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On Convergence in the Methods of Strat and of Smith for Shape from Shading

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Abstract. We study the convergence of two iterative Shape from Shading methods: the methods of Strat and of Smith. We try to determine the spectral radius of the Jacobian matrix of each iteration at any possible fixed point. We show that the method of Strat diverges for any image containing at least four pixels forming a square, any reflectance map and any relative weight between the irradiance term and the integrability term. An example is provided, in which divergence occurs after a large number of iterations, even if the reconstructed surface approaches the real surface after only a few iterations. We show then by a similar way that the method of Smith diverges for any image containing at least nine pixels forming a square, any reflectance map and any relative weight between the irradiance term and the smoothing term.

1. Introduction

The first major interest in Shape from Shading dates from the early 80's. Since the pioneering work of Horn (1970), many solutions have been proposed to recover the surface orientation of an opaque object based on its reflectance map and a single view (Horn and Brooks, 1989), among which iterative methods, which are the object of this work, are the most popular. Except for very recent work (Rouy and Tourin, 1992; Dupuis and Oliensis, 1993), all the existing iterative methods are based on variational approaches. The resulting equations are solved most of the time using a Jacobi iteration (Strat, 1979; Ikeuchi and Horn, 1981; Smith, 1982; Brooks and Horn, 1985; Lee, 1985; Horn and Brooks, 1986; Malik and Maydan, 1989; Horn, 1989) or in one case the conjugate gradient method (Szelisky, 1991).

The practical efficiency of Shape from Shading methods remains in question, since they often suffer from an excessive use of suppositions (such as homogeneous reflecting properties, infinite distance light source and viewer, smoothness of the unknown surface) and frequent recourse to boundary conditions. Furthermore, it appears that, in many cases, the demonstration that a proposed method indeed provides an exact solution to the problem is very difficult or even not at all tractable.

This problem is even more serious with iterative methods, where the experimental verification of the convergence of algorithms is always questionable (unless an infinite number of iterations has been performed) and where the theoretical proof is especially hard to obtain. Among the iterative methods using a Jacobi iteration, that proposed by Lee is the only one for which convergence has been proved under certain circumstances (Lee, 1985).

This paper deals with the study of convergence of two of the first iterative methods: the method of Strat (1979) and that of Smith (1982). The convergence of the method of Strat has been rather hastily claimed in the original dissertation (Strat, 1979, p. 83), by advocating general convergence of Lagrange multipliers and Fletcher-Powell methods but to our knowledge, a rigorous demonstration of this convergence has never been made. Furthermore, experiments with different object shapes have proved that divergence generally occurs (Durou, 1988), a property which seems to have already been observed by several researchers in the field. On the other hand, the convergence of the method of Smith has already been put in doubt by Horn and Brooks (1986), although no demonstration of divergence was made.

Our conclusions are clear-cut: in the case of images containing at least four pixels forming a square, the method of Strat will almost surely diverge, whatever the irradiance map and the choice of the weight λ between the brightness error and the regularizing term; in the case of images containing at least nine pixels forming a square, the method of Smith will always diverge.

In Section 2, we briefly recall notations used for the methods of Strat and Smith. In Section 3, the main criterion concerning the convergence of iterative methods is stated (Lattès's theorem), as well as an important result concerning the cases where the Jacobian matrix of the iteration is symmetric (called "Proposition 1"). Section 4 deals with the demonstration of the divergence of the method of Strat. This section is fairly long and constitutes in fact the heart of the paper. In Section 5, a very similar reasoning leads to the proof of the divergence of the method of Smith.

2. The Methods of Strat and Smith

Given an image $E(x, y)$ of a smooth opaque object, the Shape from Shading problem consists in searching for the surfaces which would give this image under certain lighting conditions. If the surface $z(x, y)$ is characterized by a known "reflectance map" R, which describes its light reflection properties, the problem can be mathematically expressed by the so-called "irradiance equation" (1), where the unknown is the function $z(x, y)$ (Horn, 1970) :

$$E(x, y) = R\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \tag{1}$$

All the iterative methods of resolution of the Shape from Shading problem proposed in the literature have a common point: the unknown is in fact not the height, but the surface normal $\vec{\nu}$, that is, the unit vector normal to the surface. The search for such a vector, which has two degrees of liberty, is equivalent to the search for two scalar functions. Several pairs of functions have been proposed:

- $p(x, y) = \frac{\partial z}{\partial x}$ and $q(x, y) = \frac{\partial z}{\partial y}$. One can find without difficulty:

$$\vec{\nu} = \frac{1}{\sqrt{1 + p^2 + q^2}} \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix},$$

  if the Oz axis points towards the obsever.
- $f(x, y) = \frac{2p}{1+\sqrt{1+p^2+q^2}}$ and $g(x, y) = \frac{2q}{1+\sqrt{1+p^2+q^2}}$. Then:

$$\vec{\nu} = \frac{1}{4 + f^2 + g^2} \begin{pmatrix} -4f \\ -4g \\ 4 - f^2 - g^2 \end{pmatrix}$$

- $l(x, y) = -\frac{p}{\sqrt{1+p^2+q^2}}$ and $m(x, y) = -\frac{q}{\sqrt{1+p^2+q^2}}$. Then:

$$\vec{\nu} = \begin{pmatrix} l \\ m \\ \sqrt{1 - l^2 - m^2} \end{pmatrix}$$

It is easier to give a geometric interpretation of these representations of $\vec{\nu}$ (Fig. 1). The only vectors $\vec{\nu}$ pointing the northern hemisphere on the Gaussian sphere correspond to surface points visible by the observer.

Each of these representations of $\vec{\nu}$ presents weaknesses:

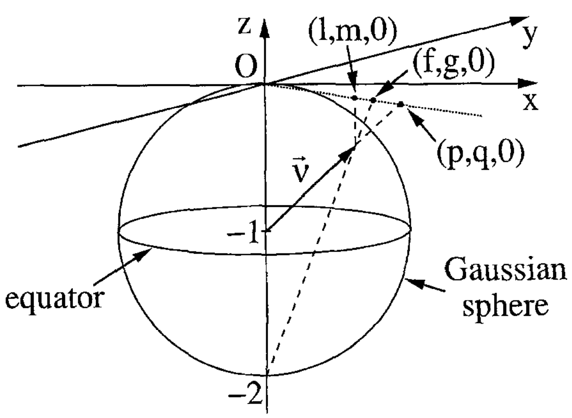

Figure 1. Geometric interpretation of (p, q), (f, g) and (l, m).

—The representation by the pair (p, q) does not allow use of the occluding boundary, where p or q is infinite;
—The (f, g) representation does not have this weakness, since on the occluding boundary:

$$f^2 + g^2 = 4$$

This representation imposes nevertheless, for each visible point, the following condition:

$$f^2 + g^2 \leq 4$$

—The (l, m) representation is similar, since on the occluding boundary:

$$l^2 + m^2 = 1$$

The associated condition is:

$$l^2 + m^2 \leq 1$$

Let us note that the calculation of the height can be easily made by integration of the normal.

2.1. *Strat's Method*

Strat was the first to propose an iterative method for solving the irradiance equation (Strat, 1979). He proposed to calculate the normal $\vec{\nu}$ with the (p, q) representation. If one searches the solutions of the irradiance equation (Eq. 1) on a domain D containing N pixels, designated by their discrete coordinates (i, j), then the N following equations at least must be solved:

$$R(p_{i,j}, q_{i,j}) = E_{i,j} \qquad (2, i, j)$$

To make the problem well-posed, that is, containing at least as many equations as unknowns, Strat adds equations based on the following remark: the variation of the height along a closed contour γ is equal to zero. That means:

$$\oint_\gamma dz = 0 \quad \Leftrightarrow \quad \oint_\gamma (p\,dx + q\,dy) = 0$$

Suppose the pixels are positioned on a square-mesh, where the distance between nearest neighbours is δ. For a pixel (i, j) of D, Strat chooses the following closed contour:

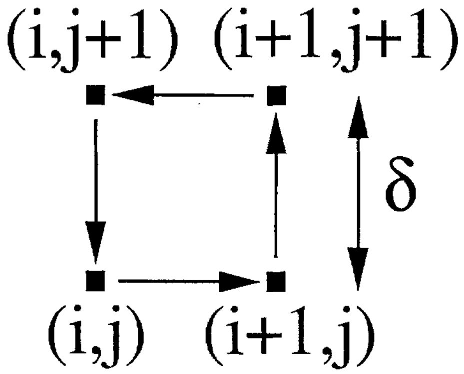

Figure 2. Strat's contour.

