\Gamma)$. Since $\vdash_{PDL_0^{\cap}} [\gamma; [\delta]\phi?; \delta]\phi$ (using axiom A_1), it follows that $[\gamma; [\delta]\phi?; \delta]\phi \in S$. Since $\gamma \in \ker(\Gamma)$, $\delta \in \ker(\Delta)$ and $[\delta]\phi \in U$, hence $\phi \in [\Gamma; U?; \Delta]S$. Since $[\Gamma; U?; \Delta]S \subseteq T$, therefore $\phi \in T$, a contradiction. $\square$

Lemma 6.13. *Let* U *be a theory,* $\Gamma(U?)$ *be a large program with* $U?$ *one of its large tests and* S, T *be theories. If* T *is maximal and* $[\Gamma(U?)]S \subseteq T$ *then for all formulas* ϕ*,* $[\Gamma([\phi?]U?)]S \subseteq T$ *or* $[\Gamma([\neg\phi?]U?)]S \subseteq T$*.*

Proof:
Suppose T is maximal and $[\Gamma(U?)]S \subseteq T$, we demonstrate for all formulas ϕ, $[\Gamma([\phi?]U?)]S \subseteq T$ or $[\Gamma([\neg\phi?]U?)]S \subseteq T$. Let ϕ be a formula such that $[\Gamma([\phi?]U?)]S \not\subseteq T$ and $[\Gamma([\neg\phi?]U?)]S \not\subseteq T$. It follows that there are formulas ψ', ψ'' such that $\psi' \in [\Gamma([\phi?]U?)]S$, $\psi' \notin T$, $\psi'' \in [\Gamma([\neg\phi?]U?)]S$

and $\psi'' \not\in T$. It follows that there are formulas χ', χ'' such that $\chi' \in [\phi?]U$, $[\Gamma(\chi'?)]\psi' \in S$, $\chi'' \in [\neg\phi?]U$ and $[\Gamma(\chi''?)]\psi'' \in S$. Hence $[\phi?]\chi' \in U$ and $[\neg\phi?]\chi'' \in U$. Since $[\phi?]\chi' \rightarrow ([\neg\phi?]\chi'' \rightarrow \chi' \vee \chi'') \in U$, hence $\chi' \vee \chi'' \in U$. Since $\vdash_{PDL_0^\cap} [\Gamma(\chi'?)]\psi' \rightarrow ([\Gamma(\chi''?)]\psi'' \rightarrow [\Gamma((\chi' \vee \chi'')?)](\psi' \vee \psi''))$ (using axiom A_6), thus $[\Gamma(\chi'?)]\psi' \rightarrow ([\Gamma(\chi''?)]\psi'' \rightarrow [\Gamma((\chi' \vee \chi'')?)](\psi' \vee \psi'')) \in S$. Since $[\Gamma(\chi'?)]\psi' \in S$ and $[\Gamma(\chi''?)]\psi'' \in S$, it follows that $[\Gamma((\chi' \vee \chi'')?)](\psi' \vee \psi'') \in S$. Since $\chi' \vee \chi'' \in U$, then $\psi' \vee \psi'' \in [\Gamma(U?)]S$. Since $[\Gamma(U?)]S \subseteq T$, therefore $\psi' \vee \psi'' \in T$. Since T is maximal and $\psi' \not\in T$, then $\neg\psi' \in T$. Since $\neg\psi' \rightarrow (\psi' \vee \psi'' \rightarrow \psi'') \in T$ and $\psi' \vee \psi'' \in T$, hence $\psi'' \in T$, a contradiction. $\square$

Lemma 6.14. *Let U be a theory, $\Gamma(U?)$ be a large program with $U?$ one of its large tests and S, T be theories. If T is maximal and $[\Gamma(U?)]S \subseteq T$ then there is a maximal theory U' such that $[\Gamma(U'?)]S \subseteq T$ and $U \subseteq U'$.*

Proof:
Suppose T is maximal and $[\Gamma(U?)]S \subseteq T$, we demonstrate there is a maximal theory U' such that $[\Gamma(U'?)]S \subseteq T$ and $U \subseteq U'$. Let us suppose that the set of all formulas is arranged in some determinate order $\phi_0, \phi_1, \ldots$. Now, we define an infinite sequence $U'_0, U'_1, \ldots$ of theories such that for all positive integers i, the following conditions are satisfied:

$C_1(i)$ For all positive integers j, if $i > j$ then $\phi_j \in U'_i$ or $\neg\phi_j \in U'_i$;
$C_2(i)$ $[\Gamma(U'_i?)]S \subseteq T$;
$C_3(i)$ $U \subseteq U'_i$.

Let $U'_0 = U$. Note that the conditions $C_1(0)$, $C_2(0)$ and $C_3(0)$ are satisfied. Let i be a positive integer. Given U'_i such that the conditions $C_1(i)$, $C_2(i)$ and $C_3(i)$ are satisfied, we let U'_{i+1} be $[\phi_i?]U'_i$ if $[\Gamma([\phi_i?]U'_i?)]S \subseteq T$. If $[\Gamma([\phi_i?]U'_i?)]S \not\subseteq T$ then, by lemma 6.13 and the fact that T is maximal, $[\Gamma([\neg\phi_i?]U'_i?)]S \subseteq T$ and we let U'_{i+1} be $[\neg\phi_i?]U'_i$. By lemma 5.3, it follows that the conditions $C_1(i+1)$, $C_2(i+1)$ and $C_3(i+1)$ are satisfied. Now we put let $U' = U'_0 \cup U'_1 \cup \ldots$. $\square$

6.2. Two lemmas

Let us now introduce our strong diamond lemma and our strong composition lemma. The lemma below contains a fact which helps to prove the strong diamond lemma.

