Preference structures: first-order characterization and modal logic

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Abstract

Preferences structures are informational systems that enable us to represent the objects prefered or detested by users. Given a preference structure relations reflecting relationships between users disclose the implicit information derived from preferences and detestations. Next we introduce a modal logic that allow us to represent and make inference from this implicit information.

Keywords: Preference structures, first-order characterization, modal logic.

1 Introduction

Consider a file from the opinion poll presented in tables 1– 4 where us_1, us_2, us_3 and us_4 denote users and ob_1, ob_2, ob_3, ob_4, ob_5, ob_6, ob_7 and ob_8 denote objects. In all tables the first column lists the set of all users. In tables 1 and 2 subsequent columns contain the objects most prefered and prefered by users whereas in tables 3 and 4 subsequent columns contain the objects most detested and detested by users. This file is an information system which describe users in terms of their prefered or detested objects. For instance user us_2 is very keen on objects ob_1, ob_2, ob_3, ob_4 and ob_5 whereas user us_3 does not like object ob_8 that much. In this paper we present preferences structures, i.e. informational systems that enable us to formally represent the objects prefered or detested by users. The notion of preference structure employed in this paper is a special instance of the notion of information system introduced by Pawlak [10] and furthered by Vakarelov [14]. A distinction between various forms of information systems is discussed by Demri and Orłowska [5].

Table 1: Most prefered objects.

Users	Most prefered objects
us_1	$\emptyset$
us_2	$\{ob_1, ob_2, ob_3, ob_4, ob_5\}$
us_3	$\{ob_1\}$
us_4	$\{ob_7, ob_8\}$

Table 2: Prefered objects.

Users	Prefered objects
us_1	$\{ob_1, ob_2, ob_3\}$
us_2	$\{ob_1, ob_2, ob_3, ob_4, ob_5, ob_6, ob_7\}$
us_3	$\{ob_1, ob_2, ob_3\}$
us_4	$\{ob_6, ob_7, ob_8\}$

Preference structures are collections of information that enable us to formally represent the most essential concepts related to the notions of preference and detestation. They employ the notions of objects and users whereas the basic pieces of information are given by preference functions and detestation functions. More formally given a positive integer n a structure $\mathcal{S} = (Ob, Us, f)$ is called a preference structure iff Ob is a nonempty set of objects, Us is a nonempty set of users and f is a total function which for each $s \in \{+, -\}$, for each $i \in \{1, \ldots, n\}$ and for each $x \in Us$ assigns a subset $f_i^s(x) \subseteq Ob$. Every set $f_i^s(x)$

Table 3: Most detested objects.

Users	Most detested objects
us_1	$\emptyset$
us_2	$\emptyset$
us_3	$\{ob_8\}$
us_4	$\{ob_1, ob_2\}$

Table 4: Detested objects.

Users	Detested objects
us_1	$\{ob_5, ob_6, ob_7, ob_8\}$
us_2	$\{ob_8\}$
us_3	$\{ob_6, ob_7, ob_8\}$
us_4	$\{ob_1, ob_2, ob_3\}$

can be viewed as the given set of objects of a user x corresponding to the pair (s, i) where s denotes either preference or detestation and i denotes the level of preference or detestation: $f_i^+(x)$ is the set of objects prefered by user x at level i whereas $f_i^-(x)$ is the set of objects detested by user x at level i. We want the subsets $f_i^s(x)$ to reflect the intended meanings of preferences and detestations. Thus a preference structure $\mathcal{S} = (Ob, Us, f)$ will be defined to be standard iff for all $i, j \in \{1, \ldots, n\}$ and for all $x \in Us$, if $i \leq j$ then $f_i^+(x) \subseteq f_j^+(x)$, $f_i^+(x) \cap f_j^-(x) = \emptyset$, $f_i^-(x) \subseteq f_j^-(x)$ and $f_i^-(x) \cap f_j^+(x) = \emptyset$. We should consider for example with $n = 2$ the standard preference structure $\mathcal{S} = (Ob, Us, f)$ where $Ob = \{ob_1, ob_2, ob_3, ob_4, ob_5, ob_6, ob_7, ob_8\}$, $Us = \{us_1, us_2, us_3, us_4\}$ and f is the total function defined by tables 1– 4, i.e. for all $x \in Us$, $f_1^+(x)$ is the set of all objects most prefered by x, $f_2^+(x)$ is the set of all objects prefered by x, $f_1^-(x)$ is the set of all objects most detested by x and $f_2^-(x)$ is the set of all objects detested by x.

