On the Consistency Problem for the $\mathcal{INDU}$ Calculus

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Abstract

In this paper, we further investigate the consistency problem for the qualitative temporal calculus $\mathcal{INDU}$ introduced by Pujari et al. *[10]. We prove the intractability of the consistency problem for the subset of preconvex relations. On the other hand, we show the tractability of strongly preconvex relations. Furthermore, we also define another interesting set of relations for which the consistency problem can be decided by a method similar to the usual path-consistency method.*

1. Introduction

Temporal reasoning is a central task for numerous applications in many areas such as natural language understanding, specification and verification of programs and systems, scheduling, *etc.* In the field of qualitative reasoning about temporal data, the framework proposed by Allen [1], the Interval Algebra ($\mathcal{IA}$), is one of the best-known models.
Allen considers as basic temporal entities intervals of the time line and bases the $\mathcal{IA}$ calculus on 13 qualitative binary relations which correspond to all possible configurations between the four end-points of two intervals. In the $\mathcal{IA}$ calculus, temporal information can be represented using constraint networks (interval networks) whose variables correspond to intervals and whose constraints are expressed by disjunctions of the basic relations (interval relations). The consistency problem for interval networks is NP-complete. A large amount of research in the recent past has been devoted to the study and characterization of tractable subclasses of the interval algebra (see for example [9, 3]). Now all tractable subclasses of $\mathcal{IA}$ are known.
More recently, a new qualitative formalism, called $\mathcal{INDU}$ has been proposed by Pujari *et al.* [10, 5, 6]. $\mathcal{INDU}$ also considers intervals as temporal entities, but it adds information about the relative durations of the intervals considered to the information expressed by Allen's relations. The resulting calculus has 25 basic relations corresponding to a refinement of Allens's basic relations. Each one of the 25 basic relations of $\mathcal{INDU}$ can therefore be represented as a pair consisting of one of Allen's basic relations and of a basic relation of the Point Algebra ($<$, $>$ or $=$), which expresses the relation between the durations.
From a structural point of view, $\mathcal{INDU}$ and $\mathcal{IA}$ look very similar. This first impression, however, is quite deceptive. The real fact is that there exist numerous differences between the two formalisms. In particular, contrary to the relations of $\mathcal{IA}$, the relations of $\mathcal{INDU}$ are not closed for the composition operation. We can also show that the consistency problem for $\mathcal{INDU}$ networks whose constraints are singleton relations cannot be decided by means of the well known path-consistency method.
In this paper, we are mainly interested in the consistency problem for $\mathcal{INDU}$ networks. Our aim is to characterize several important tractable sets for this problem. To this end we define the set of convex relations of $\mathcal{INDU}$ (in a way which is different from that used by Pujari *et al.*), the set of preconvex relations, and a subset of the latter, the set of strongly preconvex relations [2]. On the negative side, we prove that the consistency problem for $\mathcal{INDU}$ networks whose constraints are preconvex relations is NP-complete. On the positive side, we show that strongly preconvex relations can be expressed as conjunctions of Horn clauses [4] and, as a consequence, that the corresponding consistency problem is tractable. We also show that the usual method based on path-consistency cannot be used for strongly preconvex relations. On the other hand, we define an interesting subclass of $\mathcal{INDU}$ relations for which the consistency problem can be decided by means of that method.

2. The $\mathcal{INDU}$ calculus: an extension of $\mathcal{IA}$

2.1. The $\mathcal{INDU}$ relations

The framework introduced by Pujari *et al.* [10], called $\mathcal{INDU}$, is an extension of the well-known Allen's calculus

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[1], *viz.* the Interval Algebra ($\mathcal{IA}$).

$\mathcal{IA}$ considers the intervals of the line and is based onto 13 binary relations. Each relation corresponds to a particular relative position between two intervals. An interval can be before (b), meet (m), overlap (o), contain (di), start (s), finish (f), be equal to (eq), another interval (the remaining six relations correspond to the converses of the first six relations). We denote by IA the set of these 13 relations: $\mathsf{IA} = \{b, bi, m, mi, o, oi, s, si, f, fi, d, di, eq\}$. Vilain *et al.* have defined a similar but less expressive formalism, the Point Algebra ($\mathcal{PA}$). $\mathcal{PA}$ is based onto three basic relations $\mathsf{PA} = \{<, >, =\}$, corresponding to the three possible relative positions between two points of the line.

$\mathcal{INDU}$ is based on 25 binary relations between intervals of the line. Each one is defined as a pair of relations. The first relation is a basic relation of IA and denotes the relative position satisfied by the two intervals, the second one is a basic relation of $\mathsf{PA}$and corresponds to the relation satisfied by the durations of the two intervals. Each basic relation of $\mathcal{INDU}$ will be denoted by an expression i^p where $i \in \mathsf{IA}$ and $p \in \mathsf{PA}$ Two intervals x and y satisfy i^p, denoted by $x\ i^p\ y$, iff $x\ i\ y$ and $(x^+ - x^-)\ p\ (y^+ - y^+)$, with x^-, y^- (resp. x^+, y^+) the lower (resp. upper) endpoints of x and y. For example, the intervals[1] $(1, 2)$ and $(2, 4)$ satisfy the basic relation $m^<$. We will denote by INDU the set of the 25 basic $\mathcal{INDU}$ relations, $\mathsf{INDU} = \{b^<, b^>, b^=, bi^<, bi^>, bi^=, m^<, m^>, m^=, mi^<, mi^>, mi^=, o^<, o^>, o^=, oi^<, oi^>, oi^=, s^<, si^>, d^<, di^>, f^<, fi^>, eq^=\}$. Any two intervals satisfy one and only one relation of INDU. In the sequel, we will also consider other basic relations which we call *virtual*. These particular basic relations are the unsatisfiable relations i^p. This set corresponds to $\{eq^<, eq^>, d^=, d^>, di^=, di^<, f^=, f^>, fi^=, fi^<, s^=, s^>, si^=, si^<\}$.