Lemma 6.15. *Let $\phi_1, \ldots, \phi_K$ be formulas, $\gamma(\phi_1?, \ldots, \phi_K?)$ be a program with $\phi_1?, \ldots, \phi_K?$ the sequence of all its tests and S, T be theories. If $[\gamma(\phi_1?, \ldots, \phi_K?)]S \subseteq T$ then $[\gamma([\phi_1?]PDL_0^\cap?, \ldots, [\phi_K?]PDL_0^\cap?)]S \subseteq T$.*

Proof:
To keep things concrete, we will suppose $K = 1$ and leave the other cases to the reader. Suppose $[\gamma(\phi?)]S \subseteq T$, we demonstrate $[\gamma([\phi?]PDL_0^\cap?)]S \subseteq T$. If $[\gamma([\phi?]PDL_0^\cap?)]S \not\subseteq T$ then there is a formula ψ such that $\psi \in [\gamma([\phi?]PDL_0^\cap?)]S$ and $\psi \not\in T$. It follows that there is a formula χ such that $\chi \in [\phi?]PDL_0^\cap$ and $[\gamma(\chi?)]\psi \in S$. Hence $\vdash_{PDL_0^\cap} [\phi?]\chi$. It follows that $\vdash_{PDL_0^\cap} \phi \rightarrow \chi$. By item (3) of lemma 4.5, therefore $\vdash_{PDL_0^\cap} [\gamma(\chi?)]\psi \rightarrow [\gamma(\phi?)]\psi$. It follows that $[\gamma(\chi?)]\psi \rightarrow [\gamma(\phi?)]\psi \in S$. Since $[\gamma(\chi?)]\psi \in S$, then $[\gamma(\phi?)]\psi \in S$. Thus $\psi \in [\gamma(\phi?)]S$. Since $[\gamma(\phi?)]S \subseteq T$, consequently $\psi \in T$, a contradiction. $\square$

We are now ready to prove the strong diamond lemma.

Lemma 6.16. (Strong diamond lemma) *Let γ be a program, ϕ be a formula and S be a theory. If $[\gamma]\phi \not\in S$ then there is a maximal large program Γ and there is a maximal theory T such that $f(\gamma) \in \ker(\Gamma)$, $\phi \not\in T$ and $[\Gamma]S \subseteq T$.*

Proof:
Suppose $[\gamma]\phi \not\in S$, we demonstrate there is a maximal large program Γ and there is a maximal theory T such that $f(\gamma) \in \ker(\Gamma)$, $\phi \not\in T$ and $[\Gamma]S \subseteq T$. By lemma 5.8, there is a maximal theory T such that $[f(\gamma)]S \subseteq T$ and $\phi \not\in T$. Let us assume for the sake of the argument that $f(\gamma)$ contains exactly one occurrence of a test, say, $\psi?$. By lemma 6.15, it follows that $[f(\gamma)([\psi?]PDL_0^{\cap}?)]S \subseteq T$. By lemma 6.14 and the fact that T is maximal, thus there is a maximal theory U such that $[f(\gamma)(U?)]S \subseteq T$ and $[\psi?]PDL_0^{\cap} \subseteq U$. Now we put $\Gamma = f(\gamma)(U?)$. By item (2) of lemma 5.3, $\psi \in [\psi?]PDL_0^{\cap}$. Since $[\psi?]PDL_0^{\cap} \subseteq U$, then $\psi \in U$. Consequently $f(\gamma) \in \ker(\Gamma)$. $\square$

Now for the strong composition lemma. Once again, we need a preliminary lemma.

Lemma 6.17. *Let Γ, Δ be large programs and S, T be theories. If $[\Gamma; \Delta]S \subseteq T$ then $[\Gamma; PDL_0^{\cap}?; \Delta]S \subseteq T$.*

Proof:
Suppose $[\Gamma; \Delta]S \subseteq T$, we demonstrate $[\Gamma; PDL_0^{\cap}?; \Delta]S \subseteq T$. If $[\Gamma; PDL_0^{\cap}?; \Delta]S \not\subseteq T$ then there is a formula ϕ such that $\phi \in [\Gamma; PDL_0^{\cap}?; \Delta]S$ and $\phi \not\in T$. It follows that there are programs γ, δ and there is a formula ψ such that $\gamma \in \ker(\Gamma)$, $\delta \in \ker(\Delta)$, $\vdash_{PDL_0^{\cap}} \psi$ and $[\gamma; \psi?; \delta]\phi \in S$. Consequently $\vdash_{PDL_0^{\cap}} [\gamma]\psi$. It follows that $[\gamma]\psi \in S$. Since $\vdash_{PDL_0^{\cap}} [\gamma; \psi?; \delta]\phi \rightarrow ([\gamma]\psi \rightarrow [\gamma; \delta]\phi)$ (using axiom A_1), it follows that $[\gamma; \psi?; \delta]\phi \rightarrow ([\gamma]\psi \rightarrow [\gamma; \delta]\phi) \in S$. Since $[\gamma; \psi?; \delta]\phi \in S$ and $[\gamma]\psi \in S$, then $[\gamma; \delta]\phi \in S$. Since $\gamma \in \ker(\Gamma)$ and $\delta \in \ker(\Delta)$, hence $\phi \in [\Gamma; \Delta]S$. Since $[\Gamma; \Delta]S \subseteq T$, then $\phi \in T$, a contradiction. $\square$

With the help of this lemma, it is easy to prove the strong composition lemma.

Lemma 6.18. (Strong composition lemma) *Let Γ, Δ be large programs and S, T be theories. If T is maximal, $\bot \not\in T$ and $[\Gamma; \Delta]S \subseteq T$ then there is a maximal theory U such that $[\Gamma; U?; \Delta]S \subseteq T$, $\bot \not\in U$, $[\Gamma]S \subseteq U$ and $[\Delta]U \subseteq T$.*

Proof:
Suppose T is maximal, $\bot \not\in T$ and $[\Gamma; \Delta]S \subseteq T$, we demonstrate there is a maximal theory U such that $\bot \not\in U$, $[\Gamma]S \subseteq U$ and $[\Delta]U \subseteq T$. By lemma 6.17, it follows that $[\Gamma; PDL_0^{\cap}?; \Delta]S \subseteq T$. By lemma 6.14 and the fact that T is maximal, hence there is a maximal theory U such that $[\Gamma; U?; \Delta]S \subseteq T$ and $PDL_0^{\cap} \subseteq U$. By lemma 6.12 and the fact that $\bot \not\in T$, therefore $\bot \not\in U$, $[\Gamma]S \subseteq U$ and $[\Delta]U \subseteq T$. $\square$

6.3. Large models

Consider a standard Kripke model $\mathcal{M} = (W, R, V)$. By proposition 4.1, we know that $V(x)$ is a theory for each state x. Hence it makes sense to assert that a theory S is true at state x in $\mathcal{M}$ exactly when $S \subseteq V(x)$. What about large programs? In particular, what could it mean for a state y to be reachable from state x by performing large program Γ in $\mathcal{M}$?