A preference system $\mathcal{S} = (Ob, Us, f)$ contains some implicit information about relationships among the users from the set Us determined by their prefered objects and their detested objects. Following the line of reasoning suggested by Orłowska [7, 8], Orłowska and Pawlak [9] and Vakarelov [12, 13] these information relations have the form of binary relations between users. They are traditionally called indistinguishability relations and distinguishability relations. In section 2 we introduce the indistinguishability relations of forward inclusion, indiscernibility and backward inclusion between users in preference structures. Section 3 presents the modal logic based on these binary relations. In section 4 and section 5 we address the issues of axiomatization/completeness and decidability/complexity of our modal logic. Section 6 concludes the paper and outlines open problems. We assume some familiarity with modal logic and model theory. Readers wanting more details may refer to Blackburn, de Rijke and Venema [1], Chagrov and Zakharyaschev [2] and Chang and Keisler [3].

2 First-order characterization

Let $\mathcal{S} = (Ob, Us, f)$ be a standard preference structure. For all $s, t \in \{+, -\}$ and for all $i, j \in \{1, \ldots, n\}$, define in Us the unary relation of emptiness Δ_i^s and the binary relations of forward inclusion $\leq_{ij}^{st}$, of indiscernibility $\equiv_{ij}^{st}$ and of backward inclusion $\geq_{ij}^{st}$ by

- $\Delta_i^s(x)$ iff $f_i^s(x) = \emptyset$,
- $x \leq_{ij}^{st} y$ iff $f_i^s(x) \subseteq f_j^t(y)$,
- $x \equiv_{ij}^{st} y$ iff $f_i^s(x) = f_j^t(y)$,
- $x \geq_{ij}^{st} y$ iff $f_i^s(x) \supseteq f_j^t(y)$.

In the above example we have $\Delta_1^+(us_1)$, $us_1 \leq_{21}^{++} us_2$, $us_1 \equiv_{22}^{++} us_3$ and $us_1 \geq_{21}^{+-} us_4$. The relational system $F(\mathcal{S}) = (Us, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ is the standard preference frame corresponding to $\mathcal{S}$. We show first that

Lemma 1 *Let $s, t, u \in \{+, -\}$, $i, j, k \in \{1, \ldots, n\}$ and $x, y, z \in Us$. Then*

(C_1) *If $i \leq j$ then $x \leq_{ij}^{ss} x$,*

(C_2) *$x \equiv_{ii}^{ss} x$,*

(C_3) *If $i \geq j$ then $x \geq_{ij}^{ss} x$,*

(C_4) *$x \leq_{ij}^{st} y \Rightarrow y \geq_{ji}^{ts} x$,*

(C_5) *$x \equiv_{ij}^{st} y \Rightarrow y \equiv_{ji}^{ts} x$,*

(C_6) $x \geq_{ij}^{st} y \Rightarrow y \leq_{ji}^{ts} x$,

(C_7) $x \leq_{ij}^{st} y$ *&* $y \leq_{jk}^{tu} z \Rightarrow x \leq_{ik}^{su} z$,

(C_8) $x \equiv_{ij}^{st} y$ *&* $y \equiv_{jk}^{tu} z \Rightarrow x \equiv_{ik}^{su} z$,

(C_9) $x \geq_{ij}^{st} y$ *&* $y \geq_{jk}^{tu} z \Rightarrow x \geq_{ik}^{su} z$,

(C_{10}) $\Delta_i^s(x)$ *&* $\neg\Delta_j^t(y) \Rightarrow x \leq_{ij}^{st} y$,

(C_{11}) $\neg\Delta_i^s(x)$ *&* $\Delta_j^t(y) \Rightarrow x \not\leq_{ij}^{st} y$,

(C_{12}) $\Delta_i^s(x)$ *&* $\Delta_j^t(y) \Rightarrow x \equiv_{ij}^{st} y$,

(C_{13}) $\neg\Delta_i^s(x)$ *&* $\Delta_j^t(y) \Rightarrow x \geq_{ij}^{st} y$,

(C_{14}) $\Delta_i^s(x)$ *&* $\neg\Delta_j^t(y) \Rightarrow x \not\geq_{ij}^{st} y$,

(C_{15}) $x \equiv_{ij}^{st} y \Rightarrow x \leq_{ij}^{st} y$ *&* $x \geq_{ij}^{st} y$.