From the 25 basic relations of INDU, 2^{25} relations are defined. Each one corresponds to an element of the power set of INDU, *i.e.* 2^{INDU}. Let $r \in 2^{\mathsf{INDU}}$, and x and y be two intervals, $x\ r\ y$ iff $x\ a\ y$ for some $a \in r$. For instance, $(1, 2)$ and $(3, 4)$ satisfy the relation $\{b^<, b^=, m^=, d^<\}$. The particular relation $\{\}$ is called the empty relation. Each relation $r \in 2^{\mathsf{INDU}}$ can be seen as the disjunction of the basic relations which it contains. Similarly, 2^{IA} defines 2^{13} interval relations and 2^{PA} defines 2^3 relations.

In some cases, the constraint "two intervals x and y must satisfy a relation $r \in \mathsf{INDU}$" can be expressed by a conjunction of Horn clauses. A Horn clause [4] is a disjunction of literals (inequations or inequalities) of the following forms: $c_1u_1 + \ldots + c_nu_n \leq c_{n+1}u_{n+1} + \ldots + c_{n+k}u_{n+k}$ or $c_1u_1 + \ldots + c_nu_n \neq c_{n+1}u_{n+1} + \ldots + c_{n+k}u_{n+k}$, with at most one literal using $\leq$ (each c_i is a rational number, each u_i is a rational variable). The basic relations belonging to INDU can be expressed by a conjunction of unitary Horn clauses (one literal by Horn clause). For example, the satisfaction of $m^<$ by two intervals $x = (x^-, x^+)$ and $y = (y^-, y^+)$ can be expressed by the following conjunction of unitary Horn clauses: $(x^- \leq x^+) \wedge (x^- \neq x^+) \wedge (y^- \leq y^+) \wedge (y^- \neq y^+) \wedge (x^+ \leq y^-) \wedge (y^- \leq x^+) \wedge (x^+ - x^- \leq y^+ - y^-) \wedge (x^+ - x^- \neq y^+ - y^-)$. A ORD Horn clause is a Horn clause which contains only literals of the forms $u \neq v$ or $u \leq v$, with u, v two endpoints. All relations of 2^{PA} can be expressed be a conjunction of ORD Horn clauses, those of 2^{IA} having this property are called the ORD Horn relations [9].

In the same way as for the basic relations of IA, the basic relations of $\mathcal{INDU}$ can be represented by regions of the plane equipped with an orthogonal basis. Each interval $x = (x^-, x^+)$ is represented by a point with coordinates (x^-, x^+). Given an interval of reference $y = (y^-, y^+)$, the region representing the basic relation $a \in \mathsf{INDU}$, denoted by $\mathsf{Reg}(a, y)$, corresponds to the set of points $\{x = (x^-, x^+) : x\ a\ y\}$. We obtain 25 convex regions partitioning the half plane $\mathsf{H} = \{(x, y) : y > x\}$ (see Fig. 1). The geometrical representation of $r \in 2^{\mathsf{INDU}}$ is the region defined by $\mathsf{Reg}(r, y) = \bigcup_{a \in r}\{\mathsf{Reg}(a, y)\}$. When the choice of the point of reference y is not significant, we will denote this region by $\mathsf{Reg}(r)$.

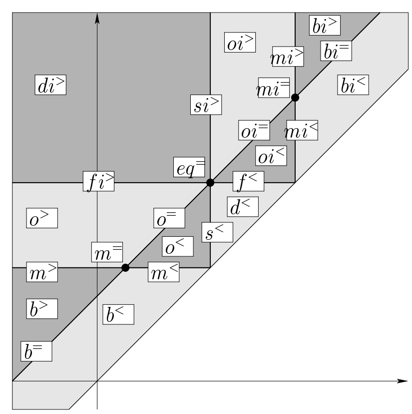

Figure 1. Geometrical representation of the basic relations of $\mathcal{INDU}$.

2.2. The fundamental operations

As for relations $\mathcal{IA}$ and $\mathcal{PA}$ we define operations allowing us to manipulate relations of $\mathcal{INDU}$. Considering a relation of 2^{INDU} as a usual binary relation defined on intervals, the operations union ($\cup$), intersection ($\cap$), converse

[1] We will consider the closed intervals of the rational number line. Such an interval will be denoted by (a, b) with a and b two rational numbers satisfying $a < b$.

