Tractability Results in the Block Algebra

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Abstract

In this paper we define the notion of a block algebra, which is based upon a spatial application of Allen's interval algebra. In the p-dimensional Euclidean space, where $p \geq 1$, we consider only blocks whose sides are parallel to the axes of some orthogonal basis. The block algebra consists of a set of relations (the block relations) together with the fundamental operations of composition, converse and intersection. The 13^p basic relations of this algebra constitute the exhaustive list of the relations possibly holding between two blocks. We are interested in the problem of testing the consistency of a set of spatial constraints between blocks, i.e. a block network. The consistency question for block networks is NP-complete. We first extend the notions of convexity and preconvexity to the block algebra. Similarly to the interval algebra case, convexity leads to a tractable set whereas, contrary to the interval algebra case, preconvexity leads to an intractable set. Nevertheless we characterize a tractable subset of the preconvex relations: the strongly preconvex relations. Moreover we show that strong preconvexity and ORD-Horn representability are the same.

Keywords: Spatial reasoning, qualitative constraints, consistency problem, interval algebra, block algebra.

1 Introduction

Qualitative temporal and spatial reasoning with constraints are important aspects of reasoning in many areas of Artificial Intelligence: geographical information systems, natural language understanding, specification and verification of programs and systems, temporal and spatial databases, temporal and spatial planning, document analysis, etc. Concerning temporal reasoning, Allen's interval algebra [1] is one of the most well-known and widely used formalisms. Allen takes intervals as primitive temporal entities and considers 13 basic relations between these intervals (see Figure 1). These basic relations represent all the possible relative positions of two intervals on the rational line. Temporal information is represented by binary constraint networks called interval networks, each constraint being defined by an interval relation (a disjunction of basic interval relations). Since the consistency problem of interval networks is NP-complete, a great deal of research has been done to find sets of interval relations whose consistency can be decided in polynomial time. In particular the convex subclass [18, 5] and the ORD-Horn subclass [17] are tractable sets whose consistency problem can be solved by the path-consistency method. Ligozat [12, 13] gives an alternative characterization of ORD-Horn relations in terms of preconvex relations. Preconvexity is defined from concepts such as the topological closure, the convex closure and the dimension of a relation. It is easy to determine whether an interval relation is preconvex. Moreover, Ligozat has produced an interesting proof of the tractability of the preconvex relations.

On the other hand, as far as qualitative spatial reasoning is concerned, a well-known approach is the region-based calculus called Region Connection Calculus (RCC) proposed by Randell, Cui and Cohn [21]. This approach is based on binary topological relations between spatial regions. Solving constraint networks in RCC is also an NP-complete problem. Re-

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cently, Renz and Nebel [23, 22] characterized tractable subclasses of region relations.

The real applications of RCC are numerous, however we note that with this calculus we cannot express relations such as directional relations (like 'a spatial entity is on the left of another spatial entity' or 'a spatial entity is above another spatial entity', etc.). That is the reason why — following our work in [2, 3] — we extend the interval algebra to any dimension $p \geq 1$. This kind of approach has been used during the past few years [16, 11, 20, 19, 24], nevertheless no complexity problems were examined in that line of work.

We consider blocks whose sides are parallel to the axes of some orthogonal basis in the p-dimensional Euclidean space. The formalism that we obtain — the block algebra formalism — provides a framework for representing and reasoning about spatial relations between those blocks. In the sequel, the term *block* will denote this kind of blocks. We will confine ourselves to the issue of the consistency of block networks which consist of sets of constraints between a finite number of blocks. Adapting the line of reasoning suggested by Ligozat [13] we extend the notions of convexity and preconvexity to block relations. We prove that, similarly to the interval algebra case, convexity leads to a tractable set whereas, contrary to the interval algebra case, preconvexity leads to an intractable set. Observing this, we define a subset of the block relations — the strongly preconvex relations — which we show to be tractable and for which the consistency problem can be decided by means of the path-consistency method and a new method: the weak path-consistency method. To close our study, we show that strongly preconvex relations coincide with ORD-Horn relations in the sense of Nebel and Bürckert [17].

Section 2 is devoted to some reminders about the interval algebra and to the definition of the block algebra. In Section 3 we focus our attention on the notion of convexity while in Section 4 the main matter is the notion of preconvexity. Then in Section 5 we define the strongly preconvex block relations; we prove that these relations coincide with ORD-Horn ones in Section 6 and that they correspond to a tractable set in Section 7.

2 From the interval algebra to the block algebra

2.1 The interval algebra

An interval in Allen's framework is an ordered pair of rational numbers, with the first number being strictly less than the second one. We use X, Y, Z, etc. to denote intervals in that sense. For every ordered pair X of rational numbers, the first number will be denoted by X^- and the second one by X^+. Given an interval Y, it defines a partition of the set of all rational numbers into five zones numbered from 0 to 4: zone 0 is $]-\infty, Y^-[$; zone 1 is $\{Y^-\}$; zone 2 is $]Y^-, Y^+[$; zone 3 is $\{Y^+\}$ and zone 4 is $]Y^+, +\infty[$. Given another interval X, the relative location of X with respect to Y is given by the zones to which X^- and X^+ belong. This yields thirteen possible relative locations between two intervals, each one being represented by a basic relation in Allen's sense [1], see Figure 1. Let $\mathcal{I}$ be the set of these basic relations. We denote by a, b, c, etc. elements of $\mathcal{I}$. Given a basic relation a, we will denote the zone to which X^- belongs by $zone^-(a)$ and the one to which X^+ belongs by $zone^+(a)$. Moreover, XaY will mean that the intervals X and Y are in relation a. The subsets of $\mathcal{I}$ define binary relations (denoted by α, β, γ, etc.) making it possible to represent indefinite information between two intervals. Indeed, each subset α is interpreted as the disjunction of its elements: if X and Y are intervals, $X\alpha Y$ iff there is a basic relation $a \in \alpha$ such that XaY.

Let IA be the algebra whose underlying set is the set $2^{\mathcal{I}}$ of all the subsets of $\mathcal{I}$ and whose

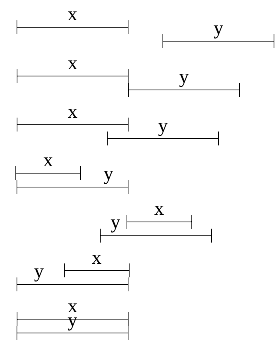
Relation a	Converse	Graphic Illustration	zone$^-$(a), zone$^+$(a)	Topological Closure
X b Y (before)	bi		zone 0, zone 0	{b,m}
X m Y (meets)	mi		zone 0, zone 1	{m}
X o Y (overlaps)	oi		zone 0, zone 2	{m,o,fi,s,eq}
X s Y (starts)	si		zone 1, zone 2	{s,eq}
X d Y (during)	di		zone 2, zone 2	{d,s,f,eq}
X f Y (finishes)	fi		zone 2, zone 3	{f,eq}
X eq Y (equals)	eq		zone 1, zone 3	{eq}

FIGURE 1. The set $\mathcal{I}$ of the basic relations between intervals

underlying operations are: the unary operation $^{-1}$ of converse, the unary operation $-$ of complement, the binary operation $\cup$ of union, the binary operation $\cap$ of intersection and the binary operation $\circ$ of composition. To define the operations of converse and composition, Allen [1] first defines a converse table and a composition table on the thirteen basic relations:

- given intervals X, Y and a basic relation a such that $Y a X$, the basic relation a^{-1} represents the position of X with respect to Y;
- given intervals X, Y, Z and basic relations a, b such that $X a Z$ and $Z b Y$, the binary relation $a \circ b$ represents the set of all possible positions of X with respect to Y.

Second, Allen defines the operations of converse and composition on $2^{\mathcal{I}}$:

$$\alpha^{-1} = \{a^{-1} : a \in \alpha\} \text{ and } \alpha \circ \beta = \bigcup \{a \circ b : a \in \alpha \text{ and } b \in \beta\}.$$

It is worth noting that if X and Y are intervals, $X \alpha^{-1} Y$ iff $Y \alpha X$, and $X(\alpha \circ \beta)Y$ iff there is an interval Z such that $X \alpha Z$ and $Z \beta Y$. This proves a simple but fundamental result: IA is a relational algebra. We conclude with the notion of dimension introduced by Ligozat [13] to define the set of preconvex interval relations. The dimension of a, denoted by $\dim(a)$, is:

- 2 if a forces no boundary equality ($a \in \{b, bi, d, di, o, oi\}$);
- 1 if a forces one boundary equality ($a \in \{m, mi, s, si, f, fi\}$);
- 0 if a forces two boundary equalities ($a = eq$).

The notion of dimension is extended to the relations of $2^{\mathcal{I}}$ as follows: the dimension of $\alpha \in 2^{\mathcal{I}}$, denoted by $\dim(\alpha)$, is $\sup\{\dim(a) : a \in \alpha\}$.

2.2 The block algebra

Let a fixed positive integer p be the dimension of the space. We consider only blocks whose sides are parallel to the axes of some orthogonal basis in the p-dimensional Euclidean space. We will use x, y, z, etc. to denote blocks. A block x is totally characterized by the tuple

$(X_1, \ldots, X_p)$ of the p intervals corresponding to its orthogonal projections onto the p axes. The interval X_i is the orthogonal projection of x onto the ith axis, with $i \in \{1, \ldots, p\}$. Then we can define the positions between blocks as follows: in order to formalize the position of x with respect to y, we have to examine the positions of X_1, X_2, etc. with respect to Y_1, Y_2, etc. To be more precise, the position of x with respect to y is a tuple $a = (a_1, \ldots, a_p)$ of p basic relations between intervals where a_i specifies the position of X_i with respect to Y_i for every $i \in \{1, \ldots, p\}$. Thus we obtain a set of 13^p basic relations, denoted by $\mathcal{B}_p$. For $p = 1$, $\mathcal{B}_p = \mathcal{I}$. We will use a, b, c, etc. for these basic relations. In Figure 2 are represented some basic relations between 2-blocks. Note that, similarly to what holds for $\mathcal{I}$, the set $\mathcal{B}_p$ forms a jointly exhaustive and pairwise disjoint set of binary relations, i.e. any two blocks satisfy one and only one basic relation of $\mathcal{B}_p$. Given that our aim is to represent partial information

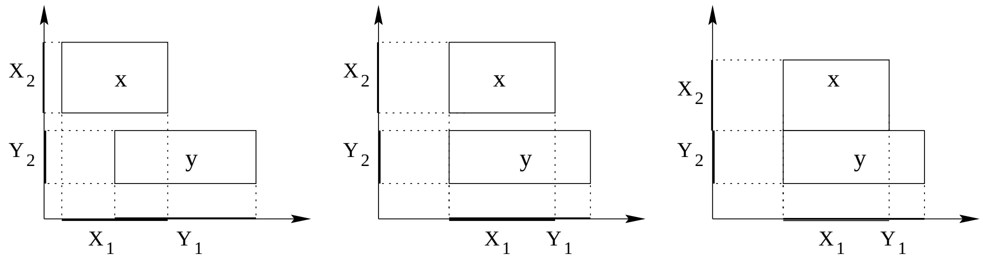


FIGURE 2. The basic relations (o, bi), (s, bi) and (s, mi) satisfied between two 2-blocks x and y

about the spatial locations of objects, we allow the binary relation between two blocks to be any subset of the set $\mathcal{B}_p$. We will use α, β, γ, etc. to denote such subsets. The relation of $2^{\mathcal{B}_p}$ containing all the basic relations is called the universal relation of $2^{\mathcal{B}_p}$. By definition, if x and y are blocks, $x\alpha y$ iff there is a basic relation $a \in \alpha$ such that xay. In the sequel we will often use the operation of Cartesian product, denoted by $\times$ and defined by : for every $\alpha \in 2^{\mathcal{B}_p}$ and $\alpha \in 2^{\mathcal{B}_q}$ with $p, q > 0$, $\alpha \times \beta = \{(a_1, \ldots, a_p, b_1, \ldots, b_q) : (a_1, \ldots, a_p) \in \alpha, (b_1, \ldots, b_q) \in \beta\}$.