The resulting equation is:

$$\begin{aligned}
&\delta\left[\frac{p_{i+1,j} + p_{i,j}}{2} + \frac{q_{i+1,j+1} + q_{i+1,j}}{2} \right.\\
&\qquad \left. - \frac{p_{i,j+1} + p_{i+1,j+1}}{2} - \frac{q_{i,j} + q_{i,j+1}}{2}\right] = 0\\
&\Leftrightarrow \frac{\delta}{2}[p_{i+1,j} + p_{i,j} + q_{i+1,j+1} + q_{i+1,j}\\
&\qquad - (p_{i,j+1} + p_{i+1,j+1} + q_{i,j} + q_{i,j+1})] = 0\\
&\Leftrightarrow e_{i,j} = 0 \qquad (3, i, j)
\end{aligned}$$

Strat chooses the system constituted by the N equations $(2, i, j)$, and by the equations $(3, i, j)$ obtained for pixels (i, j) such that (i, j), $(i + 1, j)$, $(i, j + 1)$ or $(i + 1, j + 1)$ belong to D (let us name D' this set: $D \subset D'$). The discrete error associated with this system is:

$$\varepsilon = \delta^2 \sum_{(i,j)\in D} [R(p_{i,j}, q_{i,j}) - E_{i,j}]^2 + \frac{\lambda}{\delta^2} \sum_{(i,j)\in D'} e_{i,j}^2$$

In this expression, λ is a strictly positive constant. The coefficients δ^2 and $\frac{1}{\delta^2}$ have been introduced so that the error does not depend (at least not too much) on the pixel density. Since ε must be extremal in relation to the $2N$ unknowns $p_{i,j}$ and $q_{i,j}$, one obtains:

$$\begin{cases}
\delta^2(E_{i,j} - R(p_{i,j}, q_{i,j}))R_p(p_{i,j}, q_{i,j})\\
\qquad = \frac{\lambda}{2\delta}(e_{i,j} + e_{i-1,j} - e_{i-1,j-1} - e_{i,j-1})\\
\delta^2(E_{i,j} - R(p_{i,j}, q_{i,j}))R_q(p_{i,j}, q_{i,j})\\
\qquad = \frac{\lambda}{2\delta}(e_{i-1,j} + e_{i-1,j-1} - e_{i,j-1} - e_{i,j})
\end{cases}$$

The partial derivative of R in relation to p (resp. q) is denoted by R_p (resp. R_p). Strat proposed the following

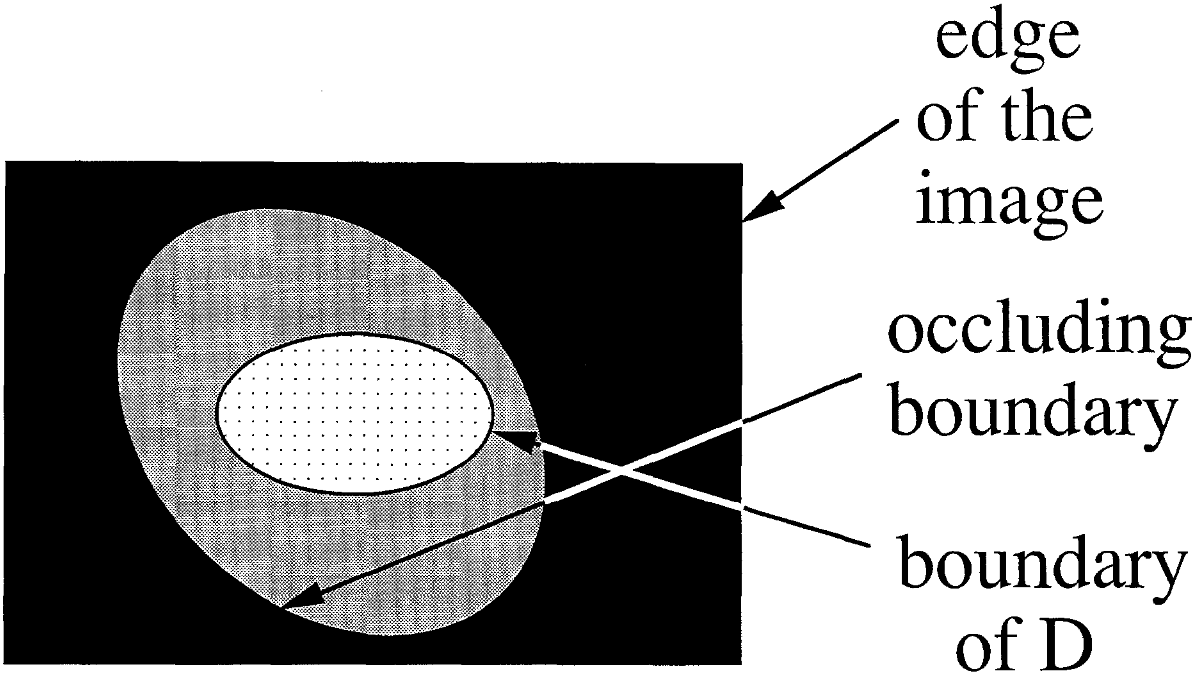

Figure 3. There must be no pixel of the occluding boundary in D.

iterative scheme for solving these equations:

$$\begin{cases} p_{i,j}^{k+1} = \bar{p}_{i,j}^k - \tilde{q}_{i,j}^k + \dfrac{\delta^2}{\lambda} \\ \qquad \times \left(E_{i,j} - R\left(p_{i,j}^k, q_{i,j}^k\right)\right) R_p\left(p_{i,j}^k, q_{i,j}^k\right) \\ q_{i,j}^{k+1} = \bar{q}_{i,j}^k - \tilde{p}_{i,j}^k + \dfrac{\delta^2}{\lambda} \\ \qquad \times \left(E_{i,j} - R\left(p_{i,j}^k, q_{i,j}^k\right)\right) R_q\left(p_{i,j}^k, q_{i,j}^k\right) \end{cases}$$

In these expressions, the following notations have been adopted:

- k denotes the iteration step;
- For any variable v:

$$\tilde{v}_{i,j}^k = \tfrac{1}{4}(v_{i+1,j+1} - v_{i-1,j+1} + v_{i-1,j-1} - v_{i+1,j-1})$$

- $\bar{p}_{i,j} = \frac{1}{4}[(p_{i+1,j+1} - 2p_{i+1,j} + p_{i+1,j-1}) + 2(p_{i,j+1} + p_{i,j-1}) + (p_{i-1,j+1} - 2p_{i-1,j} + p_{i-1,j-1})]$
- $\bar{q}_{i,j} = \frac{1}{4}[(q_{i+1,j+1} - 2q_{i,j+1} + q_{i-1,j+1}) + 2(q_{i+1,j} + q_{i-1,j}) + (q_{i+1,j-1} - 2q_{i,j-1} + q_{i-1,j-1})]$.

A first weakness with this method is that it requires the knowledge of the normal along the boundary of D and, because of the use of the (p, q) representation, none of these pixels may belong to the occluding boundary (figure 3).

But the knowledge of p and q on such a contour is generally not available, in the domain of Shape from Shading. Furthermore, we will see later that Strat's iterative scheme presents a much more serious weakness: it is hopelessly divergent.

2.2. *Smith's Method*

Smith uses the (l, m) representation of the normal $\overset{\perp}{v}$ (Smith, 1982). This representation allows us to use pixels on the occluding boundary as the boundary of D.

The irradiance equation can be written, at a pixel (i, j) of D:

$$R(l_{i,j}, m_{i,j}) = E_{i,j} \qquad (4, i, j)$$

To complete these N equations, Smith chooses the following smoothing equations (Δ denotes the Laplacian operator):

$$\begin{cases} \Delta l = 0 \\ \Delta m = 0 \end{cases}$$

which can be written in this discrete form:

$$\begin{cases} \frac{1}{\delta^2}(l_{i,j+1} + l_{i+1,j} + l_{i,j-1} \\ \quad + l_{i-1,j} - 4l_{i,j}) = 0 & (5, i, j) \\ \frac{1}{\delta^2}(m_{i,j+1} + m_{i+1,j} + m_{i,j-1} \\ \quad + m_{i-1,j} - 4m_{i,j}) = 0 & (6, i, j) \end{cases}$$

Smith chooses the system containing the equations $(4, i, j)$, where (i, j) belongs to D, and the equations $(5, i, j)$ and $(6, i, j)$ using at least one pixel of D (let us name D'' the set of such pixels (i, j)). It is obvious that D'' contains pixels which are on a two-pixel-wide boundary of D. The knowledge of $\vec{v}$ on the occluding boundary is as difficult to justify as l and m are generally not known on a two-pixel-wide boundary of D. This is a first obvious weakness of Smith's method. Let us introduce the following discrete error:

$$\begin{aligned} \varepsilon = & \sum_{(i,j)\in D} [R(l_{i,j}, m_{i,j}) - E_{i,j}]^2 \\ & + \frac{1}{\delta^4} \sum_{(i,j)\in D''} [(l_{i,j+1} + l_{i+1,j} + l_{i,j-1} + l_{i-1,j} - 4l_{i,j})^2 \\ & + (m_{i,j+1} + m_{i+1,j} + m_{i,j-1} + m_{i-1,j} - 4m_{i,j})^2] \end{aligned}$$