Definition 6.5. A large model is a triple $\mathcal{M} = (W, R, V)$ where W is a nonempty set of states, R is a function from the set of all maximal large programs into the set of all binary relations on W and V is a function from W into the set of all maximal theories. State y can be reached from state x by performing maximal large program Γ in large model $\mathcal{M} = (W, R, V)$ iff $xR(\Gamma)y$. Maximal theory S is true at state x in large model $\mathcal{M} = (W, R, V)$ iff $S \subseteq V(x)$. A large model $\mathcal{M} = (W, R, V)$ will be defined to be standard if it satisfies the following conditions:

- $\bot \notin V(x)$;
- $[\gamma]\phi \in V(x)$ iff, for all states y, if there is a maximal large program Γ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\phi \in V(y)$;
- $R(\Gamma; S?; \Delta) = \{(x, y)\colon$ there is a state z such that $xR(\Gamma)z$, $S \subseteq V(z)$ and $zR(\Delta)y\}$;
- $R(\Gamma; S?) = \{(x, y)\colon xR(\Gamma)y$ and $S \subseteq V(y)\}$;
- $R(S?; \Delta) = \{(x, y)\colon S \subseteq V(x)$ and $xR(\Delta)y\}$;
- $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$;
- $R(S?) = \{(x, y)\colon x = y$ and $S \subseteq V(y)\}$;
- If $\Gamma \preceq \Delta$ then $R(\Gamma) \subseteq R(\Delta)$.

It follows immediately from definition 6.5 that

Lemma 6.19. *Let $\mathcal{M} = (W, R, V)$ be a large model. If $\mathcal{M} = (W, R, V)$ is standard then:*

- $\phi \to \psi \in V(x)$ *iff if* $\phi \in V(x)$ *then* $\psi \in V(x)$*;*
- $\neg\phi \in V(x)$ *iff* $\phi \notin V(x)$*;*
- $\top \in V(x)$*;*
- $\phi \vee \psi \in V(x)$ *iff* $\phi \in V(x)$ *or* $\psi \in V(x)$*;*
- $\phi \wedge \psi \in V(x)$ *iff* $\phi \in V(x)$ *and* $\psi \in V(x)$*;*
- $\phi \leftrightarrow \psi \in V(x)$ *iff* $\phi \in V(x)$ *iff* $\psi \in V(x)$*;*
- $\langle\gamma\rangle\phi \in V(x)$ *iff there is a state y and there is a maximal large program Γ such that* $f(\gamma) \in \ker(\Gamma)$*,* $xR(\Gamma)y$ *and* $\phi \in V(y)$.

Now we define what it means for a formula to be satisfied and to be valid in a large model.

Definition 6.6. Formula ϕ is satisfied in large model $\mathcal{M} = (W, R, V)$ iff there is a state x such that $\phi \in V(x)$ whereas ϕ is valid in $\mathcal{M}$ iff for all states x, $\phi \in V(x)$.

Next lemma explains the link between standard Kripke models and standard large models.

Lemma 6.20. *Let $\mathcal{M} = (W, R, V)$ be a standard large model. If R' is the function from the set of all programs into the set of all binary relations on W defined as follows:*

- *If* $x \in W$ *and* $y \in W$ *then* $xR'(\gamma)y$ *iff there is a maximal large program* Γ *such that* $f(\gamma) \in \ker(\Gamma)$ *and* $xR(\Gamma)y$*;*

then $\mathcal{M}' = (W, R', V)$ *is a standard Kripke model.*

Proof:
The proof is by an easy verification. □

An ingenious related result is the following.

Proposition 6.1. *If* ϕ *is valid in all standard Kripke models then* ϕ *is valid in all standard large models.*

Proof:
Virtually immediate by lemma 6.20. □

What we have in mind is to demonstrate that if formula ϕ is valid in all standard large models then ϕ is provable in $PDL_0^{\cap}$. In this respect the intermediate concept of subordination model will be needed.

7. Subordination models

In a certain sense, the concept of subordination model is a weak form of the concept of large model.

Definition 7.1. A subordination model is a triple $\mathcal{M} = (W, R, V)$ where W is a nonempty set of states, R is a function from the set of all maximal large programs into the set of all binary relations on W and V is a function from W into the set of all maximal theories such that:

- $\bot \not\in V(x)$;
- $[\gamma]\phi \in V(x)$ only if, for all states y, if there is a maximal large program Γ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\phi \in V(y)$;
- $R(\Gamma; S?; \Delta) \supseteq \{(x, y)\colon$ there is a state z such that $xR(\Gamma)z$, $S \subseteq V(z)$ and $zR(\Delta)y\}$;
- $R(\Gamma; S?) = \{(x, y)\colon\ xR(\Gamma)y$ and $S \subseteq V(y)\}$;
- $R(S?; \Delta) = \{(x, y)\colon\ S \subseteq V(x)$ and $xR(\Delta)y\}$;
- $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$;
- $R(S?) = \{(x, y)\colon\ x = y$ and $S \subseteq V(y)\}$;
- If $\Gamma \preceq \Delta$ then $R(\Gamma) \subseteq R(\Delta)$.

Formula ϕ is true at state x in subordination model $\mathcal{M}$ iff $\phi \in V(x)$.

Now we define what it means for a formula to be satisfied and to be valid in a subordination model.

Definition 7.2. Formula ϕ is satisfied in subordination model $\mathcal{M} = (W, R, V)$ iff there is a state x such that $\phi \in V(x)$ whereas ϕ is valid in $\mathcal{M}$ iff for all states x, $\phi \in V(x)$.

Why are subordination models so interesting? The following lemma contains a fact which helps to prove the starting point of the enterprise: $PDL_0^{\cap}$ is complete with respect to the class of all subordination models.

Lemma 7.1. *Let* ϕ *be a formula such that* $\not\vdash_{PDL_0^{\cap}} \phi$. *By lemma 5.7, there is a maximal theory* S *such that* $PDL_0^{\cap} \subseteq S$ *and* $\phi \notin S$. *If* $W = \{0\}$, R *is the function from the set of all maximal large programs into the set of all binary relations on* W *defined as follows:*

- *If* $x = 0$ *and* $y = 0$ *then* $xR(\Gamma)y$ *iff* $S? \preceq \Gamma$;

and V *is the function from* W *into the set of all maximal theories defined as follows:*

- *If* $x = 0$ *then* $V(x) = S$;

then $\mathcal{M} = (W, R, V)$ *is a subordination model.*

Proof:
The proof that $\bot \notin V(x)$ is by lemma 5.1.
The proof that $[\gamma]\psi \in V(x)$ only if, for all states y, if there is a maximal large program Γ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\psi \in V(y)$ is by an easy verification.
The proof that $R(\Gamma; T?; \Delta) \supseteq \{(x, y)$: there is a state z such that $xR(\Gamma)z$, $T \subseteq V(z)$ and $zR(\Delta)y\}$ is by an easy verification.
The proof that $R(\Gamma; T?) = \{(x, y)$: $xR(\Gamma)y$ and $T \subseteq V(y)\}$ is by item 1 of lemma 6.7.
The proof that $R(T?; \Delta) = \{(x, y)$: $T \subseteq V(x)$ and $xR(\Delta)y\}$ is by item 2 of lemma 6.7.
The proof that $R(\Gamma \cap \Delta) = R(\Gamma) \cap R(\Delta)$ is by an easy verification.
The proof that $R(T?) = \{(x, y)$: $x = y$ and $T \subseteq V(y)\}$ is by lemma 6.6.
The proof that if $\Gamma \preceq \Delta$ then $R(\Gamma) \subseteq R(\Delta)$ is by an easy verification. □

With lemma 7.1 at our disposal, we can immediately prove the desired completeness result.