Let BA$_p$, or BA for short if p is fixed and implicitly determined by the context, be the algebra whose underlying set is the set $2^{\mathcal{B}_p}$ of all the subsets of $\mathcal{B}_p$ and whose underlying operations are: the unary operation $^{-1}$ of converse, the unary operation $-$ of complement, the binary operation $\cup$ of union, the binary operation $\cap$ of intersection and the binary operation $\circ$ of composition. The operations of complement, intersection and union are the usual set operations. To define the operations of converse and composition, first we define the operations of converse and composition on $\mathcal{B}_p$:

$$a^{-1} = (a_1^{-1}, \ldots, a_p^{-1}) \text{ and } a \circ b = (a_1 \circ b_1) \times \ldots \times (a_p \circ b_p);$$

and secondly we define these operations on $2^{\mathcal{B}_p}$:

$$\alpha^{-1} = \{a^{-1} : a \in \alpha\} \text{ and } \alpha \circ \beta = \bigcup \{a \circ b : a \in \alpha \text{ and } b \in \beta\}.$$

This leads to the question of whether these definitions capture the intended meaning of the operations involved. Actually, for all blocks x, y and for all block relations α, β, $x(\alpha \circ \beta)y$

iff there exists a block z such that $x\alpha z$ and $z\beta y$. Indeed, let us suppose on the one hand that $x(\alpha \circ \beta)y$. There exist $a \in \alpha$ and $b \in \beta$ such that $x\ (a_1 \circ b_1) \times \ldots \times (a_p \circ b_p)\ y$. For each $i \in \{1, \ldots, p\}$ $X_i(a_i \circ b_i)Y_i$, so on the axis i there exists an interval Z^i such that $X_i a_i Z^i$ and $Z^i b_i Y_i$. Let z be the block defined by $Z_i = Z^i$ for each $i \in \{1, \ldots, p\}$, we have $x\alpha z$ and $z\beta y$. Conversely, let us assume that there exists a block z such that $x\alpha z$ and $z\beta y$. There exist $a \in \alpha$ and $b \in \beta$ such that xaz and zby. For each $i \in \{1, \ldots, p\}$ $X_i a_i Z_i$ and $Z_i b_i Y_i$. Consequently $X_i(a_i \circ b_i)Y_i$, hence $x(a \circ b)y$ and $x(\alpha \circ \beta)y$. As an exercise, we leave the proof of the fact that $x\alpha y$ iff $y\alpha^{-1}x$ to the reader. We immediately deduce the desired result: BA is a relational algebra.

By definition, a subclass of $2^{\mathcal{B}_p}$ (resp. $2^{\mathcal{I}}$) is a set of block relations (resp. interval relations) closed under the fundamental operations of converse, composition and intersection.

Given a relation $\alpha \in 2^{\mathcal{B}_p}$ and $i \in \{1, \ldots, p\}$, we call *i-th projection* of α, denoted by α_i, the interval relation $\{a_i : a \in \alpha\}$.

EXAMPLE 2.1
Let $\alpha, \beta \in 2^{\mathcal{B}_2}$ be the two relations defined by $\alpha = \{(eq, b), (m, s)\}$ et $\beta = \{(d, b), (s, s)\}$. We have $\alpha^{-1} = \{(eq, b)^{-1}, (m, s)^{-1}\}$ $=\{(eq, bi), (mi, si)\}$, $\alpha \circ \beta = ((eq, b) \circ (d, b)) \cup ((eq, b) \circ (s, s)) \cup ((m, s) \circ (d, b)) \cup ((m, s) \circ (s, s))$= $(\{d\} \times \{b\}) \cup (\{s\} \times \{b\}) \cup (\{d, o, s\} \times \{b\}) \cup (\{m\} \times \{s\}) = \{(d, b), (s, b), (o, b), (m, s)\}$. Moreover, $\alpha_1 = \{eq, m\}$ and $\alpha_2 = \{b, s\}$.

It is worth noting that $(\alpha^{-1})_i = (\alpha_i)^{-1}$ and $(\alpha \circ \beta)_i = \alpha_i \circ \beta_i$.

Now we extend the notion of *dimension* to BA. Given a basic relation a and a relation α of BA, the dimensions of a and α, denoted by $\dim(a)$ and $\dim(\alpha)$ respectively, are given by:

$$\dim(a) = \dim(a_1) + \ldots + \dim(a_p) \text{ and } \dim(\alpha) = \sup\{\dim(a) : a \in \alpha\}.$$

Note that the dimension of a is equal to $2 \times p$ minus the number of bound equalities forced by a between the orthogonal projections.

EXAMPLE 2.2
Using Figure 2 we can easily see that $dim((o, bi)) = (2 \times 2) - 0 = 4$, $dim((s, bi))$ $=(2 \times 2) - 1 = 3$, $dim((s, mi)) = (2 \times 2) - 2 = 2$. From this, we deduce that $dim(\{(o, bi), (s, bi), (s, mi)\}) = 4$.

We now give some easy properties of these notions; the proofs are left to the reader as an exercise.

PROPOSITION 2.3
Let $\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^n$ be relations of $2^{\mathcal{I}}$. The following equalities hold:

(a) $(\alpha^1 \times \ldots \times \alpha^n)^{-1} = (\alpha^1)^{-1} \times \ldots \times (\alpha^n)^{-1}$;

(b) $(\alpha^1 \times \ldots \times \alpha^n) \cap (\beta^1 \times \ldots \times \beta^n) = (\alpha^1 \cap \beta^1) \times \ldots \times (\alpha^n \cap \beta^n)$;

(c) $(\alpha^1 \times \ldots \times \alpha^n) \circ (\beta^1 \times \ldots \times \beta^n) = (\alpha^1 \circ \beta^1) \times \ldots \times (\alpha^n \circ \beta^n)$;

(d) $dim(\alpha^1 \times \ldots \times \alpha^n) = dim(\alpha^1) + \ldots + dim(\alpha^n)$.

To end this subsection, let us stress that BA is not the Cartesian product of IA since most of the relations of $2^{\mathcal{B}_p}$ are not a Cartesian product of relations of $2^{\mathcal{I}}$.

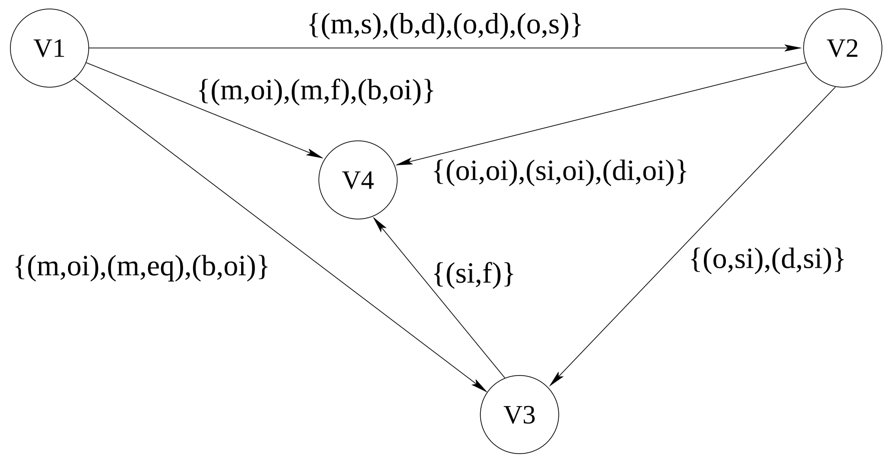


FIGURE 3. An example of block network $\mathcal{N}$

2.3 Representation with qualitative constraint networks

A network of constraints between intervals (resp. blocks) is a structure of the form (V, C), where V is a set $\{V_1, \ldots, V_n\}$ of n variables, with $n = |V|$. C is a mapping from $V \times V$ into $2^{\mathcal{I}}$ (resp. $2^{\mathcal{B}_p}$). Each $V_i \in V$ corresponds to an interval (resp. a block) and each constraint $C(V_i, V_j)$ — also denoted by C_{ij} in the sequel — corresponds to the possible relative locations between the objects represented by V_i and V_j. In what follows, we assume that all interval networks (resp. block networks) satisfy the following conditions:

$$C_{ii} = \{eq\} \text{ (resp. } C_{ii} = \{(eq, \ldots, eq)\}\text{) and } C_{ij} = C_{ji}^{-1}.$$

Note that an interval network is a block network with $p = 1$.

Given two block networks $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$, $\mathcal{N}$ is a subnetwork of $\mathcal{N}'$ iff for all $i, j \in \{1, \ldots, |V|\}$, $C_{ij} \subseteq C'_{ij}$.

We extend the notion of projection to block networks in the following way.

DEFINITION 2.4
Let $\mathcal{N} = (V, C)$ be a block network and $i \in \{1, \ldots, p\}$. The ith projection of $\mathcal{N}$, denoted by $\mathcal{N}_i$, is the interval network (V^i, C^i) defined by $V = V^i$ and for all $j, k \in \{1, \ldots, |V|\}$, $C^i_{jk} = (C_{jk})_i$.

The ith projection of a block network (V, C) corresponds to the constraints entailed on the orthogonal projections of the blocks represented by the variables of V onto the ith axis.

An instantiation of a block network (V, C) is a mapping m associating with each variable $V_i \in V$ a block denoted by $m(V_i)$ or m_i. For each $i, j \in \{1, \ldots, n\}$, with $n = |V|$, we will note by $m(V_i, V_j)$ or m_{ij} the basic relation satisfied by V_i and V_j. m is a consistent instantiation (also called a solution) iff for $i, j \in \{1, \ldots, n\}$, $m_{ij} \in C_{ij}$. Moreover, m is of maximal dimension iff $dim(m_{ij}) = dim(C_{ij})$. A block network $\mathcal{N} = (V, C)$ is consistent iff it admits a consistent instantiation. It is minimal iff for all $i, j \in \{1, \ldots, |V|\}$ and for

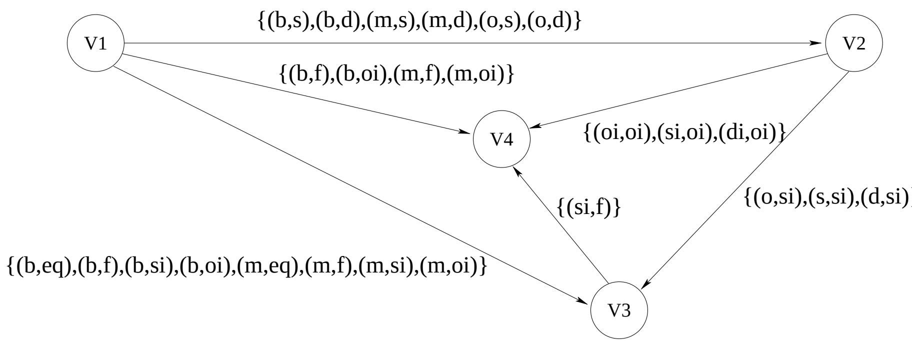


FIGURE 4. A path-consistent convex block network: $\mathcal{N}' = \mathbf{I}(\mathcal{N})$

all $a \in C_{ij}$ there exists a consistent instantiation of $\mathcal{N}$ such that $m_{ij} = a$. Furthermore, two block networks are equivalent iff they have the same variables and the same consistent instantiations.