This error can be extremal in relation to the $2N$ unknowns $l_{i,j}$ and $m_{i,j}$ only if:

$$\begin{cases} l_{i,j} = \bar{l}_{i,j} + \frac{\delta^4}{20\lambda}(E_{i,j} - R(l_{i,j}, m_{i,j}))R_l(l_{i,j}, m_{i,j}) \\ m_{i,j} = \bar{m}_{i,j} + \frac{\delta^4}{20\lambda}(E_{i,j} - R(l_{i,j}, m_{i,j}))R_m(l_{i,j}, m_{i,j}) \end{cases}$$

For any variable v, the following notation has been introduced:

$$\begin{aligned} \bar{v}_{i,j} = & \tfrac{1}{20}[8(v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1}) \\ & - 2(v_{i+1,j+1} + v_{i-1,j-1} + v_{i-1,j+1} + v_{i+1,j-1}) \\ & - (v_{i+2,j} + v_{i-2,j} + v_{i,j+2} + v_{i,j-2})] \end{aligned}$$

Smith proposed this iterative scheme:

$$\begin{cases} l_{i,j}^{k+1} = \bar{l}_{i,j}^k + \frac{\delta^4}{20\lambda}\left(E_{i,j} - R\left(l_{i,j}^k, m_{i,j}^k\right)\right)R_l\left(l_{i,j}^k, m_{i,j}^k\right) \\ m_{i,j}^{k+1} = \bar{m}_{i,j}^k + \frac{\delta^4}{20\lambda}\left(E_{i,j} - R\left(l_{i,j}^k, m_{i,j}^k\right)\right) \\ \qquad \times R_m\left(l_{i,j}^k, m_{i,j}^k\right)\right) \end{cases}$$

We will prove below that this scheme is inexorably divergent. Horn and Brooks (1986) noticed this instability and proposed as an alternative to use the Gauss-Seidel relaxation. But such a relaxation does not lend itself to parallel implementation. For this reason it would not be very tractable.

3. Convergence Criteria of an Iteration

The simplest iteration is that for solving one equation with one unknown. Thus let us consider the following iteration, where we suppose $f(x)$ to be derivable:

$$x^{k+1} = f(x^k)$$

It is convenient to plot the first bissecting line and the graph of $f(x)$, in order to foresee the behaviour of this iteration (figure 4).

In Fig. 4, we can see three fixed points x_1, x_2 and x_3. We have the following result in relation with the iteration convergence:

At any fixed point x^*:

—If $|f'(x^*)| < 1$, where $f'(x)$ denotes the derivative of $f(x)$, then the iteration does converge to x^*, for initial values x^0 constituting a neighbourhood of x^*;
—If $|f'(x^*)| > 1$, the set of initial values x^0 for which the iteration converges to x^* does not constitute a neighbourhood of x^*. This is usually described by: "the iteration diverges";
—If $|f'(x^*)| = 1$, we cannot come to a conclusion about the iteration convergence, without studying the second order derivatives of $f(x)$ at x^*.

Lattès's theorem (Lattès, 1910) is a generalisation of this result for cases of n equations with n unknowns.

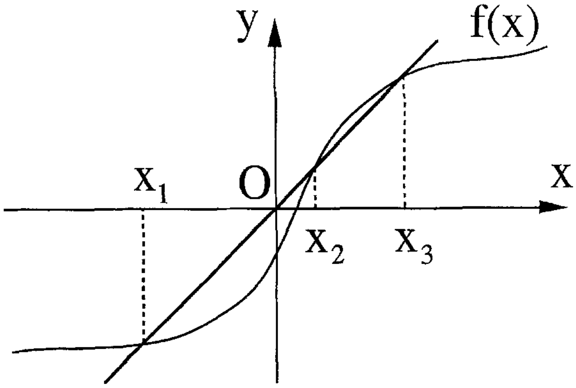

Figure 4. Visualisation of fixed points.

3.1. *Lattès's Theorem*

—If all the multiplying coefficients of a non-linear iteration defined at a certain fixed point are strictly less than one in absolute value, then the iteration converges to this fixed point, for initial values belonging to a domain which constitutes a neighbourhood of the fixed point;

—If one of these coefficients at least is strictly more than one in absolute value, then the iteration converges to the fixed point only for some initial values which do not constitute a neighbourhood of the fixed point.

If we want to solve a system of n equations with n unknowns, the iteration may be written:

$$X^{k+1} = F(X^k),$$

where k denotes the step, X a vector whose n coordinates are the n unknowns of the system, $F(X)$ a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. We suppose, and that will be always the case for us, that $F(X)$ is differentiable.

Definition. We call spectral radius of a real matrix M, $n \times n$, and note $r(M)$, the greatest modulus of the n (complex) eigenvalues of M.

For the situations we are interested in, the "multiplying coefficients" of Lattès's theorem are the eigenvalues of the Jacobian matrix of $F(X)$ at the fixed point, and so we can reformulate Lattès's theorem in the following form:

—For the non-linear iteration $X^{k+1} = F(X^k)$, if the spectral radius of the Jacobian matrix of $F(X)$ at a fixed point is strictly less than one, then the iteration converges to this fixed point, for initial values belonging to a domain which constitutes a neighbourhood of the fixed point;

—If this spectral radius is strictly more than one, then the iteration diverges at this fixed point;

—We can add that, in the case where it equals one, we cannot come to a conclusion on the convergence without knowing the second order derivatives of $F(X)$.

This shows how important is the knowledge of the Jacobian matrix, for predicting the convergence properties of an iteration. It can be shown (Durou, 1993) that, if the reflectance map is C^2, then Strat's and Smith's iterations possess real symmetric Jacobian matrices. We are going to prove a fundamental result concerning the spectral radius of such matrices.

3.2. *Proposition 1*

Definition. We call a "daughter-matrix" of a symmetric matrix M, any sub-matrix of M obtained by removing rows and columns of same indices. If M is symmetric, any daughter-matrix of M will be so too.

Proposition 1. *Every daughter-matrix of a real symmetric matrix M has a spectral radius less than or equal to the radius of M.*

Proof: Let M be a $n \times n$ real symmetric matrix. M is diagonalisable, and there exists one orthonormal base of $\mathbb{R}^n$, constituted by eigenvectors. It can be easily proved that:

$$r(M)^2 = \mathrm{Sup}\{\langle MX, MX\rangle, X \in \mathbb{R}^n, \langle X, X\rangle = 1\} \tag{7}$$

where $\langle\, ,\rangle$ denotes the Euclidean scalar product. Let $\bar{M}$ be a daughter of M. There exists one base of $\mathbb{R}^n$, obtained by permutation of the canonical base of $\mathbb{R}^n$, in which the matrix identical to M can be written:

$$M_1 = \begin{bmatrix} \bar{M} & (\bar{\bar{M}})^t \\ \bar{\bar{M}} & \bar{\bar{\bar{M}}} \end{bmatrix}$$

M_1 has the same eigenvalues, so the same spectral radius than M. M_1 is evidently symmetric. Let $\bar{X}$ be some vector of $\mathbb{R}^{\bar{n}}$, where $\bar{n}$ is the number of rows (or columns) of $\bar{M}$. Let X be the vector of $\mathbb{R}^n$, whose n first coordinates are those of $\bar{X}$, completed by zeros:

$$M_1 X = \begin{pmatrix} \bar{M} & \bar{X} \\ \bar{\bar{M}} & \bar{X} \end{pmatrix}$$

So:

$$\begin{aligned} \langle M_1 X, M_1 X\rangle &= \langle \bar{M}\,\bar{X}, \bar{M}\,\bar{X}\rangle + \langle \bar{\bar{M}}\,\bar{X}, \bar{\bar{M}}\,\bar{X}\rangle \\ &\geq \langle \bar{M}\bar{X}, \bar{M}\bar{X}\rangle \end{aligned}$$

This inequality holds for any $\bar{X}$ in $\mathbb{R}^{\bar{n}}$, thus:

$$\mathrm{Sup}\{\langle \bar{M}\,\bar{X}, \bar{M}\,\bar{X}\rangle, \bar{X} \in \mathbb{R}^{\bar{n}}, \langle \bar{X}, \bar{X}\rangle = 1\}$$

$$\le \mathrm{Sup}\{\langle M_1X, M_1X\rangle, X \in \mathbb{R}^{\bar{n}} \times \{0\}^{n-\bar{n}}, \langle X, X\rangle = 1\}$$
$$\le \mathrm{Sup}\{\langle M_1X, M_1X\rangle, X \in \mathbb{R}^{n}, \langle X, X\rangle = 1\}$$

This can be written, using (7):

$$r(\bar{M})^2 \le r(M_1)^2 = r(M)^2 \Rightarrow r(\bar{M}) \le r(M)$$

Proposition 1 is proved[1]. □

Consequence of Proposition 1. For Strat's and Smith's iterations, if the domain D contains N pixels, then the Jacobian matrix J is a $2N \times 2N$ real symmetric matrix. One can easily see that the Jacobian matrix $\bar{J}$, corresponding to a domain $\bar{D}$ included in D, is a daughter of J. Proposition 1 allows us to say:

$$r(\bar{J}) \le r(J)$$

If we can prove that the spectral radius of the Jacobian matrix is always greater than one for a certain domain $\bar{D}$, then we can conclude that this is also true for any domain D containing $\bar{D}$, and so that the iteration diverges over such a domain D, according to Lattès's theorem. This is how we are going to prove the divergence of Strat's and Smith's iterations.