Moreover

(C_{16}) $x \leq_{ij}^{s+} y$ *&* $x \leq_{ij}^{s-} y \Rightarrow \Delta_i^s(x)$,

(C_{17}) $x \leq_{ij}^{st} y$ *&* $x \geq_{ij}^{st} y \Rightarrow x \equiv_{ij}^{st} y$.

Proof: Left to the reader. ⊣

Following the approach of Vakarelov [14] lemma 1 suggests the following definition. Let $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ be a relational system. $\mathcal{F}$ will be called a prenormal preference frame iff it satisfies the conditions (C_1)—(C_{15}) from lemma 1. $\mathcal{F}$ is said to be normal iff it satisfies the conditions (C_1)—(C_{17}). We shall say that $\mathcal{F}$ is standard iff there exists a standard preference structure $\mathcal{S}$ such that $\mathcal{F}$ is isomorphic to $F(\mathcal{S})$. Lemma 1 implies that

Theorem 1 *Each standard preference frame is normal.*

Proof: By lemma 1. ⊣

Slightly less trivial is the following result.

Theorem 2 *Each normal preference frame is standard.*

Proof: Let $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ be a normal preference frame. For all $s \in \{+,-\}$, for all $i \in \{1,\ldots,n\}$ and for all $x \in W$, let

- $\overrightarrow{\leq_i^s}(x) = (\leq_{ij}^{st}(x)\colon t \in \{+,-\}$ and $j \in \{1,\ldots,n\})$,

where $\leq_{ij}^{st}(x) = \{y \in W\colon x \leq_{ij}^{st} y\}$ for any $t \in \{+,-\}$ and for any $j \in \{1,\ldots,n\}$. Let $\mathcal{S} = (Ob, Us, f)$ be the preference structure defined by

- $Ob = \{\overrightarrow{\leq_i^s}(x)\colon s,t \in \{+,-\},\ i,j \in \{1,\ldots,n\},\ x,y \in W$ and $x \not\leq_{ij}^{st} y\}$,
- $Us = W$,
- $f_i^s(x) = \{\overrightarrow{\leq_j^t}(y)\colon t,u \in \{+,-\},\ j,k \in \{1,\ldots,n\},\ y,z \in W,\ y \not\leq_{jk}^{tu} z$ and $y \leq_{ji}^{tx} x\}$ for any $s \in \{+,-\}$, for any $i \in \{1,\ldots,n\}$ and for any $x \in W$.

The reader is asked to show that $\mathcal{S}$ is standard and $\mathcal{F}$ is isomorphic to $F(\mathcal{S})$. ⊣

As a consequence normal preference frames and standard preference frames are equivalent classes of relational systems. Theorem 2 provides an informational representation of standard preference structures. It is therefore natural to develop a modal framework based on normal preference frames for studying the most essential features of information relations in standard preference structures.

3 Modal interpretation

We begin here the development of a propositional modal language for preference frames. The symbols of our language are based on a countable set Φ_0 of atomic formulas, with typical members denoted p, q, etc, parentheses (and), Boolean connectives $\neg$ and $\vee$, modal constants δ_i^s for any $s \in \{+,-\}$ and for any $i \in \{1,\ldots,n\}$, modal connective $[U]$, modal connectives $[\leq_{ij}^{st}]$, $[\equiv_{ij}^{st}]$ and $[\geq_{ij}^{st}]$ for any $s,t \in \{+,-\}$ and for any $i,j \in \{1,\ldots,n\}$. The formulas of our language, with typical members denoted A, B, etc, are

- Every atomic formula p is a formula,
- If A is a formula then $\neg A$ is a formula,
- If A and B are formulas then $(A \vee B)$ is a formula,
- δ_i^s is a formula for any $s \in \{+,-\}$ and for any $i \in \{1,\ldots,n\}$,

- If A is a formula then $[U]A$ is a formula,
- If A is a formula then $[\leq_{ij}^{st}]A$ is a formula for any $s, t \in \{+, -\}$ and for any $i, j \in \{1, \ldots, n\}$,
- If A is a formula then $[\equiv_{ij}^{st}]A$ is a formula for any $s, t \in \{+, -\}$ and for any $i, j \in \{1, \ldots, n\}$,
- If A is a formula then $[\geq_{ij}^{st}]A$ is a formula for any $s, t \in \{+, -\}$ and for any $i, j \in \{1, \ldots, n\}$.