($^{-1}$) and composition ($\circ$) can be defined in the usual way: $x(r \cap s)y$ iff $x\ r\ y$ and $x\ s\ y$; $x(r \cup s)y$ iff $x\ r\ y$ or $x\ s\ y$; $x(r \circ s)y$ iff $\exists z,\ x\ r\ z$ and $z\ s\ y$; $x\ r^{-1}\ y$ iff $y\ r\ x$. We can show that $r \cap s = \{a \in \mathsf{INDU} : a \in r$ and $a \in s\}$, $r \cup s = \{a \in \mathsf{INDU} : a \in r$ or $a \in s\}$. The converse of a singleton relation is a singleton relation, like for IA and PA and can be defined by $\{i^p\}^{-1} = \{(i^{-1})^{p^{-1}}\}$, with $i^p \in \mathsf{INDU}$ (we use operation symbols for the relations of 2^{IA} and 2^{PA} similar to the ones used for the $\mathcal{INDU}$ relations). Hence, $r^{-1} = \bigcup_{a \in r}\{a^{-1}\}$. Hence, 2^{INDU} is closed for $\cap$, $\cup$ and $^{-1}$. The $\mathcal{INDU}$ composition operation has an unusual behavior for a qualitative formalism. Indeed, unlike the relations of $\mathcal{IA}$ and of $\mathcal{PA}$, 2^{INDU} is not closed for this operation. Consider the relation $r = \{m^=\}$, the pair of intervals formed by $(1,2)$ and $(3,4)$ belongs to $r \circ r$ as $(1,2)$ and $(2,3)$ satisfy $m^=$, moreover, $(2,3)$ and $(3,4)$ satisfy $m^=$. Assuming that 2^{INDU} is closed for the composition operation, from the fact that $(1,2)$ and $(3,4)$ satisfy the relation $b^=$, $b^= \in r \circ r$. Now, given a pair of intervals (x,y) satisfying the basic relation $b^=$, an interval z such that $x\ m^=\ z$ and $z\ m^=\ y$ may not exist. For example, this is the case for $x = (0,1)$ and $y = (4,5)$. So, the composition operation is inadequate for qualitative reasoning in $\mathcal{INDU}$ since *basic building blocks* must be the basic relations. It is necessary to define a weaker operation, sometimes called *weak composition*, for which 2^{INDU} is closed, we will denote it by $\diamond$. The operation $\diamond$ can be defined by: $a \diamond b = \{c \in \mathsf{INDU} : \exists x,y,z$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$, with $a,b \in \mathsf{INDU}$, $r \diamond s = \bigcup_{a \in r, b \in s}\{a \diamond b\}$, with $r,s \in 2^{\mathsf{INDU}}$. $r \diamond s$ is the smallest relation of 2^{INDU} including $r \circ s$. Note that $\circ$ and $\diamond$ are the same operations for $\mathcal{IA}$ and $\mathcal{PA}$. The operation $\diamond$ is not associative, for instance, $(\{bi^>\}\diamond\{mi^>\})\diamond\{m^>\} = \{oi^>, mi^>, bi^>\}$ and $\{bi^>\}\diamond(\{mi^>\}\diamond\{m^>\}) = \{bi^>\}$. As a result $\diamond$ cannot be used to define a relation algebra [11] on INDU. We also define a binary operation corresponding to the Cartesian product of an interval relation and a point relation by: $r \times s = \{i^p : i \in r, p \in s\}$. This relation can contain virtual basic relations of $\mathcal{INDU}$. Note that for $i^p, j^q \in \mathsf{INDU}$, $i^p \diamond j^q = ((i \circ j) \times (p \circ q)) \cap \mathsf{INDU}$. The interval and point projections of an $\mathcal{INDU}$ relation r, denoted respectively, by r_I and r_P are defined by $r_I = \{i : i^p \in r\}$, $r_P = \{p : i^p \in r\}$. In the sequel we will say that a subset of relations of 2^{INDU} is a subclass iff it is closed for the operations $^{-1}$, $\diamond$, and $\cap$.

3. Qualitative constraint networks

3.1. Definitions

An $\mathcal{INDU}$ constraint network is a pair $\mathcal{N} = (V, C)$, where:

- V is a finite set $\{V_1, \ldots, V_n\}$ (with $n = |V|$) of variables representing intervals of the line,
- C is a mapping associating with each pair $V_i, V_j \in V$ a constraint, denoted by C_{ij}, defined by a relation of 2^{INDU}.

We assume that ${C_{ij}}^{-1} = C_{ji}$ and $C_{ii} = \{eq^=\}$.
We suppose that the constraint networks of $\mathcal{IA}$ (the interval networks) and the constraint networks of $\mathcal{PA}$ (the point networks), are defined in a similar way. An atomic network is a network whose constraints are defined by singleton relations.

Definition 1 *Let $\mathcal{N} = (V, C)$ be a constraint network of $\mathcal{INDU}$ with $n = |V|$. An instantiation m of $\mathcal{N}$ is a mapping which associates an interval (m_i) with each variable $V_i \in V$. The basic relation of* INDU *satisfied by m_i and m_j will be denoted by m_{ij}. The instantiation m will be called a consistent instantiation or a solution of $\mathcal{N}$ iff for every pair of variables $V_i, V_j \in V$, $m_{ij} \in C_{ij}$. In the case where $\mathcal{N}$ has a solution, $\mathcal{N}$ will be said to be consistent. $\mathcal{N}$ is $k-$consistent (with $k \in \{1, \ldots, n\}$) iff any partial consistent instantiation on $k-1$ variables can be extended to a new variable while remaining consistent. $\mathcal{N}$ will be said $\diamond$-closed iff for each $i,j,k \in \{1,\ldots,n\}$, $C_{ij} \subseteq C_{ik} \diamond C_{kj}$. A subnetwork $\mathcal{N}' = (V', C')$ is a network such that $V' = V$ and $C'_{ij} \subseteq C_{ij}$ for each pair of variables V_i and V_j. A network $\mathcal{N}'' = (V, C'')$ is equivalent to $\mathcal{N}$ iff $\mathcal{N}$ and $\mathcal{N}''$ have the same solutions.*

Given a constraint network, the main problem is to decide whether it admits a consistent instantiation. This problem is called the consistency problem.
Given a set $\mathcal{E} \subseteq 2^{\mathsf{INDU}}$ (closed for converse and containing the relation $\{eq^=\}$), the consistency problem for the $\mathcal{INDU}$ networks whose constraints belong to $\mathcal{E}$ will be denoted by $\mathsf{Cons}(\mathcal{E})$. The consistency problem for interval networks being NP-complete, obviously $\mathsf{Cons}(2^{\mathsf{INDU}})$ is also NP-complete.
We call $\diamond$-closure method, the method which consists in obtaining from a network $\mathcal{N} = (V, C)$ an equivalent and $\diamond$-closed subnetwork $\mathcal{N}'$ by iterating the operation $C_{ij} := C_{ij} \cap (C_{ik} \diamond C_{kj})$, for $i,j,k \in \{1,\ldots,|V|\}$, until a fixpoint is obtained. This method can be implemented in $O(n^3)$ time (with $n = |V|$) by an algorithm similar to those used to obtain equivalent path-consistent constraint subnetworks from binary constraint networks [8].

3.2. Consistency and the $\diamond$-closure method

Given that the set 2^{INDU} is not closed for the composition operation, several fundamental properties of networks of $\mathcal{IA}$ and $\mathcal{PA}$ are not true anymore in the framework of $\mathcal{INDU}$.