4. Divergence of Strat's Iteration

Let us choose $\bar{D}$ constituted by four nearest neighbours forming a square (figure 5).

If the reflectance map is C^2, we know that the Jacobian matrix $\bar{J}$ of Strat's scheme, associated with $\bar{D}$, is a 8×8 real symmetric matrix. One finds:

$$\bar{J} = \begin{bmatrix} a_{0,0} & b_{0,0} & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & -\frac{1}{4} \\ b_{0,0} & c_{0,0} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & a_{1,0} & b_{1,0} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & b_{1,0} & c_{1,0} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & a_{0,1} & b_{0,1} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & b_{0,1} & c_{0,1} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & a_{1,1} & b_{1,1} \\ -\frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & b_{1,1} & c_{1,1} \end{bmatrix}$$

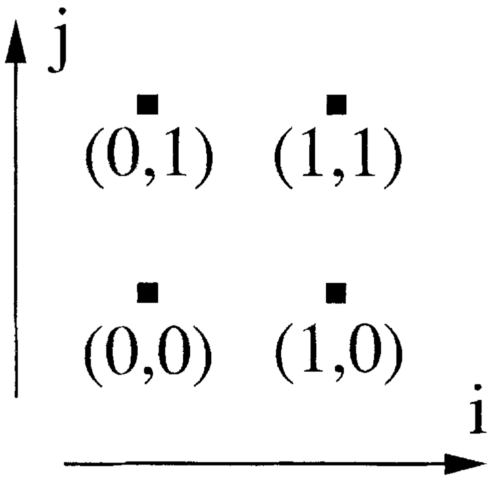

Figure 5. Definition of $\bar{D}$ (case of Strat's method).

The following notations have been taken:

$$\begin{cases} a_{i,j} = \dfrac{\delta^2}{\lambda}[(E_{i,j} - R(p_{i,j}, q_{i,j}))R_{pp}(p_{i,j}, q_{i,j}) \\ \qquad - R_p(p_{i,j}, q_{i,j})^2] \\ b_{i,j} = \dfrac{\delta^2}{\lambda}[(E_{i,j} - R(p_{i,j}, q_{i,j}))R_{pq}(p_{i,j}, q_{i,j}) \\ \qquad - R_p(p_{i,j}, q_{i,j})R_q(p_{i,j}, q_{i,j})] \\ c_{i,j} = \dfrac{\delta^2}{\lambda}[(E_{i,j} - R(p_{i,j}, q_{i,j}))R_{qq}(p_{i,j}, q_{i,j}) \\ \qquad - R_q(p_{i,j}, q_{i,j})^2] \end{cases}$$

Let $\bar{\bar{J}}$ be the daughter of $\bar{J}$ obtained by removing rows and columns with even indices:

$$\bar{\bar{J}} = \begin{bmatrix} a_{0,0} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & a_{1,0} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & a_{0,1} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & a_{1,1} \end{bmatrix}$$

We are going to prove that the spectral radius of this matrix is always greater than one.

Notations. If

$$x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

is some vector of $\mathbb{R}^4$, we denote by $M(x)$ the matrix:

$$M(x) = \begin{bmatrix} a & -\frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & b & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & c & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & d \end{bmatrix}$$

Let $r(x)$ be the spectral radius of $M(x)$. In the particular case where $a = b = c = d$, we acknowledge the following notations:

$$M(x) = m(a) \quad \text{and} \quad r(x) = \rho(a)$$

Theorem 1. *x being some vector of $\mathbb{R}^4$, $r(x)$ is strictly more than one. The spectral radius $r(x)$ equals one only when $a = b = c = d = \frac{1}{4}$.*

The demonstration of this theorem is fairly long. It is done in the five following subsections.

4.1. Calculation of $\rho(a)$

We first are going to calculate explicitely $\rho(a)$, the spectral radius of matrix $m(a)$. The eigenvalues of $m(\frac{1}{4})$ can be easily found:

$$-1; \quad \tfrac{1}{2}; \quad \tfrac{1}{2}; \quad 1$$

Thus those of $m(a)$ are:

$$\begin{aligned} &\det(m(a) - \lambda I_4) = 0 \\ &\quad \Leftrightarrow \det\left(m\left(\tfrac{1}{4}\right) + \left(a - \tfrac{1}{4} - \lambda\right) I_4\right) = 0 \\ &\quad \Leftrightarrow \lambda \in \left\{a - \tfrac{1}{4} + 1, a - \tfrac{1}{4} + \tfrac{1}{2}, \right. \\ &\qquad\qquad \left. a - \tfrac{1}{4} + \tfrac{1}{2}, a - \tfrac{1}{4} - 1\right\} \end{aligned}$$

So:

$$\rho(a) = \left|a - \tfrac{1}{4}\right| + 1$$

The minimum value of $\rho(a)$ is one. It is reached when $a = \frac{1}{4}$.

4.2. Convexity of $r(x)^2$

$M(x)$ being a real symmetric matrix, we know (equality (7)):

$$r(x)^2 = \mathrm{Sup}\{\langle M(x)X, M(x)X\rangle, X \in \mathbb{R}^4, \langle X, X\rangle = 1\}$$

Proposition 2. *Let v belong to r^4. Let $\varphi_v(x)$ be the function defined by:*

$$\begin{aligned} \mathbb{R}^4 &\to \mathbb{R} \\ x \mapsto \varphi_v(x) &= \langle M(x)v, M(x), v\rangle \end{aligned}$$

Then $\varphi_v(x)$ is convex.

Proof: It is easy to see that:

$$\begin{aligned} &\forall (x, y) \in \mathbb{R}^4 \times \mathbb{R}^4, \forall u \in [0, 1], \\ &M(ux + (1-u)y) = uM(x) + (1-u)M(y) \end{aligned}$$

So:

$$\begin{aligned} &\langle M(ux + (1-u)y)v, M(ux + (1-u)y)v\rangle \\ &\quad = u^2\langle M(x)v, M(x)v\rangle + (1-u)^2\langle M(y)v, M(y)v\rangle \\ &\quad + 2u(1-u)\langle M(x)v, M(y)v\rangle \qquad (8) \end{aligned}$$

On the other hand:

$$\langle M(x)v - M(y)v, M(x)v - M(y)v\rangle \geq 0,$$

which implies:

$$\begin{aligned} \langle M(x)v, M(y)v\rangle \leq \tfrac{1}{2}[\langle M(x)v, M(x)v\rangle \\ + \langle M(y)v, M(y)v\rangle] \quad (9) \end{aligned}$$

u belonging to [0, 1], the product $u(1-u)$ is nonnegative, and so, according to (8) and (9):

$$\begin{aligned} &\langle M(ux + (1-u)y)v, M(ux + (1-u)y)v\rangle \\ &\quad \leq \langle M(x)v, M(x)v\rangle(u^2 + u(1-u)) \\ &\quad + \langle M(y)v, M(y)v\rangle((1-u)^2 + u(1-u)) \end{aligned}$$

That means:

$$\begin{aligned} &\forall (x, y) \in \mathbb{R}^4 \times \mathbb{R}^4, \forall u \in [0, 1], \varphi_v(ux + (1-u)y) \\ &\quad \leq u\varphi_v(x) + (1-u)\varphi_v(y) \end{aligned}$$

This expresses the fact that $\varphi_v(x)$ is convex. □

Proposition 3. *Let $(f_i)_{i \in I}$ be a family, not necessarily finite, of convex functions from a set S to $\mathbb{R}$. If the following function $f(x)$:*

$$f(x) = \text{Sup}\{f_i(x), i \in I\}$$

is defined on S, then it is convex on S.