An important matter is to decide the consistency of a network (the consistency problem). An incomplete decision method for the consistency problem of a block network is the path-consistency method [15, 14]. Applied to a block network $\mathcal{N} = (V, C)$, it consists in successively replacing the constraint C_{ij} by the constraint $C_{ij} \cap (C_{ik} \circ C_{kj})$ for each $i, j, k \in \{1, \ldots, n\}$, with $n = |V|$, until a fixed point is reached. We obtain in polynomial time ($O(n^3)$) a subnetwork $\mathcal{N}' = (V, C')$ of $\mathcal{N}$ which is path-consistent, i.e. for each $i, j, k \in \{1, \ldots, n\}$:

$$C'_{ij} \subseteq C'_{ik} \circ C'_{kj}.$$

Since the path-consistency method removes from the constraints only basic relations which cannot participate in a consistent instantiation, it is sound. So $\mathcal{N}'$ is equivalent to $\mathcal{N}$; if it contains the empty constraint then both networks are inconsistent else, generally, one can say nothing about the consistency of $\mathcal{N}$: the path-consistency method is not complete for the consistency problem.

EXAMPLE 2.5
As an illustration, Figure 3 represents a block network $\mathcal{N} = (V, C)$. We did not represent the universal relation, C_{ii} and C_{ij} if C_{ji} is already represented, for all $i, j \in \{1, \ldots, |V|\}$. We note that $\mathcal{N}$ is not path-consistent. For example, $C_{12} \not\subseteq C_{13} \circ C_{32}$. Figure 4 represents a path-consistent block network $\mathcal{N}'$. $\mathcal{N}$ is a subnetwork of $\mathcal{N}'$. Moreover, the instantiation m depicted in Figure 5 is an instantiation of maximal dimension of $\mathcal{N}'$.

The projections of a path-consistent block network are always path-consistent.

PROPOSITION 2.6
Let $\mathcal{N} = (V, C)$ be a block network. If $\mathcal{N}$ is path-consistent then for each $i \in \{1, \ldots, p\}$, $\mathcal{N}_i$ is path-consistent.

PROOF. Let $k, l, m \in \{1, \ldots, |V|\}$. $C_{km} \subseteq C_{kl} \circ C_{lm}$ so $(C_{km})_i \subseteq (C_{kl} \circ C_{lm})_i$. Moreover $(C_{kl} \circ C_{lm})_i = (C_{kl})_i \circ (C_{lm})_i$. Consequently, $(C_{km})_i \subseteq (C_{kl})_i \circ (C_{lm})_i$. ∎

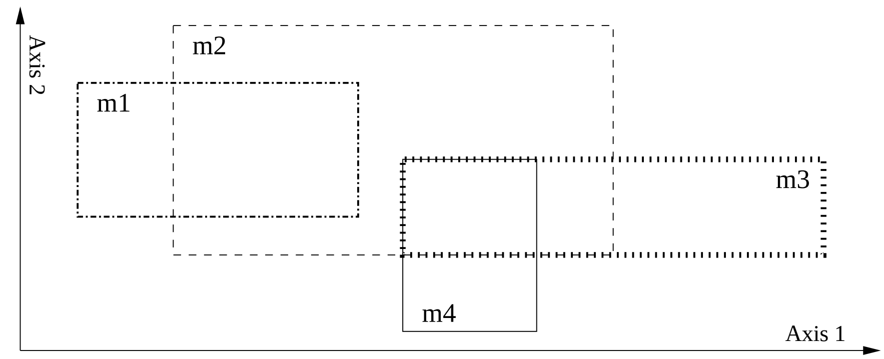


FIGURE 5. A satisfying instantiation m of maximal dimension of the block network $\mathcal{N}'$

The opposite direction of this proposition does not hold. For example, the block network in Figure 6 is not path-consistent whereas its projections are path-consistent.

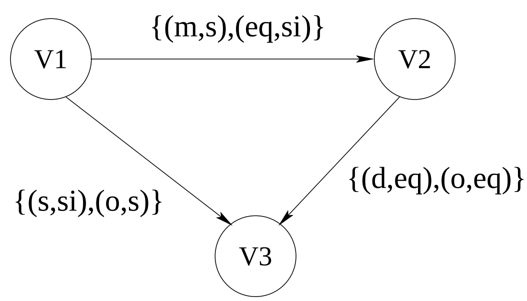


FIGURE 6. A block network which is not path-consistent

Vilain and Kautz [25] prove that deciding the consistency of an interval network is NP-complete. Therefore, the question of characterizing tractable subclasses of IA is a matter of the utmost significance. A first interesting isolated tractable subclass is the set of convex relations [6, 18]. Despite its small size this set contains useful relations including the thirteen basic relations. Nebel and Bürckert [17] give a definitive answer for the subclasses which contain all basic relations. More precisely, the subclass of ORD-Horn relations is the unique maximal tractable subclass having this property. Moreover, deciding consistency can be accomplished by using the path-consistency algorithm. Ligozat [13] produces a simple alternative characterization of the same subclass in terms of preconvex relations. A preconvex relation can be roughly described as a convex relation with some lower-dimension basic relations taken out. Ligozat [13] shows that for each path-consistent preconvex interval network a consistent instantiation of maximal dimension can be constructed. In the context of the block algebra as well, deciding the consistency of a network is NP-complete. Hence the question of characterizing tractable subclasses of the block algebra is also a matter of the first importance. By extending the notion of convexity to the block algebra we characterize a first tractable subclass of this algebra. As regards the notion of preconvexity, a problem is that if $p \geq 2$ then the coincidence between the syntactic concept of ORD-Horn representability and the geometric concept of preconvexity does not hold any longer, because the subclass of ORD-Horn relations is a proper subset of the set of all preconvex relations. A further difficulty is that the set of all preconvex relations is not a subclass in the usual sense, given that it is not closed under the operation of intersection. Actually, we prove that this set is intractable. This leads us to define a stronger notion, the concept of strong preconvexity. As a result, we prove that the issue of the consistency of a strongly preconvex network of constraints between blocks can be solved in polynomial time by means of the path-consistency algorithm.

3 The subclass of convex relations: a first tractable case

3.1 Convex interval relations

Let us recall that given two intervals X,Y and a basic relation a of $\mathcal{I}$, XaY iff X^- and X^+ belong to particular zones among *zone 0, ..., zone 4* defined by Y. Both these zones depend on a and we will denote them by $zone^-(a)$ (zone of X^-) and $zone^+(a)$ (zone of X^+). To define the set of all convex relations between intervals, it is helpful to first arrange in ascending order the 13 basic relations of IA. Following the line of reasoning suggested by Ligozat [13], let $a \preceq b$ mean that $zone^-(a) \leq zone^-(b)$ and $zone^+(a) \leq zone^+(b)$. $(\mathcal{I}, \preceq)$ is a distributive lattice called the interval lattice, see Figure 7. By definition, the interval bounded by a and b, denoted by $[a, b]$, is the following relation of $2^{\mathcal{I}}$: $\{c : a \preceq c \text{ and } c \preceq b\}$. An interval relation α is convex iff there exist $a, b \in \mathcal{I}$ such that $\alpha = [a, b]$. Let us recall that the set of all convex relations is closed under the operation of intersection. Consequently, the set of all convex relations containing α contains a least element, denoted by $\mathbf{I}(\alpha)$ and called the convex closure of α. Furthermore Ligozat [13] proves that $\mathbf{I}(\alpha^{-1}) = \mathbf{I}(\alpha)^{-1}$ and $\mathbf{I}(\alpha \circ \beta) = \mathbf{I}(\alpha) \circ \mathbf{I}(\beta)$. This implies a simple but fundamental result: the set of all convex relations is closed under the operations of converse and composition. Therefore, it constitutes a subclass of IA, the convex subclass.

The chief feature of the convex subclass is its tractability. The thing is that if a convex interval network is path-consistent then it is minimal and either it contains the empty constraint or it has a consistent instantiation of maximal dimension. As a result, the issue of the consistency of a convex interval network can be solved in polynomial time by way of the path-consistency method.

3.2 Convex block relations

In this section we extend the notion of convexity to block relations. In a natural manner, we define an order on $\mathcal{B}_p$ in the following way:

$$\forall a, b \in \mathcal{B}_p, a \preceq b \text{ iff } a_1 \preceq b_1, \ldots, a_p \preceq b_p.$$

As a product of distributive lattices, it is easily shown that $(\mathcal{B}_p, \preceq)$ is also a distributive lattice which we will call the block lattice. Note that for all $a, b \in \mathcal{B}_p$, $\inf\{a, b\}$ and $\sup\{a, b\}$ are the basic relations c and d of $\mathcal{B}_p$ such that $c_i = \inf\{a_i, b_i\}$ and $d_i = \sup\{a_i, b_i\}$ for every $i \in \{1, \ldots, p\}$. By definition, the interval bounded by a and b, denoted by $[a, b]$, is the block

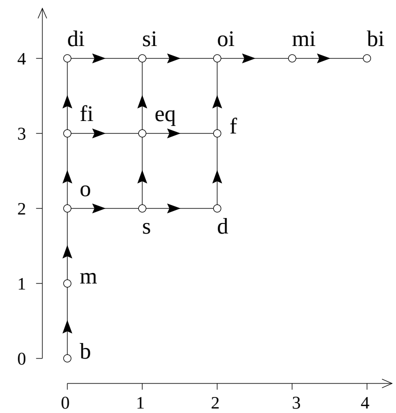


FIGURE 7. The interval lattice $(\mathcal{I}, \preceq)$

relation $\{c : a \preceq c \text{ and } c \preceq b\}$. This leads to the introduction of the following definition of *convexity*:

DEFINITION 3.1
Let $\alpha \in 2^{\mathcal{B}_p}$. α is a convex relation iff there are $a, b \in \mathcal{B}_p$ such that $\alpha = [a, b]$.

Since the block lattice is the product of the interval lattice p times by itself, a block relation α corresponds to an interval of the block lattice iff α is the Cartesian product of intervals of the interval lattice. It follows that $\alpha \in 2^{\mathcal{B}_p}$ is a convex relation iff $\alpha = \alpha_1 \times \ldots \times \alpha_p$ with for each $i \in \{1, \ldots, p\}$ α_i a convex relation of $2^{\mathcal{I}}$. From this and the fact that the set of convex interval relations forms a subclass we can easily deduce by using proposition 2.3 that convex block relations form a subclass too.

FACT 3.2
The set of all convex relations of $2^{\mathcal{B}_p}$ is closed under the operations of converse, intersection and composition.

As the convex block relations of $2^{\mathcal{B}_p}$ are closed under the operation of intersection, we can extend the convex closure to block relations: given $\alpha \in 2^{\mathcal{B}_p}$, the convex closure of α, denoted by $\mathbf{I}(\alpha)$, is the least element of the set of all the convex relations containing α. Like the convex closure of interval relations, the convex closure of block relations satisfies the following properties.

PROPOSITION 3.3
Let $\alpha, \beta \in 2^{\mathcal{B}_p}$.

(a) $\alpha \subseteq \mathbf{I}(\alpha)$ and $\mathbf{I}(\mathbf{I}(\alpha)) = \mathbf{I}(\alpha)$; (b) if $\alpha \subseteq \beta$ then $\mathbf{I}(\alpha) \subseteq \mathbf{I}(\beta)$;
(c) $\mathbf{I}(\alpha) = \mathbf{I}(\alpha_1) \times \ldots \times \mathbf{I}(\alpha_p)$; (d) $\mathbf{I}(\alpha^{-1}) = \mathbf{I}(\alpha)^{-1}$;
(e) $\mathbf{I}(\alpha \circ \beta) = \mathbf{I}(\alpha) \circ \mathbf{I}(\beta)$.

PROOF. With the definition of the convex closure the properties (a) and (b) are obvious.
To prove (c), just notice that $\mathbf{I}(\alpha_1) \times \ldots \times \mathbf{I}(\alpha_p)$ is a convex relation of $2^{\mathcal{B}_p}$ containing α and that for each convex relation γ of $2^{\mathcal{B}_p}$ if $\alpha \subseteq \gamma$ then $\mathbf{I}(\alpha_1) \times \ldots \times \mathbf{I}(\alpha_p) \subseteq \mathbf{I}(\gamma_1) \times \ldots \times \mathbf{I}(\gamma_p)$ where $\mathbf{I}(\gamma_1) \times \ldots \times \mathbf{I}(\gamma_p) = \gamma$.