A model is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ is a relational system and V is a function assigning to each atomic formula $p \in \Phi_0$ a subset $V(p)$ of W. For all formulas A and for all $x \in W$, the relation "A is true at x in $\mathcal{M}$", denoted $\mathcal{M} \models_x A$, is defined by

- $\mathcal{M} \models_x p$ iff $x \in V(p)$,
- $\mathcal{M} \models_x \neg A$ iff not $\mathcal{M} \models_x A$,
- $\mathcal{M} \models_x A \vee B$ iff $\mathcal{M} \models_x A$ or $\mathcal{M} \models_x B$,
- $\mathcal{M} \models_x \delta_i^s$ iff $\Delta_i^s(x)$,
- $\mathcal{M} \models_x [U]A$ iff for all $y \in W$, $\mathcal{M} \models_y A$,
- $\mathcal{M} \models_x [\leq_{ij}^{st}]A$ iff for all $y \in W$, if $x \leq_{ij}^{st} y$ then $\mathcal{M} \models_y A$,
- $\mathcal{M} \models_x [\equiv_{ij}^{st}]A$ iff for all $y \in W$, if $x \equiv_{ij}^{st} y$ then $\mathcal{M} \models_y A$,
- $\mathcal{M} \models_x [\geq_{ij}^{st}]A$ iff for all $y \in W$, if $x \geq_{ij}^{st} y$ then $\mathcal{M} \models_y A$.

We shall say that A is true in model $\mathcal{M}$, in symbols $\mathcal{M} \models A$, iff for all $x \in W$, $\mathcal{M} \models_x A$. A is said to be valid in relational system $\mathcal{F}$, denoted $\mathcal{F} \models A$, iff for all models $\mathcal{M}$ on $\mathcal{F}$, $\mathcal{M} \models A$. Let $\mathcal{C}$ be a class of relational systems. A will be defined to be valid in $\mathcal{C}$, in symbols $\mathcal{C} \models A$, iff for all relational systems $\mathcal{F} \in \mathcal{C}$, $\mathcal{F} \models A$. Let $\mathcal{C}_{ppf}$ be the class of all prenormal preference frames and $\mathcal{C}_{npf}$ be the class of all normal preference frames.

4 Axiomatization/completeness

We aim to give a sound and complete deductive system L_{npf} for validity in the class $\mathcal{C}_{npf}$. The proper axioms of L_{npf} are all instances of the schemata

(A_1) If $i \leq j$ then $[\leq_{ij}^{ss}]A \rightarrow A$,

(A_2) $[\equiv_{ii}^{ss}]A \rightarrow A$,

(A_3) If $i \geq j$ then $[\geq_{ij}^{ss}]A \rightarrow A$,

(A_4) $A \rightarrow [\leq_{ij}^{st}]\langle\geq_{ji}^{ts}\rangle A$,

(A_5) $A \rightarrow [\equiv_{ij}^{st}]\langle\equiv_{ji}^{ts}\rangle A$,

(A_6) $A \rightarrow [\geq_{ij}^{st}]\langle\leq_{ji}^{ts}\rangle A$,

(A_7) $[\leq_{ik}^{su}]A \rightarrow [\leq_{ij}^{st}][\leq_{jk}^{tu}]A$,

(A_8) $[\equiv_{ik}^{su}]A \rightarrow [\equiv_{ij}^{st}][\equiv_{jk}^{tu}]A$,

(A_9) $[\geq_{ik}^{su}]A \rightarrow [\geq_{ij}^{st}][\geq_{jk}^{tu}]A$,

(A_{10}) $\delta_i^s \wedge [\leq_{ij}^{st}]A \rightarrow [U](\delta_j^t \vee A)$,

(A_{11}) $\neg\delta_i^s \rightarrow [\leq_{ij}^{st}]\neg\delta_j^t$,

(A_{12}) $\delta_i^s \wedge [\equiv_{ij}^{st}]A \rightarrow [U](\neg\delta_j^t \vee A)$,

(A_{13}) $\neg\delta_i^s \wedge [\geq_{ij}^{st}]A \rightarrow [U](\neg\delta_j^t \vee A)$,

(A_{14}) $\delta_i^s \rightarrow [\geq_{ij}^{st}]\delta_j^t$,

(A_{15}) $[\leq_{ij}^{st}]A \vee [\geq_{ij}^{st}]A \rightarrow [\equiv_{ij}^{st}]A$.