Proposition 1 *Let $\mathcal{N}$ be an $\mathcal{INDU}$ network. A 3-consistent network $\mathcal{N}'$ equivalent to $\mathcal{N}$ may not exist.*

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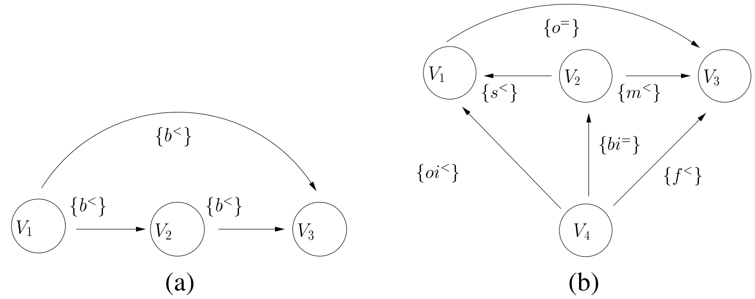


Figure 2. (a) a not 3-consistent atomic network, (b) an atomic network which is 3-consistent and not consistent.

The atomic $\mathcal{INDU}$ network depicted in Fig. 2 (a) is $\diamond$-closed and consistent but not 3-consistent. Consider the partial solution $m(V_1) = (1, 2)$, $m(V_3) = (3, 5)$. It cannot be extended to V_2. This network is consistent and does not admit an equivalent 3-consistent network. Nevertheless, there are 3-consistent $\mathcal{INDU}$ networks, in particular some atomic networks. What about their consistency ? Actually, it is possible to exhibit 3-consistent atomic networks of $\mathcal{INDU}$ which are not consistent, see Fig. 2 (b). These networks are $\diamond$-closed. Consequently, we can state the following property:

Proposition 2 *A $\diamond$-closed (atomic) $\mathcal{INDU}$ network which does not contain the empty constraint is not necessarily consistent.*

For $\mathcal{IA}$ it has been established that the complexity of $\mathsf{Cons}(\mathcal{E})$, with $\mathcal{E} \in 2^{\mathsf{IA}}$, is the same as the complexity of $\mathsf{Cons}(\overline{\mathcal{E}})$, with $\overline{\mathcal{E}}$ the closure of $\mathcal{E}$ for the operations converse, intersection and composition. The proof of this result is based on the fact that from an interval network whose constraints belong to $\overline{\mathcal{E}}$ we can always build an equivalent network from $\mathcal{E}$. By substituting $\diamond$ for the composition operation, we can no longer establish this property for $\mathcal{INDU}$. This comes from the fact that if $x\ (r \diamond s)\ y$ then a third interval z such that $x\ r\ z$ and $z\ s\ y$ may not exist (a necessary property for building of the equivalent network on $\mathcal{E}$ from the network on $\overline{\mathcal{E}}$). Nevertheless, we can establish a weaker property:

Proposition 3 *Let $\mathcal{E} \in 2^{\mathsf{INDU}}$ (closed for the converse operation and containing the singleton relation $\{eq^=\}$). $\mathsf{Cons}(\mathcal{E})$ has the same time complexity as $\mathsf{Cons}(\mathcal{E}^*)$, where * denotes the closure for the intersection operation.*

To prove this proposition, the idea is to replace the constraint $x\ (r \cap s)\ y$ by the constraints $x\ r\ y$, $x\ s\ z$ and $y\ \{eq^=\}\ z$, with z a new variable.
In the sequel of this paper, we are going to characterize several sets of $\mathcal{INDU}$ for which the consistency problem is polynomial. Several cases of tractability can be obtained in a direct way from the tractable cases of $\mathcal{IA}$:

Proposition 4 *Let $\mathcal{E} \subseteq 2^{\mathsf{IA}}$ be a set for which the consistency problem is polynomial. Let $\mathcal{E}' \subseteq 2^{\mathsf{INDU}}$ be defined by $\mathcal{E}' = \{(r \times \{<, =, >\}) \cap \mathsf{INDU} : r \in \mathcal{E}\}$. $\mathsf{Cons}(\mathcal{E}')$ is polynomial.*

Let us now establish less trivial tractability cases.

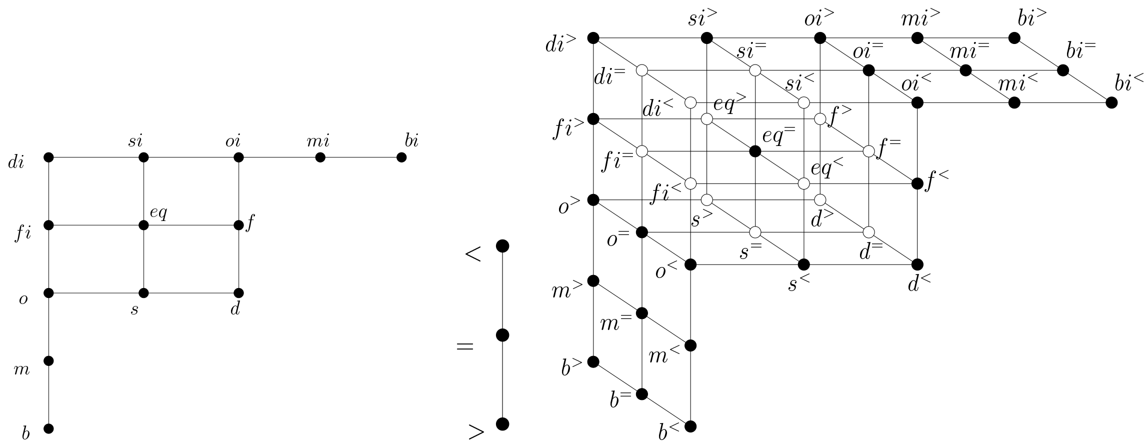
4. Convex relations in the $\mathcal{INDU}$ Calculus

4.1. Definition and representation

The convex relations of $\mathcal{IA}$ (resp. $\mathcal{PA}$) correspond to the intervals of the interval lattice (resp. the point lattice) (see Fig. 3 (a) and (b)). In a natural way, we define the $\mathcal{INDU}$ lattice by the Cartesian product of the interval lattice and the point lattice (see Fig. 3). This lattice is also defined on the virtual basic relations of $\mathcal{INDU}$. We define the set of convex relations of 2^{INDU}, denoted by $\mathcal{C}$, in the following way:

Definition 2 *Let $r \in 2^{\mathsf{INDU}}$. $r \in \mathcal{C}$ iff $r = [min, max] \cap \mathsf{INDU}$ with $[min, max]$ an interval of the $\mathcal{INDU}$ lattice.*

Remark that $r \in 2^{\mathsf{INDU}}$ is convex iff $r = (s \times t) \cap \mathsf{INDU}$ with s and t convex relations of 2^{IA} and 2^{PA}. It follows that from a geometrical point of view, a relation r of $\mathcal{INDU}$ is a convex relation when its geometrical representation in the plane $\mathsf{Reg}(r)$ satisfies the following equality: $\exists h \in \mathcal{H}, \mathsf{Reg}(r) = (Proj_1(\mathsf{Reg}(r)) \times Proj_2(\mathsf{Reg}(r))) \cap h$, with $\mathcal{H} = \{\mathsf{Reg}(\mathsf{INDU}), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{<\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{<, =\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{>\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{>, =\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{=\}))\}$ and with $Proj_1$ (resp. $Proj_2$) denoting the projection function on the horizontal axis (resp. vertical axis).
Like for the convex relations of $\mathcal{IA}$ and of $\mathcal{PA}$ we have the following property:



(a) (b) (c)

Figure 3. (a) the interval lattice, (b) the point lattice, (c) the $\mathcal{INDU}$ lattice.

Proposition 5 *Let $r \in$ INDU. r is a convex relation iff r can be equivalently expressed by a conjunction of unitary Horn clauses Φ such that if $u \neq v \in \Phi$ then $u \leq v \in \Phi$ or $v \leq u \in \Phi$ (with u and v denoting endpoints or differences of endpoints).*

Remark that a convex relation of $\mathcal{INDU}$ cannot always be represented by a conjunction of unitary ORD Horn clauses. Pujari *et al.* enumerate 227 convex relations. On the other hand, we enumerate 240 convex relations, this difference arising from the fact that Pujari *et al.* use a lattice which does not take into account the virtual basic relations. The set $\mathcal{C}$ is closed for $^{-1}$, $\cap$, but not closed for $\diamond$. The closure for the intersection and converse follows directly from the definition. The following example shows the instability of $\mathcal{C}$ for $\diamond$: $\{b^<\}\diamond\{d^<, o^<, o^>, o^=, s^<\}$ is the non-convex relation $\{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$. Some relations of $\mathcal{C}$ can be expressed by conjunctions of ORD Horn clauses. These relations, denoted by the set $\mathcal{C}_{IA}$, correspond to convex relations of $\mathcal{IA}$. $\mathcal{C}_{IA}$ can be defined by:

Proposition 6 *Let $r \in 2^{\mathsf{INDU}}$. $r \in \mathcal{C}_{IA}$ iff r satisfies one of the equivalent properties:*

- *$r = (s \times \{<, =, >\}) \cap$ INDU with s a convex interval relation,*
- *$r = [a^<, b^>] \cap$ INDU, with $[a^<, b^>]$ an interval of the $\mathcal{INDU}$ lattice ($a, b \in$ IA).*

Given that the set $\mathcal{C}_{IA}$ corresponds to the convex interval relations, the operation $\diamond$ on $\mathcal{C}_{IA}$ corresponds to the composition operation $\circ$. Moreover, it follows that $\mathcal{C}_{IA}$ is a subclass.

4.2. Tractability of the convex $\mathcal{INDU}$ relations

The convex $\mathcal{INDU}$ relations can be represented by conjunctions of unitary Horn clauses, consequently the consistency problem of the convex $\mathcal{INDU}$ networks ($\mathcal{INDU}$ networks whose constraints are convex) is polynomial. We can translate this kind of network into conjunctions of Horn clauses and apply a resolution algorithm such as the algorithm proposed by Koubarakis [4]. Notice that we can also use the Simplex algorithm or Kachian's algorithm for solving these particular constraints (as a consequence of Prop. 5).

Proposition 7 Cons($\mathcal{C}$) *is a polynomial problem.*

It is well known that the path-consistency method can be used to solve the consistency problem of the convex interval networks. Hence we have the following property:

Proposition 8 Cons($\mathcal{C}_{IA}$) *can be decided by the $\diamond$-closure method.*

5. The preconvex $\mathcal{INDU}$ relations

The maximal tractable set of $\mathcal{IA}$ containing the 13 singleton interval relations is the set of preconvex interval relations, which is identical with the set of ORD Horn interval relations. To define the preconvex relations of $\mathcal{INDU}$ we use the definition given by Ligozat [7] by extending the notions of convex closure and dimension to $\mathcal{INDU}$. The dimension of an interval relation corresponds to the dimension of the geometrical representation of this region in the plane. This dimension is the maximal dimension of the dimensions of its basic relations. In a similar way, we give the following definition:

Definition 3 *Let* $a \in \mathsf{INDU}$. *The dimension of* a, *denoted by* $dim(a)$, *is the dimension of* $\mathsf{Reg}(a)$. *If* $r \in 2^{\mathsf{INDU}}$ *is a non-empty relation,* $dim(r) = max\{dim(a) : a \in r\}$.

We define $dim(\{\})$ as -1. As an illustration, $dim(\{m^=, o^<\}) = max\{1, 2\} = 2$.
The convex closure of an $\mathcal{INDU}$ relation r, denoted by $I(r)$, is the smallest convex relation of $\mathcal{C}$ containing r. Notice that this particular relation exists since the set $\mathcal{C}$ is closed for $\cap$.

Definition 4 *Let* $r \in 2^{\mathsf{INDU}}$. $I(r) = \bigcap\{s \in \mathcal{C} : r \subseteq s\}$.

The convex closure of $\mathcal{INDU}$ can be computed from the convex closures in $\mathcal{IA}$ and $\mathcal{PA}$:

Proposition 9 *Let* $r \in 2^{\mathsf{INDU}}$. $I(r) = (I(r_I) \times I(r_P)) \cap \mathsf{INDU}$.

Now, we can define the preconvex relations of $\mathcal{INDU}$.