Now, let us prove properties (d) and (e). From (c) we have the equality $\mathbf{I}(\alpha^{-1}) = \mathbf{I}((\alpha^{-1})_1) \times \ldots \times \mathbf{I}((\alpha^{-1})_p)$. As $(\alpha^{-1})_i = (\alpha_i)^{-1}$ for each $i \in \{1, \ldots, p\}$ and as the property is true for IA we deduce that $\mathbf{I}(\alpha^{-1}) = \mathbf{I}(\alpha_1)^{-1} \times \ldots \times \mathbf{I}(\alpha_p)^{-1}$. From (c) and Proposition 2.3 (a) we can conclude that the second term of this equality is the relation $\mathbf{I}(\alpha)^{-1}$. From (c) we have $\mathbf{I}(\alpha \circ \beta) = \mathbf{I}((\alpha \circ \beta)_1) \times \ldots \times \mathbf{I}((\alpha \circ \beta)_p)$. As $(\alpha \circ \beta)_i = \alpha_i \circ \beta_i$ for each $i \in \{1, \ldots, p\}$ and as the property is true for IA we have $\mathbf{I}(\alpha \circ \beta) = (\mathbf{I}(\alpha_1) \circ \mathbf{I}(\beta_1)) \times \ldots \times (\mathbf{I}(\alpha_p) \circ \mathbf{I}(\beta_p))$. From Proposition 2.3 (c) and (c) we obtain the desired equality: $\mathbf{I}(\alpha \circ \beta) = \mathbf{I}(\alpha) \circ \mathbf{I}(\beta)$. ∎

EXAMPLE 3.4
Consider the relation α of $2^{\mathcal{B}_2}$ defined by $\alpha = \{(s, oi), (s, bi), (fi, oi)\}$. The convex closure of α, $\mathbf{I}(\alpha)$, is equal to the convex relation $\mathbf{I}(\alpha_1) \times \mathbf{I}(\alpha_2) = \mathbf{I}(\{s, fi\}) \times \mathbf{I}(\{oi, bi\}) = [o, eq] \times [oi, bi] = \{o, s, fi, eq\} \times \{oi, mi, bi\} = \{(o, oi), (o, mi), (o, bi), (s, oi), (s, mi), (s, bi), (fi, oi), (fi, mi), (fi, bi), (eq, oi), (eq, mi), (eq, bi)\}$. This relation corresponds to the interval $[(o, oi), (eq, bi)]$.

We now extend the notion of convex closure to constraint networks.

DEFINITION 3.5
Let $\mathcal{N} = (V, C)$ be a block network. The convex closure of $\mathcal{N}$, denoted by $\mathbf{I}(\mathcal{N})$, is the convex block network $\mathcal{N}' = (V', C')$ defined by $V' = V$ and $C'_{ij} = \mathbf{I}(C_{ij})$ for all $i, j \in \{1, \ldots, |V|\}$.

Figure 4 represents the convex closure of the block network $\mathcal{N}$ of Figure 3.

Consider $\alpha, \beta, \gamma \in 2^{\mathcal{B}_p}$. If $\alpha \subseteq \beta \circ \gamma$ then $\mathbf{I}(\alpha) \subseteq \mathbf{I}(\beta \circ \gamma)$ (Proposition 3.3 (b)). Therefore $\mathbf{I}(\alpha) \subseteq \mathbf{I}(\beta) \circ \mathbf{I}(\gamma)$ (Proposition 3.3 (e)). Hence we get the following proposition:

PROPOSITION 3.6
Let $\mathcal{N} = (V, C)$ be a block network. If $\mathcal{N}$ is path-consistent then $\mathbf{I}(\mathcal{N})$ is also path-consistent.

We are now ready to prove that the consistency problem of the block networks whose constraints are convex relations (convex block networks) is polynomial like the consistency problem of convex interval networks. First, let us consider convex path-consistent block networks.

PROPOSITION 3.7
If a convex block network is path-consistent then either it contains the empty constraint or it has a consistent instantiation of maximal dimension.

PROOF. Let us recall that the property is true for $p = 1$. Let $\mathcal{N} = (V, C)$ be a path-consistent convex network which does not contain the empty constraint. It is easy to see that each projection $\mathcal{N}_i$, with $i \in \{1, \ldots, p\}$, is a convex interval network without the empty constraint. Moreover, since $\mathcal{N}$ is path-consistent $\mathcal{N}_i$ is also path-consistent (Proposition 2.6). Hence, each interval network $\mathcal{N}_i$ admits a consistent instantiation m^i of maximal dimension. Let m be the instantiation of $\mathcal{N}$ defined by the condition: for each $j \in \{1, \ldots, |V|\}$ m_j is the block $(m_j^1, \ldots, m_j^p)$. It is easily verified that m is a consistent instantiation of $\mathcal{N}$. Also, from Proposition 2.3 (c) we deduce that it is an instantiation of maximal dimension. ∎

Note that using a similar proof we can also prove that a path-consistent convex block network is minimal. Moreover, a consistent instantiation of maximal dimension of a path-consistent convex interval network $\mathcal{N} = (V, C)$ without the empty constraint can be built in polynomial time ($O(|V|)^2$) by means of quadratic algorithms such as CSPAN [6]. Hence, from this and the previous proof we deduce that for path-consistent convex block networks we can use quadratic algorithms to find a consistent instantiation of maximal dimension too.

Using the previous proposition we can establish the following important result:

THEOREM 3.8
The consistency problem of a convex block network can be solved in polynomial time by means of the path-consistency algorithm.

PROOF. Let $\mathcal{N}$ be a convex block network. By applying the path-consistency method on $\mathcal{N}$ we obtain an equivalent path-consistent convex subnetwork $\mathcal{N}'$. If the latter contains the empty relation, then both $\mathcal{N}'$ and $\mathcal{N}$ are not consistent. Otherwise, from Proposition 3.7 we deduce the consistency of $\mathcal{N}'$ and by way of equivalence the consistency of $\mathcal{N}$. ∎

Hence we have characterized a first tractable fragment of BA: the subclass of convex relations. This subclass contains $82^p + 1$ relations out of the 2^{13^p} relations of $2^{\mathcal{B}_p}$. Despite its small size this subclass is very important because of its expressiveness. Indeed, this fragment makes it possible to express 'directional relations' such as 'a block is on the right of another block' (the convex relation $\{mi, bi\} \times \mathcal{I}$ for $p = 2$), 'a block is below another block' (the convex relation $\mathcal{I} \times \mathcal{I} \times \{m, b\}$ for $p = 3$), etc.

4 Preconvex relations and tractability

To continue with our quest of tractable fragments of BA we are going to extend the notion of preconvexity to block relations. We will see that despite the tractability of preconvex relations in the interval algebra we obtain an intractable set of relations in the block algebra (for the case $p > 1$).

First, we will make some short reminders about preconvex interval relations, then we will define and study the set of preconvex block relations.

4.1 Tractability of preconvex interval relations

To define the set of all the preconvex relations between intervals Ligozat introduces the operation of topological closure of an interval relation [13]. This operation can be defined in different ways, one of them being the following one: the topological closure of a basic relation $a \in \mathcal{I}$, denoted by $\mathbf{C}(a)$, is the set of all the basic relations b such that there exists a sequence of basic relations $s_1, \ldots, s_m$ such that $a = s_1$, $b = s_m$ and for each $i \in \{1, \ldots, m-1\}$, $dim(s_i) > dim(s_{i+1})$ and s_i, s_{i+1} are two neighbouring basic relations in the interval lattice. The topological closures of the basic relations of $\mathcal{I}$ are summed up in Figure 1. The operation of topological closure is extended to the relations of $2^{\mathcal{I}}$ as follows: the topological closure of $\alpha \in 2^{\mathcal{I}}$, denoted by $\mathbf{C}(\alpha)$, is the interval relation $\bigcup\{\mathbf{C}(a) : a \in \alpha\}$. Then Ligozat defines the preconvex relations as follows: α is preconvex iff $\mathbf{C}(\alpha)$ is convex. Moreover he proves that the following properties are equivalent:

- α is preconvex;
- $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$;
- $\dim(\mathbf{I}(\alpha) \setminus \alpha) < \dim(\alpha)$;
- $\dim(\mathbf{I}(\alpha) \setminus \alpha) < \dim(\mathbf{I}(\alpha))$.

In addition, Ligozat proves that the set of all preconvex interval relations is closed under the operation of intersection. Moreover, $\mathbf{C}(\alpha^{-1}) = \mathbf{C}(\alpha)^{-1}$ and $\mathbf{C}(\alpha \circ \beta) \supseteq \mathbf{C}(\alpha) \circ \mathbf{C}(\beta)$.

As a result, the set of all preconvex relations is closed under the operations of converse and composition. Therefore it constitutes a subclass of IA, the preconvex subclass. The chief feature of the preconvex subclass is its tractability. This one follows from the fact that if a preconvex interval network is path-consistent then either it contains the empty constraint or it admits a consistent instantiation of maximal dimension. As a result, the consistency problem of a preconvex interval network can be solved in polynomial time using the path-consistency method.

4.2 *Non–tractability of preconvex block relations*

Now let us define the preconvex relations of $2^{\mathcal{B}_p}$. First, we define *the topological closure* of $a \in \mathcal{B}_p$, denoted by $\mathbf{C}(a)$, as the set of the basic relations b such that $b_i \in \mathbf{C}(a_i)$ for every $i \in \{1, \ldots, p\}$, i.e. as the relation $\mathbf{C}(a_1) \times \ldots \times \mathbf{C}(a_p)$ of $2^{\mathcal{B}_p}$. This definition captures the required notion since, similarly to the interval algebra case, for each $a \in \mathcal{B}_p$ we have $b \in \mathbf{C}(a)$ iff there exists a sequence of basic relations of $\mathcal{B}_p$ $s_1, \ldots, s_m$ such that $a = s_1$, $b = s_m$ and for each $i \in \{1, \ldots, m-1\}$, $dim(s_i) > dim(s_{i+1})$ and s_i, s_{i+1} are two neighbouring basic relations in the block lattice. As each block relation is a set of basic relations, we extend the operation of topological closure as follows: the topological closure of $\alpha \in 2^{\mathcal{B}_p}$, denoted by $\mathbf{C}(\alpha)$, is $\bigcup\{\mathbf{C}(a) : a \in \alpha\}$.

EXAMPLE 4.1
Consider the relation α of $2^{\mathcal{B}_2}$ defined by $\alpha = \{(s, m), (s, bi)\}$. The topological closure of α, $\mathbf{C}(\alpha)$, is equal to the relation $(\mathbf{C}(s) \times \mathbf{C}(m)) \cup (\mathbf{C}(s) \times \mathbf{C}(bi))= (\{s, eq\} \times \{m\}) \cup (\{s, eq\} \times \{bi, mi\})= \{(s, m), (eq, m), (s, bi), (s, mi), (eq, bi), (eq, mi)\}$.

The topological closure satisfies many properties among which several will be useful in the sequel.

PROPOSITION 4.2
Let $\alpha^1, \ldots, \alpha^n \in 2^{\mathcal{I}}$ and $\alpha, \beta \in 2^{\mathcal{B}_p}$. Then:

(a) $\alpha \subseteq \mathbf{C}(\alpha)$, $\mathbf{C}(\mathbf{C}(\alpha)) = \mathbf{C}(\alpha)$;
(b) if $\alpha \subseteq \beta$ then $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\beta)$;
(c) $dim(\alpha) = dim(\mathbf{C}(\alpha))$;
(d) $dim(\mathbf{C}(\alpha) \setminus \alpha) < dim(\alpha)$;
(e) $\mathbf{C}(\alpha^{-1}) = \mathbf{C}(\alpha)^{-1}$;
(f) $\mathbf{C}(\alpha^1 \times \ldots \times \alpha^n) = \mathbf{C}(\alpha^1) \times \ldots \times \mathbf{C}(\alpha^n)$;
(g) $\mathbf{C}(\alpha \circ \beta) \supseteq \mathbf{C}(\alpha) \circ \mathbf{C}(\beta)$.