A formula A is called provable in L_{npf}, denoted $\vdash_{L_{npf}} A$, iff it can be derived from tautologies, the above axioms and instances of the schemata

- $[U](A \rightarrow B) \rightarrow ([U]A \rightarrow [U]B)$,
- $[\leq_{ij}^{st}](A \rightarrow B) \rightarrow ([\leq_{ij}^{st}]A \rightarrow [\leq_{ij}^{st}]B)$,
- $[\equiv_{ij}^{st}](A \rightarrow B) \rightarrow ([\equiv_{ij}^{st}]A \rightarrow [\equiv_{ij}^{st}]B)$,
- $[\geq_{ij}^{st}](A \rightarrow B) \rightarrow ([\geq_{ij}^{st}]A \rightarrow [\geq_{ij}^{st}]B)$,
- $[U]A \rightarrow [\leq_{ij}^{st}]A \wedge [\equiv_{ij}^{st}]A \wedge [\geq_{ij}^{st}]A$,
- $[U]A \rightarrow A$,
- $A \rightarrow [U]\langle U\rangle A$,
- $[U]A \rightarrow [U][U]A$,

by applying the inference rules

- Given A and $A \to B$, infer B,
- Given A, infer $[U]A$,
- Given A, infer $[\leq_{ij}^{st}]A$,
- Given A, infer $[\equiv_{ij}^{st}]A$,
- Given A, infer $[\geq_{ij}^{st}]A$.

We now prove two simple theorems. First, soundness of L_{npf}-provability with respect to $\mathcal{C}_{ppf}$-validity.

Theorem 3 *Let A be a formula. If $\vdash_{L_{npf}} A$ then $\mathcal{C}_{ppf} \models A$.*

Proof: Obviously, prenormal preference frames satisfy the conditions which are needed to validate the axioms of L_{npf} and inference rules of L_{npf} are correct with respect to validity in prenormal preference frames. As a result, every formula provable in L_{npf} is valid in $\mathcal{C}_{ppf}$. $\dashv$
Second, inclusion of the notion of $\mathcal{C}_{ppf}$-validity in the notion of $\mathcal{C}_{npf}$-validity.

Theorem 4 *Let A be a formula. If $\mathcal{C}_{ppf} \models A$ then $\mathcal{C}_{npf} \models A$.*

Proof: Each normal preference frame is prenormal. Hence every formula valid in $\mathcal{C}_{ppf}$ is valid in $\mathcal{C}_{npf}$. $\dashv$
Slightly less trivial is the following result stating completeness of L_{npf}-provability with respect to $\mathcal{C}_{npf}$-validity.

Theorem 5 *Let A be a formula. If $\mathcal{C}_{npf} \models A$ then $\vdash_{L_{npf}} A$.*

Proof: Suppose A is not provable in L_{npf}, we demonstrate A is not valid in $\mathcal{C}_{npf}$. Seeing that A is not provable in L_{npf}, there is a maximal set x_A of formulas such that $A \not\in x_A$. Let $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ be a generated subframe of the canonical frame for L_{npf} containing x_A. It is a simple matter to check that the proper axioms (A_1)—(A_{15}) of L_{npf} are all Sahlqvist schemata. Hence they are first-order definable. To be more precise they correspond to the conditions (C_1)—(C_{15}) from lemma 1. By the Sahlqvist completeness theorem it follows that the relational system $\mathcal{F}$ is a prenormal preference frame. We now claim that

Claim 1: Every prenormal preference frame is a bounded morphic image of a prenormal preference frame satisfying the condition (C_{17}),

Claim 2: Every prenormal preference frame satisfying the condition (C_{17}) is a bounded morphic image of a normal preference frame.

From these two claims and from the fact that A is not valid in $\mathcal{F}$ it follows that A is not valid in $\mathcal{C}_{npf}$. Firstly let us proceed to the proof of claim 1. Given any prenormal preference frame $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$, let $\mathcal{F}' = (W', \Delta_i^{s\prime}, \leq_{ij}^{st\prime}, \equiv_{ij}^{st\prime}, \geq_{ij}^{st\prime})$ be the relational system defined by