Proposition 7.1. *If* ϕ *is valid in all subordination models then* $\vdash_{PDL_0^{\cap}} \phi$.

It follows from proposition 6.1 and proposition 7.1 that we have reduced the task of proving the completeness of $PDL_0^{\cap}$ with respect to the class of all standard Kripke models to the task of showing how to transform any subordination model into a standard large model satisfying the same formulas. One remark is in order here. Given a subordination model $\mathcal{M} = (W, R, V)$, it may contain imperfections:

- Diamond imperfections, i.e. triples of the form (γ, ϕ, x) where γ is a program, ϕ is a formula and x is a state such that $[\gamma]\phi \notin V(x)$ and, for all states y, if there is a maximal large program Γ such that $f(\gamma) \in \ker(\Gamma)$ and $xR(\Gamma)y$ then $\phi \in V(y)$;
- Composition imperfections, i.e. 5-tuples of the form $(\Gamma, S, \Delta, x, y)$ where Γ, Δ are maximal large programs, S is a maximal theory and x, y are states such that $xR(\Gamma; S?; \Delta)y$ and, for all states z, $x\overline{R(\Gamma)}z$ or $S \not\subseteq V(z)$ or $z\overline{R(\Delta)}y$.

Lemma 7.2 and lemma 7.3 state that every imperfection can be repaired.

Lemma 7.2. *Let $\mathcal{M} = (W, R, V)$ be a subordination model and (γ, ϕ, x) be a diamond imperfection of $\mathcal{M}$. Since $[\gamma]\phi \notin V(x)$, then, by lemma 6.16, there is a maximal large program Γ and there is a maximal theory S such that $f(\gamma) \in \ker(\Gamma)$, $\phi \notin S$ and $[\Gamma]V(x) \subseteq S$. If $W' = W \cup \{y\}$ where y is a new state, R' is the function from the set of all maximal large programs into the set of all binary relations on W' defined as follows:*

- *If $z \in W$ and $t \in W$ then $zR'(\Delta)t$ iff $zR(\Delta)t$;*
- *If $z \in W$ and $t = y$ then $zR'(\Delta)t$ iff there is a maximal large program Δ' such that $zR(\Delta')x$ and $V(z)?; \Delta'; V(x)?; \Gamma; S? \preceq \Delta$;*
- *If $z = y$ and $t = y$ then $zR'(\Delta)t$ iff $S? \preceq \Delta$;*

and V' is the function from W' into the set of all maximal theories defined as follows:

- *If $z \in W$ then $V'(z) = V(z)$;*
- *If $z = y$ then $V'(z) = S$;*

then $\mathcal{M}' = (W', R', V')$ is a subordination model called local completion of $\mathcal{M}$ with respect to the diamond imperfection (γ, ϕ, x).

Proof:
The proof that $\bot \notin V'(z)$ is by lemma 5.1.
The proof that $[\delta]\psi \in V'(z)$ only if, for all states t, if there is a maximal large program Δ such that $f(\delta) \in \ker(\Delta)$ and $zR'(\Delta)t$ then $\psi \in V'(t)$ is by an easy verification.
The proof that $R'(\Delta; T?; \Lambda) \supseteq \{(z, t)$: there is a state u such that $zR'(\Delta)u$, $T \subseteq V'(u)$ and $uR'(\Lambda)t\}$ is by an easy verification.
The proof that $R'(\Delta; T?) = \{(z, t)$: $zR'(\Delta)t$ and $T \subseteq V'(t)\}$ is by item 1 of lemma 6.7 and item 1 of lemma 6.8.
The proof that $R'(T?; \Lambda) = \{(z, t)$: $T \subseteq V'(z)$ and $zR'(\Lambda)t\}$ is by item 2 of lemma 6.7 and item 2 of lemma 6.8.
The proof that $R'(\Delta \cap \Lambda) = R'(\Delta) \cap R'(\Lambda)$ is by an easy verification.
We demonstrate $R'(T?) = \{(z, t)$: $z = t$ and $T \subseteq V'(t)\}$. If $R'(T?) \neq \{(z, t)$: $z = t$ and $T \subseteq V'(t)\}$ then, obviously, $R'(T?) \not\subseteq \{(z, t)$: $z = t$ and $T \subseteq V'(t)\}$. Consequently, there is a state z and there is a state t such that $zR'(T?)t$ and $z \neq t$ or $T \not\subseteq V'(t)$. Since $zR'(T?)t$, hence $[T?]V'(z) \subseteq V'(t)$. By item (2) of lemma 6.9 and the fact that $V'(z)$ is maximal, $\bot \notin V'(t)$ and $\dim(T?) = 0$, it follows that $V'(t)? \preceq T?$. By lemma 6.6, then $T \subseteq V'(t)$. Since $z \neq t$ or $T \not\subseteq V'(t)$, thus $z \neq t$. Since $zR'(T?)t$, then $z \in W$, $t \in W$ and $zR(T?)t$ or $z \in W$, $t = y$ and there is a maximal large program Δ' such that $zR(\Delta')x$ and $V(z)?; \Delta'; V(x)?; \Gamma; S? \preceq T?$. In the first case, $z \in W$, $t \in W$ and $zR(T?)t$. Hence, $z = t$, a contradiction. In the second case, $z \in W$, $t = y$ and there is a maximal large program Δ' such that $zR(\Delta')x$ and $V(z)?; \Delta'; V(x)?; \Gamma; S? \preceq T?$. By item (2) of lemma 6.4, hence $\dim(\Gamma) = 0$. By lemma 6.1 and the fact that $\gamma \in \ker(\Gamma)$, then $\dim(\gamma) = 0$. If $V(x) \neq S$ then there is a formula ψ such that $\psi \in V(x)$ and $\psi \notin S$ or $\psi \notin V(x)$ and $\psi \in S$. Since $\dim(\gamma) = 0$, thus, $\vdash_{PDL_0^\cap} \psi \rightarrow [\gamma]\psi$ and $\vdash_{PDL_0^\cap} \neg\psi \rightarrow [\gamma]\neg\psi$ (using axiom A_3). It follows that $\psi \rightarrow [\gamma]\psi \in V(x)$ and $\neg\psi \rightarrow [\gamma]\neg\psi \in V(x)$. In the first case, $\psi \in V(x)$ and $\psi \notin S$. Since $\psi \rightarrow [\gamma]\psi \in V(x)$, hence, $[\gamma]\psi \in V(x)$. Thus, with the