PROOF. Using the fact that these properties are true for IA, i.e. in the case $p = 1$, their proofs are not very hard. Let us just prove property (g). Let $a \in \mathbf{C}(\alpha) \circ \mathbf{C}(\beta)$. There exist $b, c \in \mathcal{B}_p$ belonging to respectively α and β such that $a \in \mathbf{C}(b) \circ \mathbf{C}(c)$. Since $\mathbf{C}(b) \circ \mathbf{C}(c) = (\mathbf{C}(b_1) \times \ldots \times \mathbf{C}(b_p)) \circ (\mathbf{C}(c_1) \times \ldots \times \mathbf{C}(c_p)) = (\mathbf{C}(b_1) \circ \mathbf{C}(c_1)) \times \ldots \times (\mathbf{C}(b_p) \circ \mathbf{C}(c_p))$ we have $a_i \in \mathbf{C}(b_i) \circ \mathbf{C}(c_i)$ for each $i \in \{1, \ldots, p\}$. As the property is true for $p = 1$ we deduce that $a_i \in \mathbf{C}(b_i \circ c_i)$. Thus $a \in \mathbf{C}(b_1 \circ c_1) \times \ldots \times \mathbf{C}(b_p \circ c_p)$. Moreover, from (f) this term is equal to $\mathbf{C}((b_1 \circ c_1) \times \ldots \times (b_p \circ c_p))$. Consequently $a \in \mathbf{C}(b \circ c)$. Hence we can conclude that $a \in \mathbf{C}(\alpha \circ \beta)$. ■

We now have all the elements required to define *preconvexity*[1] in the block algebra in a way similar to its definition in the interval algebra.

DEFINITION 4.3
Let α be a relation of $2^{\mathcal{B}_p}$. α is preconvex iff $\mathbf{C}(\alpha)$ is convex.

[1] In [3] we use the term of weak preconvexity instead of preconvexity for the block algebra.

EXAMPLE 4.4
The relation α given in Example 4.1 is not preconvex, while the relation $\beta = \{(d,d),(eq,s),(s,eq)\}$ is preconvex since $\mathbf{C}(\beta)$ is equal to the convex relation $[s,f] \times [s,f]$ of $2^{\mathcal{B}_2}$.

The set of convex block relations is contained in the set of preconvex block relations, because of the following:

PROPOSITION 4.5
Let $\alpha \in 2^{\mathcal{B}_p}$. If α is convex then $\mathbf{C}(\alpha)$ is also convex.

PROOF. Let us assume that α is convex. We have $\alpha = \alpha_1 \times \ldots \times \alpha_p$ where for each $i \in \{1,\ldots,p\}$ α_i is a convex relation of $2^{\mathcal{I}}$. It follows from Proposition 4.2 (f) that $\mathbf{C}(\alpha) = \mathbf{C}(\alpha_1) \times \ldots \times \mathbf{C}(\alpha_p)$. As the property is true for interval relations we know that for each $i \in \{1,\ldots,p\}$ $\mathbf{C}(\alpha_i)$ is a convex interval relation. Consequently $\mathbf{C}(\alpha)$ is a convex block relation. ∎

Note that the Cartesian product of a preconvex block relation $\alpha \in 2^{\mathcal{B}_p}$ and a preconvex block relation $\beta \in 2^{\mathcal{B}_q}$ is always a preconvex block relation of $2^{\mathcal{B}_{p+q}}$. Before proceeding, let us prove the following technical lemma which relates the notions of convexity, topological closure and dimension.

LEMMA 4.6
Let $\alpha \in 2^{\mathcal{B}_p}$. If α is a convex relation then for each $a \in \alpha$ there exists $b \in \alpha$ such that $a \in \mathbf{C}(b)$ and $dim(b) = dim(\alpha)$.

PROOF. Let us suppose that α is a convex relation of $2^{\mathcal{B}_p}$. For the case $p = 1$ by an exhaustive study of the 83 convex relations of IA we can prove the property (see [7] for more details).

Now let us prove the property for any p. Let $a \in \alpha$. We know that α_i is a convex interval relation and that $a_i \in \alpha_i$ for each $i \in \{1,\ldots,p\}$. Hence, for each i there exists a basic relation $b^i \in \mathcal{I}$ such that $a_i \in \mathbf{C}(b^i)$ and $dim(b^i) = dim(\alpha_i)$. Let $b \in \mathcal{B}_p$ be defined by $b_i = b^i$. It is easy to see that $b \in \alpha$, $a \in \mathbf{C}(b)$ and $dim(b) = dim(\alpha)$ (Proposition 2.3 (d)). ∎

Now we have everything required to prove the following equivalence.

PROPOSITION 4.7
Let $\alpha \in 2^{\mathcal{B}_p}$. The following properties are equivalent:

(a) α is a preconvex relation; (b) $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$;
(c) $dim(\mathbf{I}(\alpha) \setminus \alpha) < dim(\alpha)$; (d) $dim(\mathbf{I}(\alpha) \setminus \alpha) < dim(\mathbf{I}(\alpha))$.

PROOF.

- $(a) \Rightarrow (b)$: as $\mathbf{C}(\alpha)$ is a convex relation containing α and $\mathbf{I}(\alpha)$ is the smallest convex relation containing α, we can assert that $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$.
- $(b) \Rightarrow (c)$: if $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$ then $(\mathbf{I}(\alpha) \setminus \alpha) \subseteq (\mathbf{C}(\alpha) \setminus \alpha)$. We know that $dim(\mathbf{C}(\alpha) \setminus \alpha) < dim(\alpha)$ (Proposition 4.2 (d)). Hence $dim(\mathbf{I}(\alpha) \setminus \alpha) < dim(\alpha)$.
- $(c) \Rightarrow (d)$: $\alpha \subseteq \mathbf{I}(\alpha)$, consequently $dim(\alpha) \leq dim(\mathbf{I}(\alpha))$. Thus, if $dim(\mathbf{I}(\alpha) \setminus \alpha) < dim(\alpha)$ then $dim(\mathbf{I}(\alpha) \setminus \alpha) < dim(\mathbf{I}(\alpha))$.
- $(d) \Rightarrow (b)$: let $a \in \mathbf{I}(\alpha)$. Because of Lemma 4.6 there exists $b \in \mathbf{I}(\alpha)$ such that $dim(b) = dim(\mathbf{I}(\alpha))$ and $a \in \mathbf{C}(b)$. If (d) is true then we are sure that $b \in \alpha$, consequently $a \in \mathbf{C}(\alpha)$. Hence $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$.

- $(b) \Rightarrow (a)$: let us assume that $\mathbf{I}(\alpha) \subseteq \mathbf{C}(\alpha)$. Hence $\mathbf{C}(\mathbf{I}(\alpha)) \subseteq \mathbf{C}(\mathbf{C}(\alpha))$. Consequently $\mathbf{C}(\mathbf{I}(\alpha)) \subseteq \mathbf{C}(\alpha)$. In the general case $\mathbf{C}(\alpha) \subseteq \mathbf{C}(\mathbf{I}(\alpha))$. Hence $\mathbf{C}(\alpha) = \mathbf{C}(\mathbf{I}(\alpha))$. Moreover $\mathbf{C}(\mathbf{I}(\alpha))$ is a convex relation since $\mathbf{I}(\alpha)$ is convex (Proposition 4.5). We can conclude that α is a preconvex block relation. ∎

As a consequence of property (c), a relation $\alpha \in 2^{\mathcal{B}_p}$ is a preconvex relation iff to compute its convex closure, we only have to add to it basic relations of dimension strictly less than its own. Until now, preconvex block relations have the same behaviour than preconvex interval ones. In the following theorem we characterize a first important difference.

THEOREM 4.8
The preconvex relations of $2^{\mathcal{B}_p}$ are closed under the operations of converse and composition. But for $p > 1$ they are not closed under the operation of intersection.

PROOF. Let $\alpha, \beta \in 2^{\mathcal{B}_p}$ be two preconvex relations. Since $\mathbf{C}(\alpha)$ is convex and $\mathbf{C}(\alpha^{-1}) = \mathbf{C}(\alpha)^{-1}$ we know that $\mathbf{C}(\alpha^{-1})$ is convex too. Consequently α^{-1} is a preconvex relation. With a proof similar to Ligozat's one for IA we have $\mathbf{I}(\alpha \circ \beta) \subseteq \mathbf{I}(\alpha) \circ \mathbf{I}(\beta) \subseteq \mathbf{C}(\alpha) \circ \mathbf{C}(\beta) \subseteq \mathbf{C}(\alpha \circ \beta)$ (Proposition 3.3 and Proposition 4.2). Therefore $\alpha \circ \beta$ is a preconvex relation (Proposition 4.7). This proves the first two claims.

We now exhibit a counterexample for intersection. Let us consider the following two relations of $2^{\mathcal{B}_2}$: $\alpha = \{(d,d),(eq,s),(s,eq)\}$ and $\beta = \{(s,s),(eq,s),(s,eq),\ (eq,eq)\}$. $\mathbf{C}(\alpha)$ is the convex relation $\mathbf{C}((d,d)) = \{d,s,f,eq\} \times \{d,s,f,eq\}$, consequently α is a preconvex relation. Moreover β is a convex relation, therefore a preconvex relation. Let $\gamma = \alpha \cap \beta = \{(s,eq),(eq,s)\}$. γ is not preconvex since $\mathbf{C}(\gamma)$ is equal to the non–convex relation $\{(s,eq),(eq,s),(eq,eq)\}$. This counterexample can be easily adapted for every $p > 2$. ∎

As a consequence of the previous theorem it is not true that in the general case ($p > 1$) the set of all preconvex relations constitutes a subclass of the block algebra. Despite this we can assert the following property.

PROPOSITION 4.9
If a preconvex block network is path-consistent then either it contains the empty constraint or it has a consistent instantiation of maximal dimension.

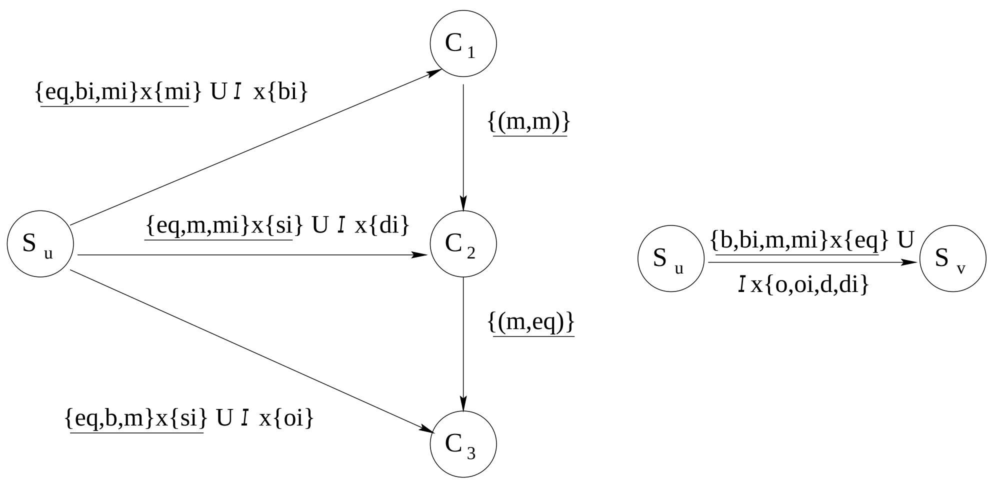
PROOF. Let $\mathcal{N} = (V, C)$ be a path-consistent block network which does not contain the empty constraint. $\mathbf{I}(\mathcal{N})$ is a path-consistent network (Proposition 3.6) which does not contain the empty constraint either. Consequently, $\mathbf{I}(\mathcal{N})$ admits a consistent instantiation m of maximal dimension (Proposition 3.7). Since for all $i, j \in \{1, \ldots, |V|\}$ C_{ij} is preconvex, $dim(\mathbf{I}(C_{ij}) \setminus C_{ij}) < dim(C_{ij})$. Consequently $m_{ij} = dim(C_{ij})$ and $m_{ij} \in C_{ij}$. Therefore m is also a consistent instantiation of maximal dimension of $\mathcal{N}$. ∎

It may be asserted, however, that

THEOREM 4.10
The consistency problem for preconvex block networks is NP-complete for $p > 1$.