Definition 5 *Let* $r \in 2^{\mathsf{INDU}}$. r *is a preconvex relation iff* $r = \{\}$ *or* $dim(I(r) \setminus r) < dim(r)$.

We will denote by $\mathcal{P}$ the set of preconvex $\mathcal{INDU}$ relations. $\mathcal{P}$ contains 88096 relations. The convex $\mathcal{INDU}$ relations belong to this set. The set $\mathcal{P}$ is closed for $^{-1}$, but not closed for the operations $\cap$ and $\diamond$. Consider the preconvex $\mathcal{INDU}$ relations $r = \{eq^=, b^<, b^=, o^<\}$, $s = \{eq^=, b^>, b^=, o^>\}$, $t = \{b^<\}$ and $u = \{d^<, o^<, o^>\}$. Then the relations $r \cap s = \{eq^=, b^=\}$ and $t \diamond u = \{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$ are not preconvex relations.

5.1. Intractability of the preconvex $\mathcal{INDU}$ relations

In this section we prove that the consistency problem for $\mathcal{P}$ is NP-complete. In order to do so, we define a polynomial reduction from the 3-coloring problem of a graph to $\mathsf{Cons}(\mathcal{P}^*)$.

Proposition 10 $\mathsf{Cons}(\mathcal{P}^*)$ *is a NP-complete problem.*

Proof. Let $G = (S, A)$ be a non-oriented graph, with S a set of vertices and A a set of edges between these vertices. We build an $\mathcal{INDU}$ network $\mathcal{N} = (V, C)$ in the following way: V is a set of variables corresponding to the union of $Col = \{Col_1, Col_2, Col_3\}$ and $V_S = \{S_1, \ldots, S_n\}$ with $n = |S|$. Each variable of Col is associated with a color. Each variable $S_i \in V_S$ is associated with a vertex $s_i \in S$. The constraints of $\mathcal{N}$ between the three variables of Col are given in Fig. 4 (a). Those between two variables S_i and S_j such that $(s_i, s_j) \in A$ (resp. $\notin A$) are given in Fig. 4 (b) (resp. (c)). We can check that these constraints belong to $\mathcal{P}^*$. For example, the relation $\{m^=, eq^=, mi^=\}$ is the intersection of the preconvex relations $\{o^>, di^>, oi^>, m^=, eq^=, mi^=\}$ and $\{o^<, d^<, o^<, m^=, eq^=, mi^=\}$. We can prove that $G = (S, A)$ is 3-colorable iff $\mathcal{N}$ is consistent. Given a solution

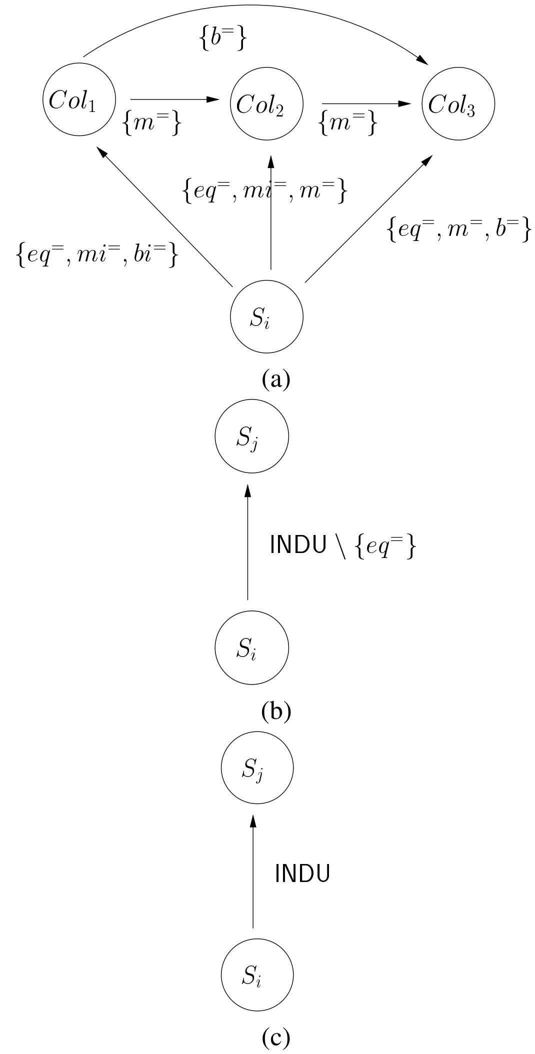


Figure 4. The constraints of $\mathcal{N} = (V, C)$

of the 3-coloring problem for G, it is sufficient to assign to each variable S_i the interval corresponding to the color assigned to the vertex s_i. Conversely, to obtain a solution of the 3-coloring problem for G from a solution of $\mathcal{N}$, it suffices to assign to the vertex s_i the color corresponding to the interval assigned to S_i. $\dashv$
From this proposition and Proposition 3 we can assert:

Theorem 1 $\mathsf{Cons}(\mathcal{P})$ *is a NP-complete problem.*

6. The strongly preconvex $\mathcal{INDU}$ relations

Balbiani *et al.* have proved that for the strongly preconvex generalized interval relations the consistency problem is polynomial [2]. Following the line of reasoning given by Balbiani *et al.*, we define the strongly preconvex relations of $\mathcal{INDU}$.

Definition 6 *Let $r \in 2^{\mathsf{INDU}}$. r is strongly preconvex iff for each convex relation $s \in \mathcal{C}$, the relation $r \cap s$ is a preconvex relation.*

$\mathcal{F}$ will denote the set of the strongly preconvex relations of $\mathcal{INDU}$. It has 45792 elements.

Proposition 11 *The set $\mathcal{F}$ is closed[2] for the operations $^{-1}$, $\cap$, but not closed for $\diamond$.*

Consider the strongly preconvex relations $r = \{b^<\}$ and $s = \{d^<, o^<, o^>\}$, $r \diamond s$ is the relation $\{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$ which is not a strongly preconvex relation.

6.1. Tractability of the strongly preconvex $\mathcal{INDU}$ relations

This section is devoted to the nice properties of the strongly preconvex $\mathcal{INDU}$ relations in relation to the consistency problems.