fact that $\gamma \in \ker(\Gamma)$ and $[\Gamma]V(x) \subseteq S$, $\psi \in S$, a contradiction. In the second case, $\psi \not\in V(x)$ and $\psi \in S$. Since $V(x)$ is maximal, then $\neg\psi \in V(x)$. Since $\neg\psi \rightarrow [\gamma]\neg\psi \in V(x)$, then $[\gamma]\neg\psi \in V(x)$. Hence, with the fact that $\gamma \in \ker(\Gamma)$ and $[\Gamma]V(x) \subseteq S$, $\neg\psi \in S$. Since $\psi \rightarrow (\neg\psi \rightarrow \phi) \in S$ and $\psi \in S$, thus $\phi \in S$, a contradiction. It follows that $V(x) = S$. If $[\Gamma \cap V(x)?]V(x) \not\subseteq V(x)$ then there is a formula ψ such that $\psi \in [\Gamma \cap V(x)?]V(x)$ and $\psi \not\in V(x)$. Hence, there is a program γ' and there is a formula ψ' such that $\gamma' \in \ker(\Gamma)$, $\psi' \in V(x)$ and $[\gamma' \cap \psi'?]\psi \in V(x)$. By lemma 6.1 and the fact that $\dim(\Gamma) = 0$, thus, $\dim(\gamma') = 0$. Therefore, $\gamma';\psi'? \preceq \gamma' \cap \psi'?$ and $\vdash_{PDL_0^{\cap}} [\gamma' \cap \psi'?]\psi \rightarrow [\gamma'](\psi' \rightarrow \psi)$ (using axiom A_1 and axiom A_4). It follows that $[\gamma' \cap \psi'?]\psi \rightarrow [\gamma'](\psi' \rightarrow \psi) \in V(x)$. Since $[\gamma' \cap \psi'?]\psi \in V(x)$, hence, $[\gamma'](\psi' \rightarrow \psi) \in V(x)$. Since $\gamma' \in \ker(\Gamma)$, thus, $\psi' \rightarrow \psi \in [\Gamma]V(x)$. Since $[\Gamma]V(x) \subseteq S$ and $V(x) = S$, therefore, $\psi' \rightarrow \psi \in V(x)$. Since $\psi' \in V(x)$, consequently, $\psi \in V(x)$, a contradiction. It follows that $[\Gamma \cap V(x)?]V(x) \subseteq V(x)$. By item (2e) of lemma 6.3 and with the fact that $V(x)$ is maximal and $\bot \not\in V(x)$, hence, $V(x)? \preceq \Gamma$. Thus, $xR(\Gamma)x$. Since $V(x) = S$, therefore, $\phi \in S$, a contradiction.
The proof that if $\Delta \preceq \Lambda$ then $R'(\Delta) \subseteq R'(\Lambda)$ is by an easy verification. □

Lemma 7.3. *Let $\mathcal{M} = (W, R, V)$ be a subordination model and $(\Gamma, S, \Delta, x, y)$ be a composition imperfection of $\mathcal{M}$. Since $xR(\Gamma;S?;\Delta)y$, then $[\Gamma;S?;\Delta]V(x) \subseteq V(y)$ and, by lemma 6.12, $\bot \not\in S$, $[\Gamma]V(x) \subseteq S$ and $[\Delta]S \subseteq V(y)$. If $W' = W \cup \{z\}$ where z is a new state, R' is the function from the set of all maximal large programs into the set of all binary relations on W' defined as follows:*

- *If $t \in W$ and $u \in W$ then $tR'(\Lambda)u$ iff $tR(\Lambda)u$;*
- *If $t \in W$ and $u = z$ then $tR'(\Lambda)u$ iff there is a maximal large program Λ' such that $tR(\Lambda')x$ and $V(t)?;\Lambda';V(x)?;\Gamma;S? \preceq \Lambda$;*
- *If $t = z$ and $u = z$ then $tR'(\Lambda)u$ iff $S? \preceq \Lambda$;*
- *If $t = z$ and $u \in W$ then $tR'(\Lambda)u$ iff there is a maximal large program Λ'' such that $yR(\Lambda'')u$ and $S?;\Delta;V(y)?;\Lambda'';V(u)? \preceq \Lambda$;*

and V' is the function from W' into the set of all maximal theories defined as follows:

- *If $t \in W$ then $V'(t) = V(t)$;*
- *If $t = z$ then $V'(t) = S$;*

then $\mathcal{M}' = (W', R', V')$ is a subordination model called local completion of $\mathcal{M}$ with respect to the composition imperfection $(\Gamma, S, \Delta, x, y)$.

Proof:
The proof that $\bot \not\in V'(t)$ is by lemma 5.1.
The proof that $[\lambda]\phi \in V'(t)$ only if, for all states u, if there is a maximal large program Λ such that $f(\lambda) \in \ker(\Lambda)$ and $tR'(\Lambda)u$ then $\phi \in V'(u)$ is by an easy verification.
We demonstrate $R'(\Lambda;T?;\Xi) \supseteq \{(t,u)$: there is a state v such that $tR'(\Lambda)v$, $T \subseteq V'(v)$ and $vR'(\Xi)u\}$. Suppose $tR'(\Lambda)v$, $T \subseteq V'(v)$ and $vR'(\Xi)u$, we demonstrate $tR'(\Lambda;T?;\Xi)u$. There are eight cases.