PROOF. We are going to characterize a polynomial reduction of the graph 3-colourability problem to the consistency problem of the preconvex block networks (which is clearly NP) for $p = 2$. Given a graph $G = (S, A)$ (S corresponds to the vertices and A to the edges), the 3-colourability problem consists in deciding whether there exists a mapping f from S onto $\{1,2,3\}$ such that if $(u,v) \in A$ then $f(u) \neq f(v)$. This problem is NP-complete [10].

From such a graph G, let us define the block network $\mathcal{N} = (V, C)$ as follows: for each $u \in S$ a variable S_u is introduced in V, moreover we add the three variables C_1, C_2, C_3 corresponding to the three colours. Hence $|V| = |S| + 3$. Now let us define the constraints C. The constraints between C_1, C_2, C_3 and a variable $S_u \in V$ are given by Figure 8 (a) and those between S_u, S_v (with $u \neq v$) by Figure 8 (b). Implicitly, the other constraints of $\mathcal{N}$ are the universal relation $\mathcal{B}_2$. The reader can check that each constraint of $\mathcal{N}$ is a preconvex



(a) (b)

FIGURE 8. The constraints of $\mathcal{N}$

relation of $2^{\mathcal{B}_2}$, so $\mathcal{N}$ is a preconvex block network.

Now we are going to show that $\mathcal{N}$ is consistent iff G is 3-colourable. For that purpose, we will define an interval network $\mathcal{N}''$ which is consistent iff $\mathcal{N}$ is consistent. Then we will show that $\mathcal{N}''$ is consistent iff G is 3-colourable.

First, for every $S_u \in V$, by computing the compositions: $C_{S_uC_1} \circ C_{C_1C_2}$, $C_{S_uC_2} \circ C_{C_2C_3}$ and $C_{S_uC_2} \circ C_{C_2C_1}$ we see that we can simplify $C_{S_uC_2}$, $C_{S_uC_3}$ and $C_{S_uC_1}$ by resp. $\{eq, m, mi\} \times \{si\}$, $\{eq, b, m\} \times \{si\}$ and $\{eq, bi, mi\} \times \{mi\}$ to obtain an equivalent subnetwork to $\mathcal{N}$. Proceeding with our simplifications, after computing $C_{S_uC_1} \circ C_{C_1S_v}$ we can reduce $C_{S_uS_v}$ to the relation $\{b, bi, m, mi\} \times \{eq\}$ for all $S_u, S_v \in V$ with $S_u \neq S_v$. Let us denote by $\mathcal{N}' = (V, C')$ the resulting equivalent subnetwork. The constraints of C' correspond to the underlined basic relations in Figure 8. Note that $\mathcal{N}'_1$ is not a preconvex interval network. Since all the constraints of $\mathcal{N}'$ are Cartesian products of interval relations, we deduce that $\mathcal{N}'$ is consistent iff $\mathcal{N}'_1$ and $\mathcal{N}'_2$ are consistent. We note that $\mathcal{N}'_2$ is always consistent. Consequently $\mathcal{N}'$ is consistent iff $\mathcal{N}'_1$ is consistent. As $\mathcal{N}$ is equivalent to $\mathcal{N}'$, $\mathcal{N}$ is consistent iff $\mathcal{N}'_1$ is consistent. In the sequel of the proof we will denote $\mathcal{N}'_1$ by $\mathcal{N}'' = (V, C'')$.

We notice that $\mathcal{N}''$ looks like the interval network defined by van Beek [5] to prove that the k-colourability problem is polynomially reducible to the consistency problem of interval networks.

Let us show now that $\mathcal{N}''$ is consistent iff G is 3-colourable:

- Let f be a 3-colouring of G. Let us define an instantiation m of $\mathcal{N}''$ by: $m(C_1), m(C_2)$ and $m(C_3)$ are the intervals $[0,1]$, $[1,2]$ and $[2,3]$ respectively; for each $u \in S$, $m(S_u)$ is the interval $m(C_{f(u)})$. The reader can check that m is a consistent instantiation of $\mathcal{N}''$.
- Let m be a consistent instantiation of $\mathcal{N}''$. We notice that the constraints of C'' force the satisfaction of $m(C_1)\ meets\ m(C_2)$ and $m(C_2)\ meets\ m(C_3)$. Moreover each $m(S_u)$, with $S_u \in V$, always corresponds to an interval among $m(C_1), m(C_2)$ and $m(C_3)$. So we can define a mapping f from S onto $\{1,2,3\}$ by $f(u) = i$ with $m(S_u) = m(C_i)$. Without difficulty we can see that if $u \neq v$ then $m(S_u) \neq m(S_v)$ and consequently $f(u) \neq f(v)$. As a result f is a 3-colouring of G.

This proof is for $p = 2$ but it can be easily extended to any dimension $p > 2$. ∎

5 Strongly preconvex block relations

We can ascribe the non-tractability of preconvex block relations (with $p \geq 2$) to the fact that they are not closed under the operation of intersection. To obtain a larger tractable set than the set of convex relations we impose the following requirements on it: it must contains all convex relations and be a subset of preconvex relations, moreover it must be closed under intersection. These requirements lead us to characterize a new set: the set of *strongly preconvex relations*.

DEFINITION 5.1
Let $\alpha \in 2^{\mathcal{B}_p}$. α is strongly preconvex iff for each convex relation β of $2^{\mathcal{B}_p}$ $\alpha \cap \beta$ is a preconvex relation.

EXAMPLE 5.2
The relation β given in example 4.4 is not strongly preconvex since the intersection of $\{(d,d),(s,eq),(eq,s)\}$ and the convex relation $\{(s,s),(s,eq),(eq,s),(eq,eq)\}$ is the non-preconvex relation $\{(s,eq),(eq,s)\}$. On the other hand, the relation $\{(d,d),(s,s),(eq,s),(s,eq)\}$ is a strongly preconvex relation.

The convex relations of $2^{\mathcal{B}_p}$ are closed under the operation of intersection. Using this fact, the previous definition and Proposition 4.5, we can assert that all convex relations are strongly preconvex. Consequently the set of strongly preconvex relations is more expressive that the set of convex relations. Moreover, notice that the Cartesian product of p preconvex interval relations is a strongly preconvex block relation of $2^{\mathcal{B}_p}$.

The strongly preconvex relations make it possible to be more precise. For example, suppose that we know that a block x is on the right of another block y in the plane. We will use the convex relation $\{mi, bi\} \times \mathcal{I}$ to represent this information. Now we learn that x and y do not satisfy the particular atomic relations (mi,m), (mi,si), (bi,eq) and (bi,si). The spatial information about x and y will be expressed by the relation $\{mi,bi\} \times \mathcal{I} \setminus \{(mi,m),(mi,si),(bi,eq),(bi,si)\}$ which is strongly preconvex and non-convex. As another example we can notice that the quite interesting block relation of $2^{\mathcal{B}_p}$ consisting of all basic relations minus the basic relation $(eq,\ldots,eq)$ is also a strongly preconvex relation which is non-convex. This relation is useful whenever we want to express the constraint: two blocks must be different. Although we do not know the cardinality of the set of strongly preconvex relations there are numerous relations which are strongly preconvex and non–convex.

Since the universal relation $\mathcal{B}_p$ is a convex block relation we can assert that all strongly preconvex relations are preconvex. For $p > 1$ the set of strongly preconvex relations is distinct from the set of preconvex relations and most of the strongly preconvex relations are not Cartesian products of interval relations. Let us add that the set of relations which are Cartesian products of preconvex interval relations (studied in [2]) is strictly included in the set of strongly preconvex relations. Using experimental tests, we could check that the former is much smaller than the latter.

Now we must establish that our last requirement is satisfied.

THEOREM 5.3
The set of strongly preconvex relations is closed under the operations of converse and intersection.

PROOF. We just give a proof for stability with respect to intersection. The stability under the operation of converse is easier.

First, let us prove that the intersection of two strongly preconvex relations α and β is preconvex. We are going to prove that $\mathbf{C}(\alpha\cap\beta)$ is a convex relation by showing that $\mathbf{I}(\mathbf{C}(\alpha\cap\beta)) = \mathbf{C}(\alpha\cap\beta)$. Let us denote by γ the convex relation $\mathbf{I}(\mathbf{C}(\alpha\cap\beta))$. $\alpha\cap\beta \subseteq \gamma$, consequently $\alpha \cap \beta \subseteq \gamma \cap \alpha$ and $\alpha \cap \beta \subseteq \gamma \cap \beta$. Hence $\gamma \subseteq \mathbf{I}(\mathbf{C}(\gamma \cap \alpha))$ and $\gamma \subseteq \mathbf{I}(\mathbf{C}(\gamma \cap \beta))$. Let us recall that γ is a convex relation and that $\alpha,\ \beta$ are two strongly preconvex relations, hence $\gamma \cap \alpha$ and $\gamma \cap \beta$ are preconvex. Therefore $\mathbf{C}(\gamma \cap \alpha)$ and $\mathbf{C}(\gamma \cap \beta)$ are two convex relations. From this fact, we deduce that $\mathbf{I}(\mathbf{C}(\gamma\cap\alpha)) = \mathbf{C}(\gamma\cap\alpha)$ and $\mathbf{I}(\mathbf{C}(\gamma\cap\beta)) = \mathbf{C}(\gamma\cap\beta)$. Hence $\gamma \subseteq \mathbf{C}(\gamma\cap\alpha)$ and $\gamma \subseteq \mathbf{C}(\gamma\cap\beta)$. So $\mathbf{C}(\gamma) \subseteq \mathbf{C}(\gamma\cap\alpha)$ and $\mathbf{C}(\gamma) \subseteq \mathbf{C}(\gamma\cap\beta)$ (Proposition 4.2). Consequently $\mathbf{C}(\gamma) = \mathbf{C}(\gamma \cap \alpha)$ and $\mathbf{C}(\gamma) = \mathbf{C}(\gamma \cap \beta)$. Hence, $dim(\mathbf{C}(\gamma \cap \alpha)) = dim(\mathbf{C}(\gamma)) = dim(\gamma)$ and $dim(\mathbf{C}(\gamma \cap \beta)) = dim(\mathbf{C}(\gamma)) = dim(\gamma)$ (Proposition 4.2 (c)). Hence for each $a \in \gamma$ such that $dim(a) = dim(\gamma)$ we have $a \in \mathbf{C}(\gamma \cap \alpha)$ and $dim(a) = dim(\gamma \cap \alpha)$. Consequently a belongs to $\gamma \cap \alpha$ (Proposition 4.2 (d)) and thus a belongs to α. By a similar way of reasoning, we can assert that $a \in \beta$. Hence $a \in \alpha \cap \beta$. As a result $\gamma \subseteq \mathbf{C}(\alpha \cap \beta)$ (Lemma 4.6). It is obvious that $\mathbf{C}(\alpha \cap \beta) \subseteq \gamma$. Putting the preceding facts together, we conclude that $\mathbf{C}(\alpha \cap \beta) = \gamma$ and consequently that $\mathbf{C}(\alpha \cap \beta)$ is a convex relation. Therefore $\alpha \cap \beta$ is preconvex.