- $W' = \{(x, \alpha)\colon x \in W$ and $\alpha \in I\}$ where I is the set of all functions α assigning to each $s \in \{+, -\}$ and to each $i \in \{1, \ldots, n\}$ an integer α_i^s such that for all $s, t \in \{+, -\}$ and for all $i, j \in \{1, \ldots, n\}$, $i < j$ iff $\alpha_i^s < \alpha_j^t$,
- $\Delta_i^{s\prime}(x, \alpha)$ iff $\Delta_i^s(x)$,
- $(x, \alpha) \leq_{ij}^{st\prime} (y, \beta)$ iff either $\Delta_i^s(x)$, or $x \leq_{ij}^{st} y$ and $\alpha_i^s < \beta_j^t$, or $x \equiv_{ij}^{st} y$ and $\alpha_i^s = \beta_j^t$,
- $(x, \alpha) \equiv_{ij}^{st\prime} (y, \beta)$ iff either $\Delta_i^s(x)$ and $\Delta_j^t(y)$, or $x \equiv_{ij}^{st} y$ and $\alpha_i^s = \beta_j^t$,
- $(x, \alpha) \geq_{ij}^{st\prime} (y, \beta)$ iff either $\Delta_j^t(y)$, or $x \geq_{ij}^{st} y$ and $\alpha_i^s > \beta_j^t$, or $x \equiv_{ij}^{st} y$ and $\alpha_i^s = \beta_j^t$.

$\mathcal{F}'$ is obviously a prenormal preference frame satisfying the condition (C_{17}). Now, let π be the mapping from W' to W defined by $\pi(x, \alpha) = x$. The reader is asked to show that π is a bounded morphism from $\mathcal{F}'$ to $\mathcal{F}$. Secondly let us proceed to the proof of claim 2. Given any prenormal preference frame $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ satisfying the condition (C_{17}), let $\mathcal{F}' = (W', \Delta_i^{s\prime}, \leq_{ij}^{st\prime}, \equiv_{ij}^{st\prime}, \geq_{ij}^{st\prime})$ be the relational system defined by

- $W' = \{(x,\alpha)\colon x \in W \text{ and } \alpha \in I\}$ where I is the set of all functions α assigning to each $s \in \{+,-\}$ and to each $i \in \{1,\ldots,n\}$ an integer α_i^s such that for all $s,t \in \{+,-\}$ and for all $i,j \in \{1,\ldots,n\}$, $s \neq t$ iff $\alpha_i^s \neq \alpha_j^t$,

- $\Delta_i^{s\prime}(x,\alpha)$ iff $\Delta_i^s(x)$,

- $(x,\alpha)\leq_{ij}^{st\prime}(y,\beta)$ iff either $\Delta_i^s(x)$, or $x \leq_{ij}^{st} y$ and $\alpha_i^s = \beta_j^t$,

- $(x,\alpha)\equiv_{ij}^{st\prime}(y,\beta)$ iff either $\Delta_i^s(x)$ and $\Delta_j^t(y)$, or $x \equiv_{ij}^{st} y$ and $\alpha_i^s = \beta_j^t$,

- $(x,\alpha)\geq_{ij}^{st\prime}(y,\beta)$ iff either $\Delta_j^t(y)$, or $x \geq_{ij}^{st} y$ and $\alpha_i^s = \beta_j^t$.

$\mathcal{F}'$ is obviously a normal preference frame. Now, let π be the mapping from W' to W defined by $\pi(x,\alpha) = x$. The reader is asked to show that π is a bounded morphism from $\mathcal{F}'$ to $\mathcal{F}$. $\dashv$

As a consequence L_{npf} is a sound and complete deductive system for validity in the class $\mathcal{C}_{npf}$.

5 Decidability/complexity

The issue of decidability can be approached via the notion of the finite frame property. Let $\mathcal{C}_{fppf}$ be the class of all finite prenormal preference frames.

Theorem 6 *Let A be a formula. If $\mathcal{C}_{fppf} \models A$ then $\mathcal{C}_{ppf} \models A$.*

Proof: Suppose A is not valid in $\mathcal{C}_{ppf}$, we demonstrate A is not valid in $\mathcal{C}_{fppf}$. Seeing that A is not valid in $\mathcal{C}_{ppf}$, there is a prenormal preference frame $\mathcal{F} = (W, \Delta_i^s, \leq_{ij}^{st}, \equiv_{ij}^{st}, \geq_{ij}^{st})$ such that $\mathcal{F} \not\models A$. Hence there is a model $\mathcal{M} = (\mathcal{F}, V)$ on $\mathcal{F}$ such that $\mathcal{M} \not\models A$. Let Γ_A be the smallest set of formulas such that