Case 1: $t \in W$, $v \in W$ and $u \in W$. Consequently $tR(\Lambda)v$, $T \subseteq V(v)$ and $vR(\Xi)u$. Hence $tR(\Lambda;T!;\Xi)u$. As a result $tR'(\Lambda;T!;\Xi)u$.
Case 2: $t \in W$, $v \in W$ and $u = z$. Consequently $tR(\Lambda)v$, $T \subseteq V(v)$ and there is a maximal large program Ξ' such that $vR(\Xi')x$ and $V(v)?;\Xi';V(x)?;\Gamma;S? \preceq \Xi$. Thus $tR(\Lambda;T?;\Xi')x$ and $V(t)?;\Lambda;T?;\Xi';V(x)?;\Gamma;S? \preceq \Lambda;T?;\Xi$. It follows that $tR'(\Lambda;T!;\Xi)u$.
Case 3: $t \in W$, $v = z$ and $u = z$. Consequently there is a maximal large program Λ' such that $tR(\Lambda')x$ and $V(t)?;\Lambda';V(x)?;\Gamma;S! \preceq \Lambda$, $T \subseteq S$ and $S? \preceq \Xi$. It follows that $V(t)?;\Lambda';V(x)?;\Gamma;S? \preceq \Lambda;T?;\Xi$. As a result $tR'(\Lambda;T?;\Xi)u$.
Case 4: $t \in W$, $v = z$ and $u \in W$. Hence there is a maximal large program Λ' such that $tR(\Lambda')x$ and $V(t)?;\Lambda';V(x)?;\Gamma;S? \preceq \Lambda$, $T \subseteq S$ and there is a maximal large program Ξ'' such that $yR(\Xi'')u$ and $S?;\Delta;V(y)?;\Xi'';V(v)? \preceq \Xi$. Thus $tR(V(t)?;\Lambda';V(x)?;\Gamma;S?;\Delta;V(y)?;\Xi'';V(v)?)u$. Moreover $V(t)?;\Lambda';V(x)?;\Gamma;S?;\Delta;V(y)?;\Xi'';V(v)? \preceq \Lambda;T?;\Xi$. Consequently $tR(\Lambda;T?;\Xi)u$. As a result $tR'(\Lambda;T!;\Xi)u$.
Case 5: $t = z$, $v = z$ and $u = z$. Then $S? \preceq \Lambda$, $T \subseteq S$ and $S? \preceq \Xi$. Consequently $S? \preceq \Lambda;T?;\Xi$. As a result $tR'(\Lambda;T?;\Xi)u$.
Case 6: $t = z$, $v = z$ and $u \in W$. This case is similar to case 3.
Case 7: $t = z$, $v \in W$ and $u \in W$. This case is similar to case 2.
Case 8: $t = z$, $v \in W$ and $u = z$. Therefore there is a maximal large program Λ'' such that $yR(\Lambda'')v$ and $S?;\Delta;V(y)?;\Lambda'';V(v)? \preceq \Lambda$, $T \subseteq V(v)$ and there is a maximal large program Ξ' such that $vR(\Xi')x$ and $V(v)?;\Xi';V(x)?;\Gamma;S? \preceq \Xi$. It follows that $xR((\Gamma;S?;\Delta;V(y)?;\Lambda'';T?;\Xi')\cap V(x)?)x$ and $[(\Gamma;S?;\Delta;V(y)?;\Lambda'';T?;\Xi')\cap V(x)?]V(x) \subseteq V(x)$. By lemma 6.11, $[(\Delta;V(y)?;\Lambda'';T?;\Xi';V(x)?;\Gamma)\cap S?]S \subseteq S$. By item (2) of lemma 6.9, $S? \preceq \Delta;V(y)?;\Lambda'';T?;\Xi';V(x)?;\Gamma$. Since $S?;\Delta;V(y)?;\Lambda'';V(v)? \preceq \Lambda$, $T \subseteq V(v)$ and $V(v)?;\Xi';V(x)?;\Gamma;S? \preceq \Xi$, thus $S? \preceq \Lambda;T?;\Xi$. As a result $tR'(\Lambda;T?;\Xi)u$.
The proof that $R'(\Lambda;T?) = \{(t,u)\colon\ tR'(\Lambda)u \text{ and } T \subseteq V'(u)\}$ is by item 1 of lemma 6.7 and item 1 of lemma 6.8.
The proof that $R'(T?;\Xi) = \{(t,u)\colon\ T \subseteq V'(t) \text{ and } tR'(\Xi)u\}$ is by item 2 of lemma 6.7 and item 2 of lemma 6.8.
The proof that $R'(\Lambda \cap \Xi) = R'(\Lambda) \cap R'(\Xi)$ is by an easy verification.
The proof that $R'(T?) = \{(t,u)\colon\ t = u \text{ and } T \subseteq V'(u)\}$ is similar to the proof given within the context of lemma 7.2.
The proof that if $\Lambda \preceq \Xi$ then $R'(\Lambda) \subseteq R'(\Xi)$ is by an easy verification. □

8. Completeness

We already know what perfect, i.e. imperfection-free, subordination models are: standard large models. The following proposition constitutes the heart of our method. Its proof shows how to transform any subordination model into a standard large model satisfying the same formulas.

Proposition 8.1. *If ϕ is valid in all standard large models then ϕ is valid in all subordination models.*

Proof:
In the following construction we consider all the imperfections in a subordination model that we can repair with the machinery available to us at present: local completions. Let $\mathcal{M} = (W, R, V)$ be a subordination model and I be the set of all its imperfections. Let us be clear that I may be uncountable. By the enumeration principle, however, there is an ordinal α such that I is enumerated by an α-termed sequence i_0, i_1, $\ldots$ of imperfections. We use α', α'', α''', etc, for ordinals in α. Let $\mathcal{M}_0 = (W_0, R_0, V_0)$, $\mathcal{M}_1 = (W_1, R_1, V_1)$, $\ldots$ be the α-termed sequence of subordination models defined as follows. Let α' be an ordinal in α such that for all ordinals α'' in α', $\mathcal{M}_{\alpha''}$ is defined. Let $\mathcal{M}_{\alpha'}$ be the local completion of $\mathcal{M}_{\alpha''}$ with respect to the imperfection $i_{\alpha''}$ if α' is the successor ordinal of some ordinal α''. If α' is a limit ordinal then either $\alpha' = 0$ in which case we let $\mathcal{M}_{\alpha'} = \mathcal{M}$ or α' is an infinite ordinal in which case we let $\mathcal{M}_{\alpha'}$ be the structure defined as follows:

- $W_{\alpha'}$ is $\bigcup\{W_{\alpha''}$: α'' is an ordinal in $\alpha'\}$;
- $R_{\alpha'}$ is the function from the set of all maximal large programs into the set of all binary relations on $W_{\alpha'}$ defined as follows:
  - If $x \in W_{\alpha''}$ for some ordinal α'' in α' and $y \in W_{\alpha'''}$ for some ordinal α''' in α' then $xR_{\alpha'}(\Gamma)y$ iff $xR_{\alpha''\cup\alpha'''}(\Gamma)y$;
- $V_{\alpha'}$ is the function from $W_{\alpha'}$ into the set of all maximal theories defined as follows:
  - If $x \in W_{\alpha''}$ for some ordinal α'' in α' then $V_{\alpha'}(x) = V_{\alpha''}(x)$.