Secondly, let us prove that if α is a strongly preconvex relation and β a convex relation, then $\alpha \cap \beta$ is a strongly preconvex relation. For each convex relation γ we have $\beta \cap \gamma$ is convex relation (Fact 3.2) and therefore a strongly preconvex relation. Consequently from the first result in this proof we can conclude that $\alpha \cap (\beta \cap \gamma)$ is preconvex. Therefore $\alpha \cap \beta$ is a strongly preconvex relation.

Finally, let us prove that if α and β are two strongly preconvex relations then $\alpha \cap \beta$ is a strongly preconvex relation too. From the previous result, for each convex relation γ, $\beta \cap \gamma$ is strongly preconvex. Consequently, because of the first result in this proof $\alpha \cap (\beta \cap \gamma)$ is preconvex. We conclude that $\alpha \cap \beta$ is a strongly preconvex block relation. ∎

A brutal algorithm to check whether a block relation is strongly preconvex follows from the strong preconvexity definition. It consists in computing the intersection of the block relation and each convex block relation then checking whether the result is a preconvex relation. We prove that we can restrict these intersections to a subset of convex block relations.

DEFINITION 5.4
Let ψ_p be the set of convex block relations of $2^{\mathcal{B}_p}$ defined in the following way:

- $\psi_1 = \{\mathcal{B}_1, \{s, eq, si\}, \{f, eq, fi\}, \{m\}, \{mi\}, \{eq\}\}$;
- $\psi_p = \{\alpha^1 \times \ldots \times \alpha^p : \forall i \in 1, \ldots, p, \alpha^i \in \psi_1\}$, for $p > 1$.

Intuitively, each block relation belonging to ψ_p corresponds to the set of basic relations of $\mathcal{B}_p$ satisfying a set of particular boundary equalities. Note that ψ_p contains 6^p elements.

LEMMA 5.5
For each convex non-empty block relation $\alpha \in 2^{\mathcal{B}_p}$ there exists a convex relation $\beta \in \psi_p$ such that $\alpha \subseteq \beta$ and $dim(\alpha) = dim(\beta)$.

PROOF. Let us prove the property for $p = 1$. Let α be a convex interval relation. If $dim(\alpha) = 2$ then we take $\mathcal{B}_1$ for β, if $dim(\alpha) = 1$ then α is included in one of the following relations $\{s, eq, si\}$, $\{f, eq, fi\}$, $\{m\}$,$\{mi\}$, we take this relation for β, if $dim(\alpha) = 0$ then we take $\{eq\}$ for β.

Now, let us prove the proposition for $p > 1$. Let α be a convex block relation of $2^{\mathcal{B}_p}$. Since α is a convex relation, $\alpha = \alpha_1 \times \ldots \times \alpha_p$ with for every $i \in 1, \ldots, p$, α_i which is a convex interval relation. There exists $\beta^i \in \psi_1$ such that $dim(\alpha_i) = dim(\beta^i)$ and $\alpha_i \subseteq \beta^i$. Let β be defined by $\beta = \beta^1 \times \ldots \times \beta^p$. We have $\beta \in \psi_p$, $dim(\beta) = dim(\alpha)$ (Proposition 2.3) and $\alpha \subseteq \beta$. Moreover, β is a convex block relation. ∎

PROPOSITION 5.6
Let $\alpha \in 2^{\mathcal{B}_p}$. α is a strongly preconvex relation iff for each convex relation $\beta \in \psi_p$, $\alpha \cap \beta$ is a preconvex relation.

PROOF. Let $\alpha \in 2^{\mathcal{B}_p}$ such that its intersection with each relation of ψ_p is a preconvex relation. Let β a convex relation of $2^{\mathcal{B}_p}$. We are going to prove that $\mathbf{C}(\alpha \cap \beta) = \mathbf{I}(\mathbf{C}(\alpha \cap \beta))$ and hence that $\mathbf{C}(\alpha \cap \beta)$ is a convex relation. Let $\gamma = \mathbf{I}(\mathbf{C}(\alpha \cap \beta))$. γ is a convex relation, consequently from the previous lemma there exists a convex relation $\delta \in \psi_p$ such that $\gamma \subseteq \delta$ and $dim(\gamma) = dim(\delta)$. It follows that $\gamma \cap \alpha \subseteq \delta \cap \alpha$, hence $\mathbf{I}(\mathbf{C}(\gamma \cap \alpha)) \subseteq \mathbf{I}(\mathbf{C}(\delta \cap \alpha))$. The same is true for β, $\mathbf{I}(\mathbf{C}(\gamma \cap \beta)) \subseteq \mathbf{I}(\mathbf{C}(\delta \cap \beta))$. $\delta \cap \alpha$ and $\delta \cap \beta$ are preconvex relations, consequently $\mathbf{I}(\mathbf{C}(\delta \cap \alpha)) = \mathbf{C}(\delta \cap \alpha)$ and $\mathbf{I}(\mathbf{C}(\delta \cap \beta)) = \mathbf{C}(\delta \cap \beta)$ (Proposition 4.7). It follows that $\mathbf{I}(\mathbf{C}(\gamma \cap \alpha)) \subseteq \mathbf{C}(\delta \cap \alpha)$ and $\mathbf{I}(\mathbf{C}(\gamma \cap \beta)) \subseteq \mathbf{C}(\delta \cap \beta)$. Moreover, as $\alpha \cap \beta \subseteq \gamma \cap \alpha$ and $\alpha \cap \beta \subseteq \gamma \cap \beta$, it results that $\mathbf{I}(\mathbf{C}(\alpha \cap \beta)) \subseteq \mathbf{I}(\mathbf{C}(\gamma \cap \alpha))$ and $\mathbf{I}(\mathbf{C}(\alpha \cap \beta)) \subseteq \mathbf{I}(\mathbf{C}(\gamma \cap \beta))$. Consequently, $\gamma \subseteq \mathbf{C}(\delta \cap \alpha)$ and $\gamma \subseteq \mathbf{C}(\delta \cap \beta)$. Since $dim(\gamma) = dim(\delta)$, we have $dim(\mathbf{C}(\delta \cap \alpha)) = dim(\delta \cap \alpha) = dim(\delta)$ and $dim(\mathbf{C}(\delta \cap \beta)) = dim(\delta \cap \beta) = dim(\delta)$. Hence for each $a \in \gamma$ such that $dim(a) = dim(\gamma)$ we have $a \in \mathbf{C}(\delta \cap \alpha)$ and $dim(a) = dim(\delta \cap \alpha)$. Consequently a belongs to $\delta \cap \alpha$ (Proposition 4.2 (d)) and thus a belongs to α. By a similar way of reasoning, we can assert that $a \in \beta$. Hence $a \in \alpha \cap \beta$. As a result $\gamma \subseteq \mathbf{C}(\alpha \cap \beta)$ (Lemma 4.6). It is obvious that $\mathbf{C}(\alpha \cap \beta) \subseteq \gamma$. Putting the preceding facts together, we conclude that $\mathbf{C}(\alpha \cap \beta) = \gamma$ and consequently that $\mathbf{C}(\alpha \cap \beta)$ is a convex relation. We conclude that $\alpha \cap \beta$ is a preconvex relation.

From strong preconvexity definition, the opposite implication of the proposition is obvious. ∎

Consequently, in order to show that a block relation belonging to $2^{\mathcal{B}_p}$ is a strongly preconvex relation, we just make 6^p intersections with convex relations instead of the 82^p intersections required by the brutal algorithm. For example, for $p = 2$ we must just make 36 intersections instead of 6724, and for $p = 3$, just 108 instead of 551368.

6 ORD-Horn representable block relations

In the interval algebra, preconvex relations and the interval relations expressible with a conjunction of ORD-Horn clauses [17] coincide. For $p \geq 2$, since the preconvex block relations are not closed under the operation of intersection, it is no longer true. So an interesting point is the relationship between strongly preconvex block relations and the ORD-Horn representable block relations. Actually we will prove that these sets are the same. We start by recalling ORD-Horn representable interval relations.

6.1 ORD-Horn representable interval relations

ORD-Horn representable relations correspond to particular sets of clauses [17]. Clauses are built up from variables u, v, etc. — representing rational points — using arithmetical symbols such as $=$, $\leq$, $\neq$ and $\not\leq$ which represent the possible binary relations between two points on the line. A literal is any expression in the form $u \Re v$, where $\Re$ is an arithmetical symbol. The literals $u = v$ and $u \leq v$ are positive literals whereas $u \neq v$ and $u \not\leq v$ are negative literals. A unit clause is a clause with only one literal. A Horn clause is a disjunction of literals containing at most one positive literal. An ORD clause is a disjunction of literals of the form $u = v$, $u \leq v$ or $u \neq v$. We notice that $u \not\leq v$ can be expressed as the conjunction of $v \leq u$ and $v \neq u$. A constraint between two intervals X and Y expressed by a relation of IA can be equivalently given by a conjunction of ORD clauses whose variables are the endpoints of X and Y. Nebel and Bürckert [17] define the ORD-Horn relations of IA by the relations which can be equivalently represented by a conjunction of ORD Horn clauses (an ORD-Horn formula). These relations correspond to Ligozat's preconvex relations. The convex relations are ORD-Horn relations expressible by ORD-Horn formulas Φ using only unit clauses such that if $u \neq v \in \Phi$ then $u \leq v \in \Phi$ or $v \leq u \in \Phi$.

6.2 ORD-Horn representable block relations

An important question is to determine which block relations correspond to *ORD-Horn block relations*, i.e. relations expressible by ORD-Horn formulas whose variables are the endpoints of the orthogonal projections of the blocks. It is obvious that each relation of $2^{\mathcal{B}_p}$ can be represented as a conjunction of ORD clauses. Moreover, we have:

LEMMA 6.1
The convex relations of $2^{\mathcal{B}_p}$ are ORD-Horn relations which can be expressed by ORD-Horn formulas Φ containing only unit clauses.

PROOF. Let α be a convex relation of $2^{\mathcal{B}_p}$. $\alpha = \alpha_1 \times \ldots \times \alpha_p$ and for every $i \in \{1, \ldots, p\}$ α_i is a convex interval relation. So, there exists an ORD-Horn formula Φ_i representing α_i containing only unit clauses. Consequently we can take for Φ the conjunction $\wedge_{i=1}^{i=p} \Phi_i$. ∎

Now, we are in a position to prove the following lemma.

LEMMA 6.2
The strongly preconvex relations of $2^{\mathcal{B}_p}$ are ORD-Horn relations.

PROOF. Let $\alpha \in 2^{\mathcal{B}_p}$ be a strongly preconvex relation. $\mathbf{I}(\alpha)$ is a convex relation, hence it can be expressed as an ORD-Horn formula $\Phi_{\mathbf{I}(\alpha)}$. Let $a \in \mathbf{I}(\alpha) \setminus \alpha$. As α is preconvex, $dim(a) < dim(\mathbf{I}(\alpha))$. The basic relation a forces $d = 2 * p - dim(a)$ endpoint equalities $u_1^1 = u_1^2, \ldots, u_d^1 = u_d^2$. Let β_a be the convex relation corresponding to all the basic relations forcing these equalities. Remark that $dim(a) = dim(\beta_a)$. Let Φ_a be the ORD Horn clause $u_1^1 \neq u_1^2 \vee \ldots \vee u_d^1 \neq u_d^2$. Taking the conjunction $\Phi_a \wedge \Phi_{\mathbf{I}(\alpha)}$ does not allow a but also excludes the basic relations belonging to $\beta_a \cap \alpha$. We must be less restrictive. Let $b \in \beta_a \cap \alpha$. Then $b \in \mathbf{I}(\beta_a \cap \alpha)$. Suppose that $a \in \mathbf{I}(\beta_a \cap \alpha)$. Since α is strongly preconvex and β_a convex, $\beta_a \cap \alpha$ is preconvex. Moreover, $dim(a) \geq dim(\beta_a \cap \alpha)$ because $dim(a) = dim(\beta_a)$. Therefore $a \in \beta_a \cap \alpha$ (Proposition 4.7), from which a contradiction follows. Consequently $a \notin \mathbf{I}(\beta_a \cap \alpha)$. Since $\mathbf{I}(\beta_a \cap \alpha)$ is a convex relation, it can be expressed by a conjunction of unit clauses. a always falsifies a clause in this conjunction. Let us denote by cl_a such a clause. a falsifies the ORD-Horn clause $\Phi_a \vee cl_a$ whereas b satisfies it. Hence α can be expressed by the ORD-Horn formula:

$$\Phi_{\mathbf{I}(\alpha)} \bigwedge_{a \in \mathbf{I}(\alpha) \setminus \alpha} (\Phi_a \vee cl_a).$$

∎

Note that in this proof we used the fact that the strongly preconvex block relations are closed under the operation of intersection. Now let us prove the converse implication:

LEMMA 6.3
The ORD-Horn relations of $2^{\mathcal{B}_p}$ are strongly preconvex.