- A is in Γ_A,

- Γ_A is closed under subformulas,

- Γ_A is closed under Boolean connectives,

- δ_i^s is in Γ_A for any $s \in \{+,-\}$ and for any $i \in \{1,\ldots,n\}$,

- For all formulas B,
  - If $[\leq_{ij}^{st}]B$ is in Γ_A then $[\leq_{ik}^{su}]B$ and $[\leq_{jk}^{tu}]B$ are in Γ_A for any $s,t,u \in \{+,-\}$ and for any $i,j,k \in \{1,\ldots,n\}$,
  - If $[\equiv_{ij}^{st}]B$ is in Γ_A then $[\equiv_{ik}^{su}]B$ and $[\equiv_{jk}^{tu}]B$ are in Γ_A for any $s,t,u \in \{+,-\}$ and for any $i,j,k \in \{1,\ldots,n\}$
  - If $[\geq_{ij}^{st}]B$ is in Γ_A then $[\geq_{ik}^{su}]B$ and $[\geq_{jk}^{tu}]B$ are in Γ_A for any $s,t,u \in \{+,-\}$ and for any $i,j,k \in \{1,\ldots,n\}$.

Let $=_{\Gamma_A}$ be the equivalence relation on W defined by

- $x =_{\Gamma_A} y$ iff for all formulas B in Γ_A, $\mathcal{M} \models_x B$ iff $\mathcal{M} \models_y B$.

For all x in W, the equivalence class of x modulo $=_{\Gamma_A}$ is denoted $\mid x \mid$. The quotient set of W modulo $=_{\Gamma_A}$ is denoted $W_{|=_{\Gamma_A}}$. It is a simple matter to check that $W_{|=_{\Gamma_A}}$ is finite. Let $\mathcal{F}' = (W', \Delta_i^{s\prime}, \leq_{ij}^{st\prime}, \equiv_{ij}^{st\prime}, \geq_{ij}^{st\prime})$ be the relational system defined by

- $W' = W_{|=_{\Gamma_A}}$,

- $\Delta_i^{s\prime}(\mid x \mid)$ iff $\Delta_i^s(x)$,

- $\mid x \mid \leq_{ij}^{st\prime} \mid y \mid$ iff for all formulas B, if $[\leq_{ij}^{st}]B$ is in Γ_A then
  - If $\mathcal{M} \models_x [\leq_{ik}^{su}]B$ then $\mathcal{M} \models_y [\leq_{jk}^{tu}]B$,

  and if $[\geq_{ji}^{ts}]B$ is in Γ_A then
  - If $\mathcal{M} \models_y [\geq_{jk}^{tu}]B$ then $\mathcal{M} \models_x [\geq_{ik}^{su}]B$,

- $\mid x \mid \equiv_{ij}^{st\prime} \mid y \mid$ iff for all formulas B, if $[\equiv_{ij}^{st}]B$ is in Γ_A then
  - If $\mathcal{M} \models_x [\equiv_{ik}^{su}]B$ then $\mathcal{M} \models_y [\equiv_{jk}^{tu}]B$,

  and if $[\equiv_{ji}^{ts}]B$ is in Γ_A then
  - If $\mathcal{M} \models_y [\equiv_{jk}^{tu}]B$ then $\mathcal{M} \models_x [\equiv_{ik}^{su}]B$,

- $\mid x \mid \geq_{ij}^{st\prime} \mid y \mid$ iff for all formulas B, if $[\geq_{ij}^{st}]B$ is in Γ_A then

- If $\mathcal{M} \models_x [\geq^{su}_{ik}]B$ then $\mathcal{M} \models_y [\geq^{tu}_{jk}]B$,

and if $[\leq^{ts}_{ji}]B$ is in Γ_A then

- If $\mathcal{M} \models_y [\leq^{tu}_{jk}]B$ then $\mathcal{M} \models_x [\leq^{su}_{ik}]B$.

$\mathcal{F}'$ is obviously a finite prenormal preference frame. Now, let $\mathcal{M}' = (\mathcal{F}', V')$ be the model on $\mathcal{F}'$ defined by

- $V'(p) = \{\mid x \mid : x \in V(p)\}$ for each atomic formula $p \in \Gamma_A$,
- $V'(p) = \emptyset$ for each atomic formula $p \in \Phi_0 \setminus \Gamma_A$.