The reader may easily verify that $\mathcal{M}_{\alpha'}$ is a subordination model. Let $\mathcal{M}' = (W', R', V')$ be the structure defined as follows:

- W' is $\bigcup\{W_{\alpha'}$: α' is an ordinal in $\alpha\}$;
- R' is the function from the set of all maximal large programs into the set of all subsets of $W' \times W'$ defined as follows:
  - If $x \in W_{\alpha'}$ for some ordinal α' in α and $y \in W_{\alpha''}$ for some ordinal α'' in α then $xR'(\Gamma)y$ iff $xR_{\alpha'\cup\alpha''}(\Gamma)y$;
- V' is the function from W' into the set of all maximal theories defined as follows:
  - If $x \in W_{\alpha'}$ for some ordinal α' in α then $V'(x) = V_{\alpha'}(x)$.

The reader may easily verify that $\mathcal{M}'$ is a subordination model called global completion of $\mathcal{M}$. Of course for all imperfections of diamond (γ, ϕ, x) in I, if $[\gamma]\phi \notin V'(x)$ then there is a state y and there is a maximal large program Γ such that $f(\gamma) \in \ker(\Gamma)$, $xR'(\Gamma)y$ and $\phi \notin V'(y)$ and for all imperfections of composition $(\Gamma, S, \Delta, x, y)$ in I, if $xR'(\Gamma; S?; \Delta)y$ then there is a state z such that $xR'(\Gamma)z$, $S \subseteq V'(z)$ and $zR'(\Delta)y$. In other words all the imperfections in $\mathcal{M}$ are repaired in $\mathcal{M}'$. Now let $\mathcal{M}^\star_0 = (W^\star_0, R^\star_0, V^\star_0)$, $\mathcal{M}^\star_1 = (W^\star_1, R^\star_1, V^\star_1)$, $\ldots$ be the infinite sequence of subordination models defined as follows. Let $\mathcal{M}^\star_0 = \mathcal{M}$. Let n be a positive integer. Given $\mathcal{M}^\star_n$, let $\mathcal{M}^\star_{n+1}$ be the global completion of $\mathcal{M}^\star_n$. Now we put $\mathcal{M}^\star = (W^\star, R^\star, V^\star)$ the structure defined as follows:

- $W^\star$ is $\bigcup\{W_n^\star\colon n$ is a positive integer$\}$;
- $R^\star$ is the function from the set of all maximal large programs into the set of all binary relations on $W^\star$ defined as follows:
  - If $x \in W_m^\star$ and $y \in W_n^\star$ then $xR^\star(\Gamma)y$ iff $xR^\star_{\max\{m,n\}}(\Gamma)y$;
- $V^\star$ is the function from $W^\star$ into the set of all maximal theories defined as follows:
  - If $x \in W_m^\star$ then $V^\star(x) = V_m^\star(x)$.

The reader may easily verify that $\mathcal{M}^\star$ is a standard large model satisfying the same formulas as $\mathcal{M}$. □

The result that emerges from the discussion above is the following theorem.

Theorem 8.1. *The following conditions are equivalent:*

1. $\vdash_{PDL_0^\cap} \phi$;
2. ϕ *is valid in all standard Kripke models;*
3. ϕ *is valid in all standard large models;*
4. ϕ *is valid in all subordination models.*

Proof:
(1)→(2) By proposition 4.1.
(2)→(3) By proposition 6.1.
(3)→(4) By proposition 8.1.
(4)→(1) By proposition 7.1. □

Now let us consider the related problem of how to eliminate the use of $\preceq$ in our axiomatization. Let L be the smallest normal logic that contains the axioms A_1–A_3, the axiom:

A_4' $\langle\gamma \cap \delta\rangle\phi \rightarrow \langle\gamma\rangle\phi \wedge \langle\delta\rangle\phi$;

the axioms A_5–A_7 and that is closed under the following inference rule:

RI "Given $\langle\gamma\rangle p \rightarrow \langle\delta\rangle p \wedge \langle\lambda\rangle p$ for each propositional variable p, prove $\langle\gamma\rangle\phi \rightarrow \langle\delta \cap \lambda\rangle\phi$".

A formula ϕ is called L-provable, in symbols $\vdash_L \phi$, if ϕ can be derived from the axioms of L by applying the inference rules of L: modus ponens, generalization and RI. In order to show that the concept of $PDL_0^\cap$-provability and the concept of L-provability are equal, we have to demonstrate the following lemmas.

Lemma 8.1. *Let* γ, δ *be programs. If* $\gamma \preceq \delta$ *then* $\vdash_L \langle\gamma\rangle\phi \rightarrow \langle\delta\rangle\phi$.

Proof:
The proof runs by induction on the least positive integer k such that $\gamma \preceq_k \delta$. □

Lemma 8.2. *Let* γ, δ, λ *be programs. If* $\langle\gamma\rangle p \rightarrow \langle\delta\rangle p \wedge \langle\lambda\rangle p$ *is valid in all standard Kripke models for each propositional variable* p*, then* $\langle\gamma\rangle\phi \rightarrow \langle\delta \cap \lambda\rangle\phi$ *is valid in all standard Kripke models.*

Proof:
Suppose $\langle\gamma\rangle p \rightarrow \langle\delta\rangle p \wedge \langle\lambda\rangle p$ is valid in all standard Kripke models for each propositional variable p, we demonstrate $\langle\gamma\rangle\phi \rightarrow \langle\delta\cap\lambda\rangle\phi$ is valid in all standard Kripke models. If $\langle\gamma\rangle\phi \rightarrow \langle\delta\cap\lambda\rangle\phi$ is not valid in all standard Kripke models then there is a standard Kripke model $\mathcal{M} = (W, R, V)$ and there is a state x in W such that $\langle\gamma\rangle\phi \rightarrow \langle\delta\cap\lambda\rangle\phi \notin V(x)$. Consequently, $\underline{\langle\gamma\rangle\phi \in V(x)}$ and $\langle\delta\cap\lambda\rangle\phi \notin V(x)$. Hence there is a state y such that $xR(\gamma)y$, $\phi \in V(y)$ and $x\overline{R(\delta)}y$ or $x\overline{R(\lambda)}y$. Let p be a propositional variable not occurring in γ, δ, λ or ϕ, and V' be a function from W into the set of all sets of formulas such that for all states z and for all propositional variables q:

- If q occurs in γ, δ, λ or ϕ then $q \in V'(z)$ iff $q \in V(z)$;
- If q does not occur in γ, δ, λ or ϕ then $q \in V'(z)$ iff $z = y$.