PROOF. The set of the strongly preconvex relations of $2^{\mathcal{B}_p}$ is closed under the operation of intersection, so we will just prove that the ORD-Horn relations expressed by one ORD Horn clause is a strongly preconvex relation (implicitly we always have the conjunction of unit clauses corresponding to the following constraint: the lower endpoint is strictly inferior to the upper endpoint of an orthogonal projection of a block). Let Φ_α be an ORD-Horn clause $u \Re v \vee w_1^1 \neq w_2^1 \vee \ldots \vee w_1^d \neq w_2^d$, with $\Re \in \{=, \leq\}$ and d a positive integer. The case where Φ_α is a unit clause is easy, so we will suppose that this is not the case so that $d > 0$. Let us denote by α the relation corresponding to Φ_α. Now consider a convex relation β whose corresponding ORD-Horn formula is denoted by Φ_β. Consider the relation $\gamma = \alpha \cap \beta$ expressed by $\Phi_\gamma = \Phi_\alpha \wedge \Phi_\beta$. We are going to prove that γ is preconvex. Without restricting the problem, we may assume that there exists at least one basic relation of γ satisfying $w_1^i \neq w_2^i$ for each $i \in \{1, \ldots, d\}$ (consequently $w_1^1 = w_2^1 \notin \Phi_\beta$). Let us denote by Φ_δ the ORD-Horn formula formed by the conjunction of the unit clauses in Φ_β with equality and let us denote by δ the corresponding convex relation. We have $\beta \subseteq \delta$. Moreover as β is convex and contains γ we have the following inclusion: $\gamma \subseteq \mathbf{I}(\gamma) \subseteq \beta$. Let $a \in \beta$ such that a satisfies $w_1^1 \neq w_2^1$ (we know that such a basic relation exists). We have $a \in \gamma$ and $dim(a) = 2 * p$ minus the number of equalities of Φ_β, hence $dim(a) = dim(\delta)$. It follows that $dim(\gamma) = dim(\delta)$. Therefore $dim(\gamma) = dim(\mathbf{I}(\gamma))$. Now let $b \in \mathbf{I}(\gamma) \setminus \gamma$ such that $dim(b) \geq dim(\gamma)$. We have $dim(b) = dim(\gamma)$. Since $\mathbf{I}(\gamma) \subseteq \beta$, b satisfies the equalities of Φ_β and the equality $w_1^1 = w_2^1$ (else $b \in \gamma$). It follows that $dim(b) < dim(\delta)$. Consequently $dim(b) < dim(\gamma)$, a contradiction. Hence $dim(\mathbf{I}(\gamma) \setminus \gamma) < dim(\gamma)$, so γ is preconvex (Proposition 4.7). Finally, we conclude that α is strongly preconvex. ∎

From both these lemmas we get:

THEOREM 6.4
A relation $\alpha \in 2^{\mathcal{B}_p}$ is strongly preconvex iff α can be expressed by an ORD-Horn formula.

7 Tractability of the strongly preconvex block relations

Now there are two notions to define the same set of block relations: the Ord-Horn representability and the strong preconvexity. By using the first notion and the line of reasoning of Nebel and Bürckert [17] we did not succeed in proving that the set of all ORD-Horn relations is closed under the operation of composition and is a tractable set in the general case. Consequently, we have continued our study by using the second approach: the notion of strong preconvexity, with the help of which we are now in position to prove the tractability of the set of strongly preconvex relations.

The difficulty lies in the fact that it is unclear whether the set of all strongly preconvex relations is closed under the operation of composition. Consequently, applying the path-consistency method to a strongly preconvex network may yield a network which is not strongly preconvex. Hence despite Proposition 4.9 we cannot establish the tractability of strongly preconvex relations. To remedy this we introduce a new property: *weak path-consistency*, which is a weaker property than path-consistency.

DEFINITION 7.1
Let $\mathcal{N} = (V, C)$ be a block network. $\mathcal{N}$ is weakly path-consistent iff $C_{ij} \subseteq \mathbf{I}(C_{ik} \circ C_{kj})$ for each $i, j, k \in \{1, \ldots, |V|\}$.

Of course all path-consistent block networks are weakly path-consistent. However the converse does not hold. For instance the block network $\mathcal{N}$ of Figure 3 is weakly path-consistent but not path-consistent. We call the weak path-consistency method the method which consists in iterating on a block network $\mathcal{N} = (V, C)$ the weak triangulation operation: $C_{ij} := C_{ij} \cap \mathbf{I}(C_{ik} \circ C_{kj})$, until a fixed point is reached. The resulting network is a weakly path-consistent subnetwork of $\mathcal{N}$. Like the usual path-consistency method this method can be implemented in polynomial time ($O(n^3)$). Moreover, it is also sound and not complete since the weak triangulation operation is weaker than the usual one.

Now, we are going to prove a result which is stronger than Proposition 4.9.

PROPOSITION 7.2
If a preconvex block network is weakly path-consistent then either it contains the empty constraint or it has a consistent instantiation of maximal dimension.

PROOF. Let $\mathcal{N} = (V, C)$ be a weakly path-consistent preconvex block network which does not contain the empty constraint. For all $i, j, k \in \{1, \ldots, V\}$, since $C_{ij} \subseteq \mathbf{I}(C_{ik} \circ C_{kj})$ we deduce that $\mathbf{I}(C_{ij}) \subseteq \mathbf{I}(\mathbf{I}(C_{ik} \circ C_{kj}))$. Consequently $\mathbf{I}(C_{ij}) \subseteq \mathbf{I}(C_{ik}) \circ \mathbf{I}(C_{kj})$ (Proposition 3.3). Hence $\mathbf{I}(\mathcal{N})$ is path-consistent and does not contain the empty relation. Therefore $\mathbf{I}(\mathcal{N})$ admits a consistent instantiation of maximal dimension (Proposition 3.7) which is also an instantiation of maximal dimension of $\mathcal{N}$ (Proposition 4.7 (c)). ∎

As the set of strongly preconvex relations is stable under the operation of intersection with convex relations we can assert that applying the weak path-consistency method on a strongly preconvex block network gives a block network which is always strongly preconvex. From this and the previous proposition we can deduce the main result of this paper.

THEOREM 7.3
The consistency problem of a strongly preconvex block network can be solved in polynomial time by means of the weak path-consistency method.

PROOF. Applying the weak path-consistency method on a strongly preconvex block network $\mathcal{N} = (V, C)$ produces an equivalent strongly preconvex and weakly path-consistent network

$\mathcal{N}'$. Consequently, if $\mathcal{N}'$ contains the empty relation then both $\mathcal{N}'$ and $\mathcal{N}$ are inconsistent. Otherwise, as $\mathcal{N}'$ is preconvex we deduce its consistency and the one of $\mathcal{N}$ from the previous proposition. ∎

Notice that because of a result of Nebel and Bürckert [17] the smallest subclass of $2^{\mathcal{B}_p}$ containing the strongly preconvex block relations is also a tractable set.

Moreover, from the previous theorem we can assert the following proposition.

PROPOSITION 7.4
The consistency problem of a strongly preconvex block network can be solved by means of the path-consistency method.

PROOF. Applying the path-consistency method on a block network $\mathcal{N} = (V, C)$ produces an equivalent path-consistent network $\mathcal{N}'$; this one is obtained by triangulation operations characterized by a finite sequence of triples $(i_0, j_0, k_0), \ldots, (i_n, j_n, k_n)$ (where the triple (i_l, j_l, k_l) corresponds to the lth triangulation operation : $C_{i_l j_l} := C_{i_l j_l} \cap (C_{i_l k_l} \circ C_{k_l j_l})$). Let $\mathcal{N}''$ be the block network obtained by making on $\mathcal{N}$ the weak triangulation operations characterized by the same sequence of triples $(i_0, j_0, k_0), \ldots, (i_n, j_n, k_n)$ (where the triple (i_l, j_l, k_l) corresponds to the lth weak triangulation operation : $C_{i_l j_l} := C_{i_l j_l} \cap \mathbf{I}(C_{i_l k_l} \circ C_{k_l j_l})$) and then by applying the weak path-consistency method. Note that $\mathcal{N}''$ is a weak path-consistent strongly preconvex network equivalent to $\mathcal{N}$. By induction on the size of the sequence, we can prove that $\mathcal{N}'$ is a subnetwork of $\mathcal{N}''$. Consequently, if $\mathcal{N}'$ does not contain the empty relation then $\mathcal{N}''$ does not contain the empty relation too, it results that $\mathcal{N}$ is consistent. ∎

We have two polynomial methods to solve the consistency problem of a strongly block network. In practice the weak path-consistency method is faster than the path-consistency method (for the case $p > 1$). This is due to the fact that much of the weak triangulation operation can be computed in the interval algebra since $\mathbf{I}(R \circ S) = \mathbf{I}(R) \times \mathbf{I}(S) = (\mathbf{I}(R_1) \times \ldots \times \mathbf{I}(R_p)) \circ (\mathbf{I}(S_1) \times \ldots \times \mathbf{I}(S_p)) = (\mathbf{I}(R_1) \circ \mathbf{I}(S_1)) \times \ldots \times (\mathbf{I}(R_p) \circ \mathbf{I}(S_p))$. This is not the case for the usual triangulation operation and the calculation of $R \circ S$ becomes very slow when R and S contain lot of basic relations, see [7] for more details.

8 Conclusions

We have defined the block algebra as a set of relations — the block relations — together with the fundamental operations of composition, converse and intersection, for every $p \geq 1$ the dimension of the space). Adapting the line of reasoning suggested by Ligozat, we extended the concepts of convexity and preconvexity to the block algebra. In this paper, we proved that convexity leads to a tractable subset whereas for the general case, i.e. $p > 1$, the consistency problem of the preconvex block networks is an NP-complete problem. We introduced the set of strongly preconvex relations which is larger than the set of convex relations and smaller than the set of preconvex relations, and we proved its tractability. We established the fact that the path-consistency algorithm as well as the weak path-consistency algorithm constitute decision methods for the consistency problem of strongly preconvex networks. We also showed that the strongly preconvex relations and the ORD-Horn relations (in the sense of Nebel and Bürckert [17]) are the same set of relations. The problem of the maximality of this tractable subset remains an open problem. Usually to prove the maximality of a fragment of a relational algebra an extensive machine-generated analysis is used. Because of the huge size of $2^{\mathcal{B}_p}$ we cannot proceed in the same way. Another open question is concerned with the size of the set of strongly preconvex relations. We only know that there are many more strongly preconvex relations than convex block relations and block relations corresponding to the Cartesian product of preconvex interval relations. In another direction, the question also arises as to how the qualitative constraints we have been considering could be integrated into a more general setting to include metric constraints. In [8] we considered metric constraints of STPs [9] and proved that the consistency problem of the strongly preconvex block network augmented with these metric constraints is always polynomial. In conclusion let us indicate that we have implemented several algorithms (in particular the weak path-consistency method) to solve the consistency problem of strongly preconvex networks for $p \in \{1, 2, 3\}$.

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