Proof:

$$\begin{aligned}&\forall (x, y) \in S^2, \forall u \in [0, 1], \forall i \in I,\\ &f_i(ux + (1 - u)y) \leq u f_i(x) + (1 - u) f_i(y)\end{aligned}$$

u belonging to $[0, 1]$, we have both inequalities:

$$\begin{cases} f_i(x) \leq f(x) \\ f_i(y) \leq f(y) \end{cases} \Rightarrow \begin{cases} u f_i(x) \leq u f(x) \\ (1 - u) f_i(y) \leq (1 - u) f(y) \end{cases}$$

So:

$$f_i(ux + (1 - u)y) \leq u f(x) + (1 - u) f(y)$$

This can be written for any index i in I, so:

$$\begin{aligned}&\forall (x, y) \in S^2, \forall u \in [0, 1],\\ &f(ux + (1 - u)y) \leq u f(x) + (1 - u) f(y)\end{aligned}$$

This expresses the fact that $f(x)$ is convex. □

We can use this result with the function $r(x)^2$, since:

$$\begin{aligned} r(x)^2 &= \text{Sup}\{\langle M(x)v, M(x)v\rangle, v \in \mathbb{R}^4, \langle v, v\rangle = 1\} \\ &= \text{Sup}\{\varphi_v(x), v \in \mathbb{R}^4, \langle v, v\rangle = 1\}\end{aligned}$$

According to Proposition 2, the functions $\varphi_v(x)$ are all convex, and we can conclude that $r(x)^2$ is a convex function over $\mathbb{R}^4$. As a convex function over an open part of a vector space is continuous, it follows that $r(x)^2$ is continuous, and so is $r(x)$. Let us note that $r(x)$ is not necessarily convex.

4.3. *Set of Points of $\mathbb{R}^4$ Where r(x) Reaches its Minimum*

Proposition 4. *$r(x)^2$ tends to infinity when $\langle x, x\rangle$ tends to infinity.*

Proof: Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

belong to $\mathbb{R}^4$. Let $D(x)$ be the matrix:

$$D(x) = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}$$

$D(x)$ allows us to do a partition of $M(x)$:

$$M(x) = M(O) + D(x),$$

where O denotes the origin of $\mathbb{R}^4$.

Let v be some vector of $\mathbb{R}^4$. Since $M(x) - M(O) = D(x)$, we have the equality:

$$\begin{aligned}&\langle M(x)v, M(x)v\rangle + \langle M(O)v, M(O)v\rangle\\ &\quad - 2\langle M(x)v, M(O)v\rangle = \langle D(x)v, D(x)v\rangle \end{aligned} \tag{10}$$

On the other hand:

$$\langle M(x)v + M(O)v, M(x)v + M(O)v\rangle \geq 0$$

This leads to, after expansion:

$$\begin{aligned}&\langle M(x)v, M(x)v\rangle + \langle M(O)v, M(O)v\rangle\\ &\quad + 2\langle M(x)v, M(O)v\rangle \geq 0 \end{aligned} \tag{11}$$

Adding (10) and (11), it follows:

$$\begin{aligned}&\langle M(x)v, M(x)v\rangle + \langle M(O)v, M(O)v\rangle\\ &\quad \geq \tfrac{1}{2}\langle D(x)v, D(x)v\rangle\end{aligned}$$

Let us choose v as a unit vector. According to (7), we can write:

$$\begin{aligned} r(x)^2 + r(O)^2 &\geq \langle M(x)v, M(x)v\rangle + \langle M(O)v, M(O)v\rangle \\ &\geq \tfrac{1}{2}\langle D(x)v, D(x)v\rangle\end{aligned}$$

Let us choose v successively equal to the four vectors of the canonical base:

$$\forall i \in \{1, 2, 3, 4\}, r(x)^2 + r(O)^2 \geq \tfrac{1}{2}x_i^2$$

Adding these four inequalities:

$$r(x)^2 + r(O)^2 \geq \tfrac{1}{8}\left(x_i^2 + x_2^2 + x_3^2 + x_4^2\right)$$
$$\Rightarrow r(x)^2 \geq \tfrac{1}{8}\langle x, x\rangle - r(O)^2$$

Proposition 4 is demonstrated. □

Proposition 5. *$r(x)$ has at least one minimum on $\mathbb{R}^4$, that means:*

$$\exists x_0 \in \mathbb{R}^4, \forall x \in \mathbb{R}^4, r(x) \geq r(x_0)$$

Proof: According to Proposition 4, we know that:

$$\exists \tau > 0, \forall x \in \mathbb{R}^4, \langle x, x\rangle > \tau^2 \Rightarrow r(x)^2 > r(O)^2$$
$$\Rightarrow r(x) > r(O)$$

Let C be the closed bowl with center at the origin O and radius τ. The function $r(x)$ being continuous on $\mathbb{R}^4$, and C being a compact set, $r(C)$ is a compact set of r. This means that $r(x)$ has a minimum on C, that is:

$$\exists x_0 \in C, \forall x \in C, r(x) \geq r(x_0) = r_{\min}$$

We have especially:

$$r(O) \geq r(x_0)$$

Moreover:

$$\forall x \in \mathbb{R}^4 - C, \langle x, x\rangle > \tau^2$$
$$\Rightarrow r(x) > r(O) \geq r(x_0)$$

Thus x_0 is a minimum of $r(x)$ on $\mathbb{R}^4$. □

Let E be the set of points of $\mathbb{R}^4$ where $r(x)$ equals $r_{\min}$. Proposition 5 proves that E is not empty.

Proposition 6. *E is a convex set of $\mathbb{R}^4$.*

Proof: Let x_1 and x_2 be two points of E. The points of the segment $[x_1, x_2]$ can be written:

$$u x_1 + (1-u) x_2, \quad \text{where } u \in [0, 1]$$

Since $r(x)^2$ is convex on $\mathbb{R}^4$:

$$\forall u \in [0,1], r(ux_1 + (1-u)x_2)^2$$
$$\leq u r(x_1)^2 + (1-u) r(x_2)^2 = r_{\min}^2$$

Thus necessarily:

$$\forall u \in [0,1], r(ux_1 + (1-u)x_2) = r_{\min}$$

That means that segment $[x_1, x_2]$ is contained in E, and so E is convex. □

4.4. *Rotations of $\mathbb{R}^4$ for which r(x) is invariant*

x being some vector of $\mathbb{R}^4$, we define the functions $s(x)$ and $t(x)$ by the following:

$$\text{For } x = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \quad s(x) = \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} \text{ and } \quad t(x) = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}.$$

s and t are in fact rotations of $\mathbb{R}^4$, and also $s \circ t$. Let us show that:

$$r(s(x)) = r(t(x)) = r(x)$$

Let us calculate the characteristic equation of $M(x)$. We find after some algebra:

$$\begin{aligned}
&\det(M(x) - \lambda I_4) = 0 \\
&\quad\Leftrightarrow \lambda^4 - \Sigma_1\lambda^3 - \left(\Sigma_2 - \tfrac{9}{8}\right)\lambda^2 \\
&\quad + \left(-\Sigma_3 + \tfrac{9}{16}\Sigma_1 + \tfrac{1}{2}\right)\lambda \\
&\quad - \Sigma_4 - \frac{\Sigma_2}{4} - \frac{\Sigma_1}{8} - \tfrac{15}{256} + \tfrac{3}{16}(ad + bc) = 0
\end{aligned}$$

with

$$\begin{cases} \Sigma_1 = a + b + c + d \\ \Sigma_2 = ab + ac + ad + bc + bd + cd \\ \Sigma_3 = abc + abd + acd + bcd \\ \Sigma_4 = abcd \end{cases}$$

We observe that this characteristic equation is invariant with respect to the rotations s, t and consequently $s \circ t$. The matrices $M(x)$, $M(s(x))$, $M(t(x))$ and $M(s \circ t(x))$ have then the same eigenvalues, so:

$$r(x) = r(s(x)) = r(t(x)) = r(s \circ t(x))$$

4.5. Determination of the Points Where r(x) Reaches its Minimum

We already know that E is a non-empty, convex set. We want to specify it explicitely. Let x be a point in E. According to what precedes:

$$r(x) = r_{\min} = r(s(x)) = r(t(x)) = r(s \circ t(x))$$

This shows that $s(x)$, $t(x)$ and $s \circ t(x)$ are also in E, which is furthermore a convex set, then it contains also the center of mass $\bar{x}$ of the points x, $s(x)$, $t(x)$ and $s \circ t(x)$, assigned with the same coefficient $\frac{1}{4}$. Let us compute the coordinates of $\bar{x}$:

$$\begin{aligned}\bar{x} &= \frac{x + s(x) + t(x) + s \circ t(x)}{4} \\ &= \frac{1}{4}\left[\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} + \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} + \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} + \begin{pmatrix} c \\ d \\ a \\ b \end{pmatrix}\right] \\ &= \frac{a+b+c+d}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