The reader may easily verify that V' can be chosen in such a way that the Kripke model (W, R, V') is standard. It follows that $\langle\gamma\rangle p \in V'(x)$ and $\langle\delta\rangle p \notin V'(x)$ or $\langle\lambda\rangle p \notin V'(x)$. Consequently $\langle\gamma\rangle p \rightarrow \langle\delta\rangle p \wedge \langle\lambda\rangle p \notin V'(x)$, a contradiction. □

With this observed, the desired result is within reach.

Theorem 8.2. *The following conditions are equivalent:*

1. $\vdash_{PDL_0^{\cap}} \phi$;
2. $\vdash_{L} \phi$.

Proof:
(1)→(2) It suffices to prove that the axiom A_4 is L-provable, an immediate consequence of lemma 8.1.
(2)→(1) It suffices to prove that the inference rule RI preserves validity in all standard Kripke models, an immediate consequence of lemma 8.2. □

To conclude this section, three remarks are in order here. First, notice that L constitutes a solution to the problem concerned with the elimination of the use of $\preceq$ in the axiomatization of $PDL_0^{\cap}$ given in definition 4.2. Second, observe that L employs the unorthodox inference rule RI which is a simplification of the special derivation rule for intersection considered in [3, 4]. Third, the question of how to eliminate the use of $\preceq$ in our axiomatization without reintroducing unorthodox rules for the undefinable remains intact.

9. Conclusion

In this paper we have proved the completeness of $PDL_0^{\cap}$. Seeing that the technique of the canonical model cannot work for proving the completeness of $PDL_0^{\cap}$ because intersection of binary relations is not definable in the language of our modal logic, we have used a step-by-step construction based on the concepts of theory and large program. Our axiomatization is based on a recursively enumerable set of axioms, the question whether an equivalent finite set of axioms exists or not remaining unsettled. There is also the related question of how to eliminate the use of $\preceq$ in our axiomatization without reintroducing rules for the undefinable like the special derivation rule for intersection considered in [3, 4]. We have seen in section 8 that there is no easy answer to the issue at stake here. Finally it remains to see whether our approach can be extended to the full language of $PDL^{\cap}$.

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References

[1] Balbiani, P.: A new proof of completeness for a relative modal logic with composition and intersection. Journal of Applied Non-Classical Logics **11** (2001) 269–280.

[2] Balbiani, P., Fariñas del Cerro, L.: Complete axiomatization of a relative modal logic with composition and intersection. Journal of Applied Non-Classical Logics **8** (1998) 325–335.

[3] Balbiani, P., Vakarelov, D.: Iteration-free PDL with intersection: a complete axiomatization. Fundamenta Informaticæ**45** (2001) 173–194.

[4] Balbiani, P., Vakarelov, D.: PDL with intersection of programs: a complete axiomatization. Journal of Applied Non-Classical Logics, to appear.

[5] Benanav, D., Kapur, D., Narendran, P.: Complexity of matching problems. In Jouannaud, J.-P. (Editor): Rewriting Techniques and Applications. Springer-Verlag, Lecture Notes in Computer Science **202** (1985) 417–429.

[6] Blackburn, P., de Rijke, M., Venema, Y. Modal Logic. Cambridge University Press, Cambridge Tracts in Theoretical Computer Science **53** (2001).

[7] Chagrov, A., Zakharyaschev, M. Modal Logic. Oxford University Press, Oxford Logic Guides **35** (1997).

[8] Danecki, R.: Non-deterministic propositional dynamic logic is decidable. In Skowron, A. (Editor): Computation Theory. Springer-Verlag, Lecture Notes in Computer Science **208** (1985) 34–53.

[9] Fisher, M., Ladner, R.: Propositional dynamic logic of regular programs. Journal of Computer and System Sciences **18** (1979) 194–211.

[10] Gargov, G., Passy, S. A note on Boolean modal logic. In Petkov, P. (Editor): Mathematical Logic. Plenum Press (1990) 299–309.

[11] Gargov, G., Passy, S., Tinchev, T. Modal environment for Boolean speculations In Skordev, D. (Editor): Mathematical Logic and its Applications. Plenum Press (1987) 253–263.

[12] Goldblatt, R.: Axiomatising the Logic of Computer Programming. Springer-Verlag, Lecture Notes in Computer Science **130** (1982).

[13] Harel, D.: First-Order Dynamic Logic. Springer-Verlag, Lecture Notes in Computer Science **68** (1979).

[14] Harel, D.: Recurring dominoes: making the highly undecidable highly understandable. In Karpinski, M. (Editor): Foundations of Computation Theory. Springer-Verlag, Lecture Notes in Computer Science **158** (1983) 177–194.

[15] Harel, D.: Dynamic logic. In Gabbay, D., Guenthner, F. (Editors): Handbook of Philosophical Logic, Volume II, Extensions of Classical Logic. Kluwer Academic Publishers, Synthese Library **165** (1984) 497–604.

[16] Harel, D., Kozen, D., Tiuryn, J.: Dynamic Logic. MIT Press (2000).

[17] Hughes, G., Cresswell, M.: A Companion to Modal Logic. Methuen (1984).

[18] Kozen, D., Parikh, R.: An elementary proof of the completeness of PDL. Theoretical Computer Science **14** (1981) 113–118.

[19] Kozen, D., Tiuryn, J.: Logics of programs. In van Leeuwen, J. (Editor): Handbook of Theoretical Computer Science, Volume B, Formal Models and Semantics. Elsevier (1990) 789–840.

[20] Lemmon, E., Scott, D.: The “Lemmon Notes”: an Introduction to Modal Logic. Blackwell (1977).

[21] Marx, M., Venema, Y.: Multi-Dimensional Modal Logic. Kluwer Academic Publishers, Applied Logic Series **4** (1997).

[22] Parikh, R.: Propositional dynamic logics of programs: a survey. In Engeler, E. (Editor): Logic of Programs. Springer-Verlag, Lecture Notes in Computer Science **125** (1981) 102–144.

[23] Passy, S., Tinchev, T.: An essay in combinatory dynamic logic. Information and Computation **93** (1991) 263–332.

[24] Pratt, V.: A near-optimal method for reasoning about action. Journal of Computer and System Sciences **20** (1980) 231–254.

[25] Segerberg, K.: A completeness theorem in the modal logic of programs. In Traczyk, T. (Editor): Universal Algebra and Applications. Polish Scientific Publishers, Banach Center Publications **9** (1982) 31–36.