Proposition 12 *The strongly preconvex $\mathcal{INDU}$ relations can be represented by conjunctions of Horn clauses.*

Proof. Let $r \in \mathcal{F}$. As $I(r)$ is convex, there exists a conjunction of Horn clauses representing it. Let us denote by $\Phi_{I(r)}$ such a conjunction. In the general case, $\Phi_{I(r)}$ is too permissive. Indeed, a basic relation $a \in I(r) \setminus r$ is realizable w.r.t. $\Phi_{I(r)}$. We must forbid these basic relations, without forbidding the basic relations of r. Let $a \in I(r) \setminus r$. Let us define for all possible a an Horn clause, denoted by Φ_a, such that the addition of Φ_a to Φ allows to exclude the satisfaction of a without excluding the satisfaction of the atomic relations belonging to r. r is preconvex, so $dim(I(r) \setminus r) < dim(r)$, then $dim(a) < dim(r)$. It follows that $dim(a) = 1$ or 0. First, let us consider the basic relations which do not impose equality between the interval durations.

$\Phi_{m^<} = (x^+ \neq y^- \vee x^+ - x^- \geq y^+ - y^-)$,
$\Phi_{m^>} = (x^+ \neq y^- \vee x^+ - x^- \leq y^+ - y^-)$,
$\Phi_{mi^<} = (y^+ \neq x^- \vee x^+ - x^- \geq y^+ - y^-)$,
$\Phi_{mi^>} = (y^+ \neq x^- \vee x^+ - x^- \leq y^+ - y^-)$,
$\Phi_{s^<} = (x^- \neq y^- \vee x^+ - x^- \geq y^+ - y^-)$,
$\Phi_{si^>} = (x^- \neq y^- \vee x^+ - x^- \leq y^+ - y^-)$,
$\Phi_{f^<} = (x^+ \neq y^+ \vee x^+ - x^- \geq y^+ - y^-)$,
$\Phi_{fi^>} = (x^+ \neq y^+ \vee x^+ - x^- \leq y^+ - y^-)$,

Now, let us consider the basic relations imposing equality of the interval durations. These atomic relations belong to the convex relation $s = \{eq^=, b^=, bi^=, o^=, oi^=, m^=, mi^=\}$. This is why Φ_a will always contains $x^+ - x^- \neq y^+ - y^-$ (except for Φ_{eq}).

$\Phi_{b^=} = (x^+ - x^- \neq y^+ - y^- \vee x^+ \geq y^-)$,
$\Phi_{bi^=} = (x^+ - x^- \neq y^+ - y^- \vee y^+ \geq x^-)$,
$\Phi_{m^=} = (x^+ - x^- \neq y^+ - y^- \vee x^+ \neq y^-)$,
$\Phi_{mi^=} = (x^+ - x^- \neq y^+ - y^- \vee y^+ \neq x^-)$,
$\Phi_{eq^=} = (x^- \neq y^- \vee x^+ \neq y^+)$.

The cases of the basic relations $o^=$ and $oi^=$ remain. Consider the case $a = o^=$ (the case $oi^=$ is similar). Suppose that $r \cap \{b^=, m^=\} \neq \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} \neq \emptyset$. Hence $a \in I(r \cap s)$. Moreover, we know that $a \notin r$. As a consequence $dim(I(r \cap s) \setminus (r \cap s)) \geq 1$. Since $r \cap s \subseteq s$ and $dim(s) = 1$, $dim(I(r \cap s) \setminus (r \cap s)) \leq 1$. Hence, $dim(r \cap s) \leq dim(I(r \cap s) \setminus (r \cap s))$ and $r \cap s$ is not preconvex. There is a contradiction (r is a strongly preconvex relation). Hence, only three cases can hold:
• $r \cap s = \emptyset$. $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^-$,
• $r \cap \{b^=, m^=\} \neq \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} = \emptyset$. $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^- \vee x+ \leq y^-$,
• $r \cap \{b^=, m^=\} = \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} \neq \emptyset$. $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^- \vee x+ \geq y^+$.
r can be represented by the conjunction of Horn clauses $\Phi_{I(r)} \wedge \bigwedge_{a \in (I(r) \setminus r)} \Phi_a$. $\dashv$
From this, we can assert that:

Theorem 2 $\mathsf{Cons}(\mathcal{F})$ *is a polynomial problem.*

7. The tractable subclass $\mathcal{G}$

In this section we characterize a new subset of preconvex relations for which the $\diamond$-closure method gives a decision method for the consistency problem (unlike the $\mathcal{F}$ case). We will denote this set by $\mathcal{G}$. The definition of $\mathcal{G}$ was guided by our desire to obtain preconvex relations forming a subclass for which the convex closures are convex interval relations.

Definition 7 *Let $r \in 2^{\mathsf{INDU}}$. r belongs to $\mathcal{G}$ iff for each convex relation $s \in \mathcal{C}_{IA}$ $r \cap s$ is a preconvex relation and $I(r \cap s)$ is a convex relation which belongs to the set $\mathcal{C}_{IA}$.*

The set $\mathcal{G}$ forms a subclass containing 11854 relations.

Proposition 13 *The set $\mathcal{G}$ is closed for the operations $^{-1}$, $\cap$ and $\diamond$.*

From the fact that the universal relation INDU belongs to $\mathcal{C}_{IA}$, we can deduce that each relation of $\mathcal{G}$ is a preconvex relation. Moreover, we notice that some relations of $\mathcal{G}$ are not strongly preconvex. For example, the relation $\{eq^=, d^<, di^>, o^<, o^>, oi^<, oi^>, m^<, m^>, m^=, mi^<, mi^>, mi^=\}$ belongs to $\mathcal{G}$ but is not strongly preconvex: indeed its intersection with the convex relation $\{eq^=, o^=, oi^=, m^=, mi^=\}$ is not a preconvex relation.

[2] A computer-program has been used to prove this result, *idem* for the future Proposition 13.