With the preceding notations:

$$r(\bar{x}) = \rho\left(\frac{a+b+c+d}{4}\right)$$

This shows that the minimal value of $r(x)$ is also the one of $\rho(a)$, which we already know. We can conclude:

$$\begin{cases} r_{\min} = 1 \\ x_0 = \dfrac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ is in } E. \end{cases}$$

Proposition 7. *E is reduced to the single point x_0.*

Proof: Let us prove Proposition 7 by reducing it to the absurd. Let us suppose E contains one point x_1 different from x_0. The set E being convex, the segment $[x_0, x_1]$ is included in E, that is:

$$\begin{aligned}\forall u \in [x_0, x_1], r((1-u)x_0 + u x_1) &= 1 \\ &= r(x_0 + u(x_1 - x_0))\end{aligned}$$

This means that for each point of $[x_0, x_1]$, either 1 or -1 is an eigenvalue. Let us define the following two sets:

$$\begin{cases} U_1 = \{u \in [0,1], \det(M(x_0 + u(x_1 - x_0)) - I_4) = 0\} \\ U_2 = \{u \in [0,1], \det(M(x_0 + u(x_1 - x_0)) + I_4) = 0\} \end{cases}$$

One of these sets at least is infinite. Let us suppose, without loss of generality, that U_1 is infinite. That means that $\det(M(x_0 + u(x_1 - x_0)) - I_4)$, which is a u-polynomial, has an infinity of zeroes on $[0, 1]$. Inevitably, this polynomial equals zero:

$$\forall u \in \mathbb{R}, \det(M(x_0 + u(x_1 - x_0)) - I_4) = 0 \quad (12)$$

With notations introduced earlier, let us make the following partition:

$$\begin{aligned} M(x_0 + u(x_1 - x_0)) &= M(x_0) + D(u(x_1 - x_0)) \\ &= M(x_0) + u D(x_1 - x_0) \end{aligned}$$

So we can rewrite (12):

$$\forall u \in \mathbb{R}, u^4 \det\left(D(x_1 - x_0) + \frac{1}{u}(M(x_0) - I_4)\right) = 0$$

When u becomes infinite, we obtain:

$$\det(D(x_1 - x_0)) = 0 \quad (13)$$

Let us suppose x_1 to be written:

$$x_1 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Equality (13) gives:

$$\left(a - \frac{1}{4}\right)\left(b - \frac{1}{4}\right)\left(c - \frac{1}{4}\right)\left(d - \frac{1}{4}\right) = 0 \quad (14)$$

The reasoning that led us to Eq. (14) can be applied to each pair of points of E. We are going to apply it to three other pairs of points of E: $(x_1, s(x_1))$, $(x_1, t(x_1))$ and $(x_0, \bar{x}_1)$, where $\bar{x}_1$ is the center of mass of $x_1, s(x_1), t(x_1)$ and $s \circ t(x_1)$ assigned with the coefficient $\frac{1}{4}$. We know that $s(x_1), t(x_1)$ and $\bar{x}_1$ are in E, since x_1 is in E. We obtain the three respective equations:

$$\begin{cases} (b-a)(d-c)=0 & (15) \\ (d-a)(b-c)=0 & (16) \\ a+b+c+d=1 & (17) \end{cases}$$

From (15) and (16), it follows:

$$a=b=c \quad \text{or} \quad a=b=d \quad \text{or}$$
$$a=c=d \quad \text{or} \quad b=c=d$$

According to the symmetry of the system constituted by (14), (15), (16) and (17), there is no loss of generality if we suppose for instance that $a=b=c$.

Equations (14) and (17) then give:

$$\left\{ \begin{array}{l} \left(a-\frac{1}{4}\right)\left(d-\frac{1}{4}\right)=0 \\ 3a+d=1 \end{array} \right\} \Leftrightarrow a=d=\tfrac{1}{4}$$

This shows that $a=b=c=d=\frac{1}{4}$, that is, $x_1=x_0$, which is in contradiction with our assumption. Proposition 7 is proved. □

Recapitulation: We have shown that:

- $\forall x \in r, r(x) \geq 1$
- $r(x)=1 \Leftrightarrow x=\dfrac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

Theorem 1 is thus proved. □

4.6. Conclusion

Let D be a domain containing four nearest neighbours forming a square. The Jacobian matrix J of Strat's scheme over D has a daughter $\bar{\bar{J}}$, which belongs to the family $M(x)$. According to Theorem 1 and Proposition 1, we can say:

$$r(J) \geq 1$$

This will be true for each image, each reflectance map and each relative weight λ. The value 1 may be reached by $r(J)$ exceptionally, so we can use Lattès's theorem to conclude:

Proposition 8. *Over each domain D containing four nearest neighbours forming a square, Strat's iterative scheme is almost surely divergent, for each image, each reflectance map and each relative weight* λ.

Example. The following must be said in defense of Strat: the Jacobian matrix of Strat's scheme at a fixed point, though its spectral radius is strictly more than one, may have most of its eigenvalues less than one in absolute value. There may be only one eigenvalue greater than one in absolute value. In such a case, divergence will indeed take place, but possibly only after a large number of iterations. For the examples supplied in his dissertation (Strat, 1979), Strat never goes further than fifty iterations. Despite the divergence, the discrete error ε usually decreases during the first iterations, before it increases towards infinity during the divergence (see (Maître, 1981)). Let us give an example that illustrates this remark. Let us suppose we want to extract the height from the image shown in Fig. 6. This is the image of a stalagmite (it is in fact a Gaussian) seen from above, lighted at a slight angle. Figure 7 shows the behaviour of Strat's scheme applied to this image. We try to reconstruct the shape on the whole image, which is possible because there is no occluding boundary. The correct values of p and q on the image boundary have been given. The real shape is represented upper-left. The initial surface was chosen arbitrarily (Fig. 7, upper-right). After five iterations

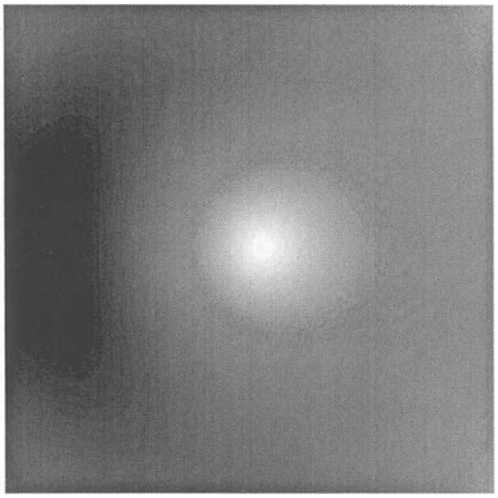

Figure 6. Image of a stalagmite seen from above, lighted at a slight angle.