The reader is asked to show that $\mathcal{M}'$ is a filtration of $\mathcal{M}$ through Γ_A. $\dashv$
As a consequence the decision problem

Problem: L_{npf}-provability,

Input: a formula A,

Output: determine whether $\vdash_{L_{npf}} A$ or not,

is decidable. Our intention is to prove that L_{npf}-provability is $PSPACE$-complete. In the first place we prove that $PSPACE$ is a lower bound.

Theorem 7 *L_{npf}-provability is PSPACE-hard.*

Proof: To demonstrate that L_{npf}-provability is $PSPACE$-hard, we have to show that some $PSPACE$-hard problem is reducible to it. To this end we use the $PSPACE$-hard problem of K-provability. Let A be a formula in the restriction of the language of L_{npf} obtained by deleting all modal constants as well as all modal connectives but $[\leq^{+-}_{1n}]$. We now claim that

Claim 1: If $\vdash_K A$ then $\vdash_{L_{npf}} A$,

Claim 2: If $\vdash_{L_{npf}} A$ then $\vdash_K A$.

From these two claims and from the fact that K-provability is $PSPACE$-hard, see Ladner [6] for details, it follows that L_{npf}-provability is $PSPACE$-hard. Firstly let us proceed to the proof of claim 1. Suppose A is provable in K, we demonstrate A is provable in L_{npf}. Seeing that A is provable in K, it can be derived from the axioms of K by applying the inference rules of K. It is a simple matter to check that all axioms of K are axioms of L_{npf} and all inference rules of K are inference rules of L_{npf}. Hence A is provable in L_{npf}. Secondly let us proceed to the proof of claim 2. Suppose A is not provable in K, we demonstrate A is not provable in L_{npf}. Seeing that A is not provable in K, there is a tree $\mathcal{F} = (W, R)$ such that $\mathcal{F} \not\models A$. Let $\mathcal{F}' = (W', \Delta^{s\prime}_i, \leq^{st\prime}_{ij}, \equiv^{st\prime}_{ij}, \geq^{st\prime}_{ij})$ be the relational system defined by

- $W' = W$,
- $\Delta^{s\prime}_i = \emptyset$,
- $x \leq^{st\prime}_{ij} y$ iff either $s = t$, $i \leq j$ and $x = y$, or $s = +$, $t = -$, $i = 1$, $j = n$ and xRy,
- $x \equiv^{st\prime}_{ij} y$ iff $s = t$, $i = j$ and $x = y$,
- $x \geq^{st\prime}_{ij} y$ iff either $s = t$, $i \geq j$ and $x = y$, or $s = -$, $t = +$, $i = n$, $j = 1$ and $xR^{-1}y$.

It is a simple matter to check that $\mathcal{F}'$ is a prenormal preference frame. Moreover $\leq^{+-\prime}_{1n} = R$. $\dashv$
In the second place we prove that $PSPACE$ is an upper bound.

Theorem 8 *L_{npf}-provability is in PSPACE.*

Proof: L_{npf}-provability can be solved by means of a polynomial-space bounded nondeterministic algorithm à la Demri [4]. Seeing that $NPSPACE = PSPACE$, see Savitch [11] for details, we therefore conclude that L_{npf}-provability is in $PSPACE$. $\dashv$
We leave the details of the proof to the extended version of this paper.

6 Conclusion

The notion of preference structure employed in this paper is a special instance of the notion of information system introduced by Pawlak [10] and furthered by Vakarelov [14].

Binary relations like forward inclusion, indiscernibility and backward inclusion have been considered by several authors, see Demri and Orłowska [5] for details. Within the context of standard preference structures one can also define the binary relations of positive similarity Σ^{st}_{ij}, of complementarity C^{st}_{ij} and of negative similarity N^{st}_{ij} by

- $x\Sigma^{st}_{ij}y$ iff $f^s_i(x) \not\subseteq Ob \setminus f^t_j(y)$,
- $xC^{st}_{ij}y$ iff $f^s_i(x) = Ob \setminus f^t_j(y)$,
- $xN^{st}_{ij}y$ iff $f^s_i(x) \not\supseteq Ob \setminus f^t_j(y)$.

Together with the unary relation of emptiness and the binary relations of forward inclusion, of indiscernibility and of backward inclusion, the characterization of these binary relationships by means of first-order conditions is not problematic, see Vakarelov [14] for details. However the issues of axiomatization/completeness and decidability/complexity of modal logics based on at least one of them are still open.

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