7.1. Tractability of $\mathcal{G}$

We are now in a position to prove the tractability of the consistency problem for the set $\mathcal{G}$. Given a solution m of a network $\mathcal{N} = (V, C)$, m will be said maximal if $dim(m_{ij}) = dim(C_{ij})$ for all $i, j \in 1, \ldots, |V[$. Firstly, we prove the following result:

Proposition 14 *Let $\mathcal{N} = (V, C)$ a convex $\mathcal{INDU}$ network whose constraints belong to $\mathcal{C}_{IA}$ ($\neq \emptyset$). If $\mathcal{N}$ is $\diamond$-closed then $\mathcal{N}$ admits a maximal solution.*

Proof. $\mathcal{C}_{IA}$ corresponds to the convex relations of $\mathcal{IA}$. Let $\mathcal{N}' = (V, C')$ be the convex interval network equivalent to $\mathcal{N}$. $\mathcal{N}'$ admits a solution $m_1, \ldots, m_n$ (with $n = |V|$) such that $a = b$, with a and b two endpoints of m_i and m_j, iff all basic relations of IA belonging to C'_{ij} imposes this equality (a maximal solution for $\mathcal{IA}$, see [7]). We can modify m to obtain a solution s having the additional property: $s_i^+ - s_i^- = s_j^+ - s_j^-$ iff $C_{ij} = \{eq^=\}$. Consider the lower endpoint m_i^-, let l be the number of endpoints located before m_i^-. We assign to s_i^- the value $l/(1+l)$. We treat in a similar way the upper endpoints. s satisfies the properties fixed previously. Hence, s is a maximal solution of $\mathcal{N} = (V, C)$. $\dashv$

Proposition 15 *Let $r, s \in$ INDU such that $I(r \diamond s)$, $I(r)$ and $I(s) \in \mathcal{C}_{IA}$. We have $I(r \diamond s) \subseteq I(r) \diamond I(s)$.*

Proof. $r \subseteq I(r)$ and $s \subseteq I(s)$. Hence $r \diamond s \subseteq I(r) \diamond I(s)$. Consequently, $I(r \diamond s) \subseteq I(I(r) \diamond I(s))$. As $\mathcal{C}_{IA}$ is closed for the operation $\diamond$, $I(r) \diamond I(s)$ is a convex relation. Hence, $I(I(r) \diamond I(s)) = I(r) \diamond I(s)$. It results that $I(r \diamond s) \subseteq I(r) \diamond I(s)$. $\dashv$

Proposition 16 *Let $\mathcal{N} = (V, C)$ be a network whose constraints belong to $\mathcal{G}$. Let $\mathcal{N}^I = (V, C^I)$ be defined by $C_{ij}^I = I(C_{ij})$ for all $i, j \in \{1, \ldots, n\}$, with $n = |V|$. If $\mathcal{N}$ is $\diamond$-closed then $\mathcal{N}^I$ is $\diamond$-closed.*

Proof. Let $V_i, V_j, V_k \in V$. $C_{ij} \subseteq C_{ik} \diamond C_{kj}$, consequently, $I(C_{ij}) \subseteq I(C_{ik} \diamond C_{kj})$. We know that $\mathcal{G}$ is closed for the operation $\diamond$. It follows that $I(C_{ik} \diamond C_{kj}) \in \mathcal{C}_{IA}$. Moreover, by definition of $\mathcal{G}$, $I(C_{ik})$ and $I(C_{kj}) \in \mathcal{C}_{IA}$. From Prop. 15, it follows that $I(C_{ik} \diamond C_{kj}) \subseteq I(C_{ik}) \diamond I(C_{kj})$. Using this result, we deduce that $I(C_{ij}) \subseteq I(C_{ik}) \diamond I(C_{kj})$. $\dashv$
Now, we can establish the main result concerning the set $\mathcal{G}$.

Theorem 3 Cons($\mathcal{G}$) *can be decided by means of the $\diamond$-closure method.*

Proof. Let $\mathcal{N} = (V, C)$ be a network containing constraints belonging to $\mathcal{G}$. By using the $\diamond$-closure method on $\mathcal{N}$ we obtain an equivalent subnetwork $\mathcal{N}' = (V, C')$. The constraints of $\mathcal{N}'$ belongs to $\mathcal{G}$ since $\mathcal{G}$ is closed for the three operations $^{-1}$, $\cap$ and $\diamond$. If $\mathcal{N}'$ contains the empty constraint, then $\mathcal{N}$ is not consistent. In the opposite case, let us show that $\mathcal{N}'$ (and consequently $\mathcal{N}$) is consistent. Let $\mathcal{N}'' = (V, C'')$ be defined by $C''_{ij} = I(C'_{ij})$. $\mathcal{N}''$ is $\diamond$-closed (Prop. 16). It admits a maximal solution m (Prop. 14). This solution m is also a maximal solution of $\mathcal{N}'$. This is due to the fact that $dim(I(C'_{ij}) \setminus C'_{ij}) < dim(C'_{ij})$ (see definition of $\mathcal{G}$), for each pair of variables V_i and V_j. $\dashv$
Hence, we have characterized a set for which the $\diamond$-closure method is complete.

8. Conclusions

The $\mathcal{INDU}$ calculus lacks many of the nice properties of Allen's calculus: its (weak) composition table does not define a relation algebra. Consistency does not imply 3-consistency, and neither does 3-consistency imply consistency, even for atomic networks: some four node networks are 3-consistent but not consistent. In spite of these negative results, we are able to characterize interesting tractable subsets of relations. To this end, we use both the syntactic approach (Horn classes) and the geometrical approach (convexity and preconvexity). While the two methods yield the same class in Allen's case, they provide us with two separate tractable subsets in the case of $\mathcal{INDU}$. Following the geometrical approach, we define the set of preconvex relations and prove that its consistency problem is NP-complete. We then characterize two subsets of preconvex relations: one is the subset of strongly preconvex relations, which is tractable (for reasons pertaining to the syntactic properties of its relations), but for which consistency cannot be decided by the usual path-consistency method. The other (incomparable) subclass is tractable and its consistency problem can be solved by the path-consistency method. This paper constitutes a first fully successful exploration of the complexity properties of the $\mathcal{INDU}$ calculus.

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