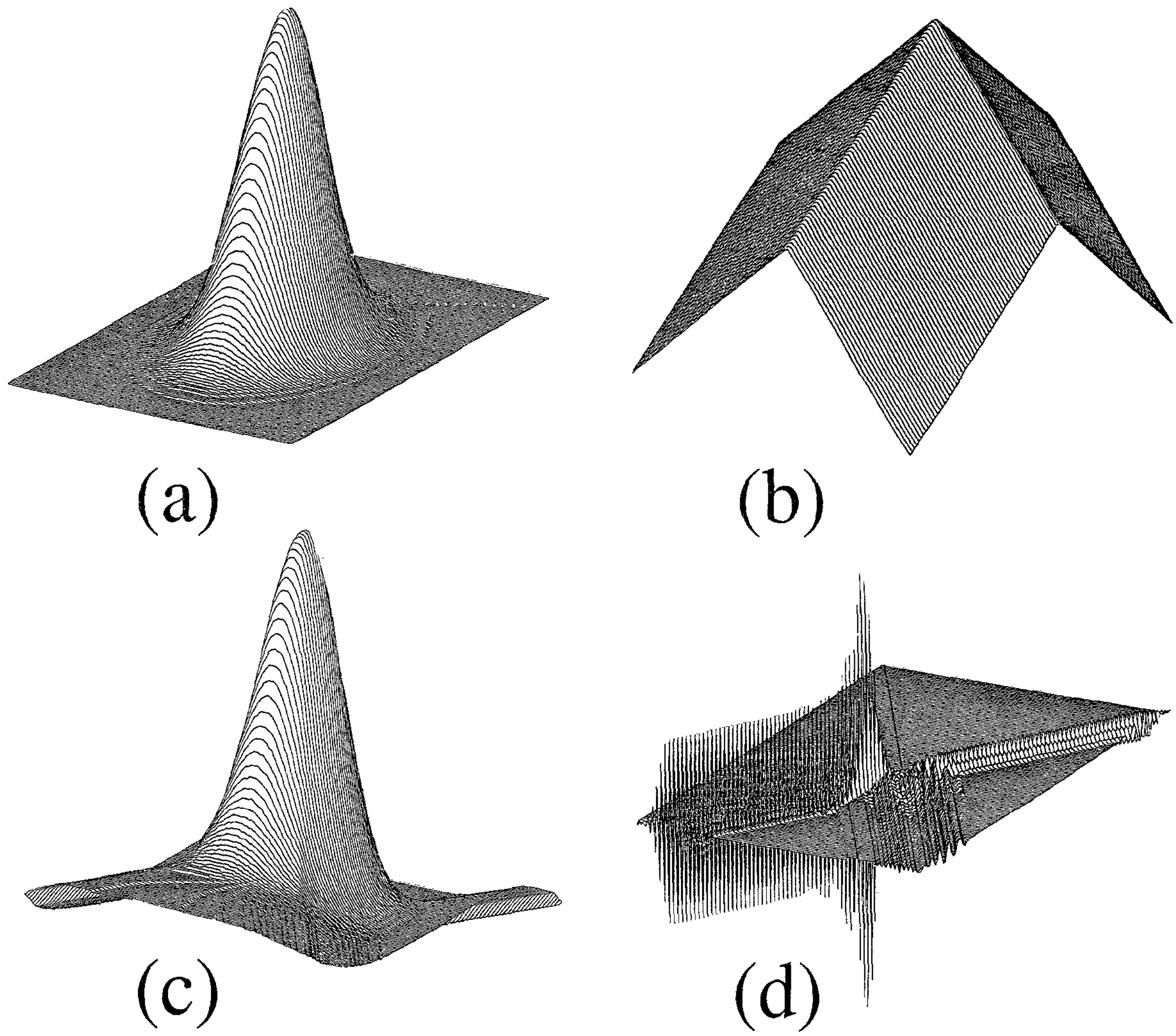

Figure 7. (a) Real shape of the stalagmite, (b) initial shape chosen arbitrarily, (c) shape obtained after five iterations and (d) after hundred iterations.

(Fig. 7, lower-left), we note that we are closer to the real shape. However, after hundred iterations, the divergence is clear (Fig. 7, lower-right).

Now we are going to study Smith's iteration. We will see that this iteration diverges too. The study will be similar to that done for Strat's iteration.

5. Divergence of Smith's Iteration

We choose the domain $\bar{D}$ constituted by nine nearest neighbours forming a square (figure 8).

The Jacobian matrix $\bar{J}$ of Smith's scheme, on $\bar{D}$, is a 18×18 real, symmetric matrix, if the reflectance map is C^2. It is not practical to write it, because of its size.

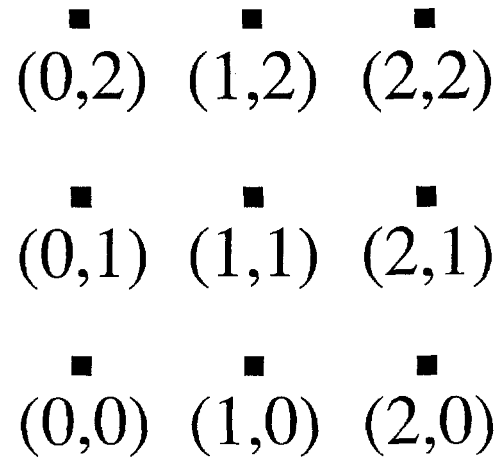

Figure 8. Definition of $\bar{D}$ (case of Smith's method).

When removing its rows and columns of even indices, one obtains a matrix $\bar{\bar{J}}$, daughter of $\bar{J}$, belonging to the family of matrices $M(x)$ we are going to study now.

Notation. $x = (a_i)_{i\in[1,9]}$ being a vector of $\mathbb{R}^9$, we denote by $M(x)$ the following matrix:

$$M(x) = \begin{bmatrix} a_1 & 0.4 & -0.05 & 0.4 & -0.1 & 0 & -0.05 & 0 & 0 \\ 0.4 & a_2 & 0.4 & -0.1 & 0.4 & -0.1 & 0 & -0.05 & 0 \\ -0.05 & 0.4 & a_3 & 0 & -0.1 & 0.4 & 0 & 0 & -0.05 \\ 0.4 & -0.1 & 0 & a_4 & 0.4 & -0.05 & 0.4 & -0.1 & 0 \\ -0.1 & 0.4 & -0.1 & 0.4 & a_5 & 0.4 & -0.1 & 0.4 & -0.1 \\ 0 & -0.1 & 0.4 & -0.05 & 0.4 & a_6 & 0 & -0.1 & 0.4 \\ -0.05 & 0 & 0 & 0.4 & -0.1 & 0 & a_7 & 0.4 & -0.05 \\ 0 & -0.05 & 0 & -0.1 & 0.4 & -0.1 & 0.4 & a_8 & 0.4 \\ 0 & 0 & -0.05 & 0 & -0.1 & 0.4 & -0.05 & 0.4 & a_9 \end{bmatrix}$$

We note as $r(x)$ the spectral radius of $M(x)$. In the case where:

$$\begin{aligned} a_1 &= a_3 = a_7 = a_9 = a \quad \text{and} \\ a_2 &= a_4 = a_6 = a_8 = b \quad \text{and} \quad a_5 = c, \end{aligned}$$

we note:

$$M(x) = m(a, b, c) \quad \text{and} \quad r(x) = \rho(a, b, c)$$

Theorem 2. *x being some vector of $\mathbb{R}^9$, $r(x)$ is strictly more than one.*

The demonstration is in any point similar to that of Theorem 1.

5.1. *Calculation of $\rho(a, b, c)$*

We are not going to calculate explicitely $\rho(a, b, c)$. We could not find its analytical expression. But the following result is satisfactory:

Proposition 9. *a, b and c being some real numbers, $\rho(a, b, c)$ verifies:*

$$\rho(a, b, c) > 1$$

The demonstration of this result is a little tedious. It can be found in Durou (1993).

5.2. *Convexity of $r(x)^2$*

The generalisation of Proposition 2 to the case of a vector of $\mathbb{R}^9$ is immediate. Proposition 3 allows us to conclude that $r(x)^2$ is convex.

5.3. *Points of $\mathbb{R}^9$ Where $r(x)$ Reaches its Minimum*

Propositions 4, 5 et 6 can be easily generalised. It can be shown that the set E constituted by points of $\mathbb{R}^9$ where $r(x)$ reaches its minimum is non-empty and convex.

5.4. *Transformations of $\mathbb{R}^9$ for which $r(x)$ is Invariable*

x being some vector of $\mathbb{R}^9$, we define the functions $s(x)$, $t(x)$ and $u(x)$ by the following:

$$\text{If} \quad x = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix}, \text{then} \quad s(x) = \begin{pmatrix} a_7 \\ a_4 \\ a_1 \\ a_8 \\ a_5 \\ a_2 \\ a_9 \\ a_6 \\ a_3 \end{pmatrix},$$

$$t(x) = \begin{pmatrix} a_9 \\ a_8 \\ a_7 \\ a_6 \\ a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} \quad \text{and} \quad u(x) = \begin{pmatrix} a_3 \\ a_6 \\ a_9 \\ a_2 \\ a_5 \\ a_8 \\ a_1 \\ a_4 \\ a_7 \end{pmatrix}$$

Proposition 10.

$$\forall x \in \mathbb{R}^9, r(x) = r(s(x)) = r(t(x)) = r(u(x))$$

To prove this result, we could of course calculate the characteristic determinant of $M(x)$, and verify that it is invariable by the transformations $s(x)$, $t(x)$ and $u(x)$. But this would be tedious. There exists a simpler way to prove this.

Proof of Proposition 10: Let $B = \{e_i, i \in [1, 9]\}$ be the canonical base of $\mathbb{R}^9$. $M(x)$ is the matrix of

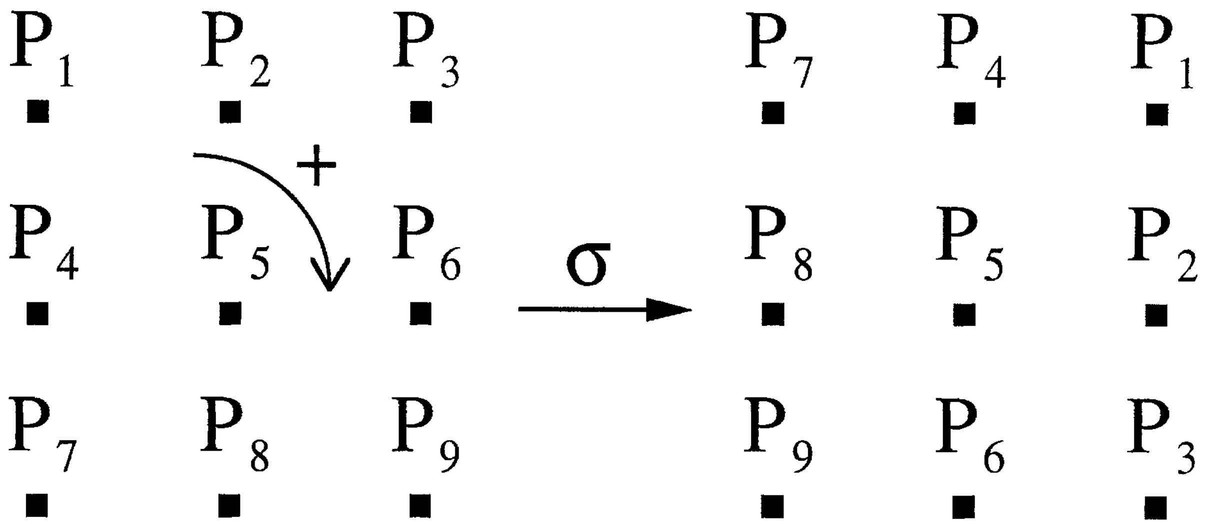

Figure 9. Illustration of the permutation σ.

an endomorphism $g(x)$ in the base B. We are going to show that $M(x)$, $M(s(x))$, $M(t(x))$ and $M(u(x))$ are identical matrices, that is $M(s(x))$, $M(t(x))$ and $M(u(x))$ are the matrices of $g(x)$ in three respective bases B_1, B_2 and B_3. Let us show this is true if:

$$\begin{cases} B_1 = \{e_7, e_4, e_1, e_8, e_5, e_2, e_9, e_6, e_3\} \\ B_2 = \{e_9, e_8, e_7, e_6, e_5, e_4, e_3, e_2, e_1\} \\ B_3 = \{e_3, e_6, e_9, e_2, e_5, e_8, e_1, e_4, e_7\} \end{cases}$$

Let us show first that the matrix of $g(x)$ in the base B_1 is $M(s(x))$. B_1 is obtained from B after a permutation σ of [1, 9]. Let us introduce the nine following points, numbered from one to nine (figure 9).

We note that the renumbering of the points after application of the permutation σ is equivalent to a rotation of $\frac{\pi}{2}$ in the direction indicated by the arrow. Let us define the coefficients $m_{i,j}$, i and j being in [1, 9], as follows:

—If P_i and P_j are nearest neighbours (example: P_1 et P_2), then:

$$m_{i,j} = 0.4$$

—If P_i and P_j are "neighbours by the corners" (example: P_1 et P_5), then:

$$m_{i,j} = -0.1$$

—If P_i and P_j are "nearest neighbours of order two" (example: P_1 et P_7), then:

$$m_{i,j} = -0.05$$

—If $P_i = P_j$, then:

$$m_{i,j} = a_i$$

—Otherwise:

$$m_{i,j} = 0$$

It is easy to verify that matrix $(m_{i,j})$ is precisely $M(x)$. The matrix of the endomorphism $g(x)$ in the base $B_1 = \sigma(B)$ is the matrix $(m'_{i,j})$, defined by:

$$m'_{i,j} = m_{\sigma(i),\sigma(j)}$$

We note that, if i and J are different, $m_{i,j}$ depends only on which sort of neighbours P_i and P_j are. Since this is invariable by the permutation σ, which is a rotation, we can affirm:

$$\forall (i, j) \in [1, 9]^2, i \neq j \Rightarrow m'_{i,j} = m_{\sigma(i),\sigma(j)} = m_{i,j}$$

On the other hand:

$$\forall i \in [1, 9], m'_{i,i} = m_{\sigma(i),\sigma(i)} = a_{\sigma(i)}$$

Thus we can conclude:

$$(m'_{i,j}) = M(s(x))$$

That means that $M(s(x))$ is the matrix of $g(x)$ in the base B_1. Let us show that the matrix of $g(x)$ in the base B_2 is $M(t(x))$. We have just to note that $B_2 = \sigma(B_1)$. So, according to what precedes, the matrix of $g(x)$ in the base B_2 is:

$$M(s \circ s(x)) = M(t(x))$$

Finally, since $B_3 = \sigma(B_2)$, the matrix of $g(x)$ in the base B_3 is:

$$M(s \circ t(x)) = M(u(x))$$

The matrices $M(x)$, $M(s(x))$, $M(t(x))$ and $M(u(x))$ are then identical. They have the same eigenvalues, and the same spectral radius. That proves Proposition 10.

□

5.5. *Minoration of r(x)*

The set E of points of r^9 where $r(x)$ reaches its minimum $r_{\min}$ is non-empty and convex. Let x be a point of E. According to what precedes:

$$r(x) = r_{\min} = r(s(x)) = r(t(x)) = r(u(x))$$

Thus x, $s(x)$, $t(x)$ and $u(x)$ are in E, and their center of mass $\bar{x}$ too:

$$\bar{x} = \frac{x + s(x) + t(x) + u(x)}{4}$$

$$= \frac{1}{4}\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} + \begin{pmatrix} a_7 \\ a_4 \\ a_1 \\ a_8 \\ a_5 \\ a_2 \\ a_9 \\ a_6 \\ a_3 \end{pmatrix} + \begin{pmatrix} a_9 \\ a_8 \\ a_7 \\ a_6 \\ a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} + \begin{pmatrix} a_3 \\ a_6 \\ a_9 \\ a_2 \\ a_5 \\ a_8 \\ a_1 \\ a_4 \\ a_7 \end{pmatrix}\right] = \begin{pmatrix} a \\ b \\ a \\ b \\ c \\ b \\ a \\ b \\ a \end{pmatrix}$$

The following notations have been taken:

$$\begin{cases} a = \dfrac{a_1 + a_3 + a_7 + a_9}{4} \\ c = a_5 \\ b = \dfrac{a_2 + a_4 + a_6 + a_8}{4} \end{cases}$$

This proves that the minimum of $r(x)$ is also the minimum, on r^3, of $\rho(a, b, c)$. According to Proposition 9, we can conclude:

$$\forall x \in r^9, r(x) > 1$$

Theorem 2 is proved.

5.6. *Conclusion*

Let D be a domain containing nine nearest neighbours forming a square. We know that the Jacobian matrix J of Smith's scheme on D has a daughter which belongs to the family $M(x)$. According to Theorem 2 and Proposition 1, we can affirm:

$$r(J) > 1$$

We can conclude, thanks to Lattès's theorem:

Proposition 11. *On each domain D containing nine nearest neighbours forming a square, Smith's iteration is divergent, for any image, any reflectance map and any relative weight λ.*

Let us note that Horn and Brooks (1986) had noticed the instability of this method, while in the same time concluding that Strat's method worked well. This is the reason why we preferred to test Strat's method rather than Smith's method (see (Durou, 1988)). To overcome this instability, Horn and Brooks proposed to use a Gauss-Seidel relaxation, which is actually known to be more stable than the Jacobi iteration in the particular case of linear equations. However, they asserted the stability of the Gauss-Seidel relaxation applied to the equations of Smith's method, but did not prove it. Moreover, as they noted, a relaxation cannot be implemented in parallel, as it is possible for an iteration.

6. Summary

The iterative Shape from Shading methods of Strat and Smith have been studied, in relation to their convergence properties. By determining the spectral radius of the Jacobian matrices of both iterations, we could prove divergence in most of the cases, since any current image contains at least nine nearest pixels forming a square. So there is no chance that these methods could be of any practical use, and they must be abandoned.

However, it must be noted that, nowhere in the literature, have they been re-used, even if several authors mention them as important, from a historical point of view. Many other iterative algorithms have been proposed since, stemming from reasonings very similar to those of Strat and Smith. This suggests that the divergence of Strat's and Smith's methods is not methodological, but purely analytical.

Moreover, there exists one iterative algorithm for Shape from Shading, proposed by Lee (1985), very similar to Smith's, for which convergence has already been proved under certain circumstances. Finally, let us stress the fact that we could not generalize our demonstration of divergence to methods other than those of Strat and Smith. The remaining problem is that for most of the existing iterative algorithms for Shape from Shading, neither divergence nor convergence has yet been proved.

Note

1. Let us note that this proposition is usually false for non-symmetric matrices. Let us take for instance the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

It is easy to see that $r(M) = 0$. For $\bar{M} = [1]$, which ia daughter of M, $r(\bar{M}) = 1$, and so:

$$r(\bar{M}) > r(M).$$

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