Update postulates without inertia

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Abstract. Starting from Katsuno and Mendelzon postulates, a new set of postulates which is minimal and complete has been defined, these postulates characterize an update operator. The update operator class which is obtained is larger than Katsuno and Mendelzon's one, since it includes updating operators which are not inert, and which allow for the existence of unreachable states. The main property is that every update operator satisfying this new set of postulates admits an underlying ranked transition graph. The rank-ordering of its transitions has not to be faithful as in Katsuno and Mendelzon system.

1 Introduction

This paper deals with reasoning about change and especially with updating. The problem is to take into account the arrival of a new piece of information concerning a system which is represented by a knowledge base. Winslett [8] and later on Katsuno and Mendelzon [7] have shown that this piece of information can be of two kinds: either the piece of information describes the system itself, then the knowledge about the system is evolving, and the knowledge base should be *revised*; or it describes the evolution of the system, and then the knowledge base should be *updated*.

Katsuno and Mendelzon introduce 9 postulates in order to characterize an update operator. They point out that having an update operator verifying those postulates is equivalent to having, for each possible world ω, a pre-ordering relation which rank-orders every states of the world with respect to ω. They also demonstrate that, semantically, in order to update a knowledge base, you can make each of its models evolve independently towards the closest (with respect to the pre-ordering mentioned above) model in agreement with the new information, and then perform the union of all the resulting models.

These postulates are essentially criticized on one point: an update operator satisfying them must always prefer inertia (staying in the same state) to strict evolution (evolving towards a different state). So, in such an approach, update operators cannot describe transient states. Another point is that an update operator of that kind is such that every state of the world is accessible from every other state. It should be interesting to have a less restrictive formalization, so the aim of this work is to propose a minimal set of update postulates characterizing more update operators.

In the first part, Katsuno and Mendelzon postulates are described, and critics are more precisely exposed, then in order to answer to those critics a new set of postulates is defined which leads to a representation of updating in terms of ternary rank-ordering relations similarly to Katsuno and Mendelzon theory.

2 Katsuno and Mendelzon update postulates

A very important result in the field of reasoning about change in artificial intelligence is Alchourrón, Gärdenfors and Makinson [1] theory based on *rationality postulates* which any revision operator should satisfy. These postulates are constraining the way in which it is possible to revise a knowledge base in order to take into account a new piece of belief. These authors have shown that the existence of a *rational* revision operator is equivalent to the existence of a rank-ordering over the set of all possible states of the world (and conversely).

Winslett [8] has clearly shown the difference between revising a knowledge base and updating it, and that AGM theory is not appropriate to deal with updating. In revision: we learn something new about the world (our knowledge increases), while in updating: we learn that the world has changed. Then, Katsuno and Mendelzon [7] have defined a set of postulates characterizing update as it has been done for revision. And similarly, they established that the existence of a rational update operator is equivalent to the existence of a ternary rank-ordering over possible worlds (this time, each state of the world ω is associated with a relation which rank-orders all the states of the world with respect to ω). Semantically an operator satisfying these postulates is equivalent to a transition graph between states of the world.

2.1 Formalization

Let Ω be the non-empty set of possible states of the world, and z be an hypothetical state of the world not belonging to Ω. An updating operator is a function which computes a final state given an initial state and an information about the final state. Here the most general case is considered, i.e., every piece of knowledge can be imprecise: for example, the initial state can be ill-known. So every information is represented by the set of possible states in which this information is true (if the knowledge is complete then the set contains only one possible state). Consequently, an update operator will be defined over subsets of possible states of the world Ω.

2.2 Katsuno and Mendelzon postulates in a set formalism

A set formalism is used instead of a logical one, because updating is syntax independent, and using sets is simpler. Let g be an update operator, $g : 2^{\Omega} \times 2^{\Omega} \to 2^{\Omega}$, X is the representation of the state of the system before the evolution and A is the new information characterizing the result of this evolution, here are Katsuno and Mendelzon postulates expressed in the set formalism:

U1: $\forall X \subseteq \Omega, \forall A \subseteq \Omega, \quad g(X, A) \subseteq A$

When the system is learned to have evolved in a way such that the resulting state can be partially described by the information A, then the representation of the system state must be updated in accordance with A.

U2: $\forall X \subseteq \Omega, \forall A \subseteq \Omega, \quad X \subseteq A \Rightarrow g(X, A) = X$

If the representation X of the system state is in accordance with the information A, which describes the system state after a change, then the representation of the system state, after updating it, will still be X. This postulate is the one that leads to always prefer inertia to spontaneous evolutions.

U3: $\forall X \subseteq \Omega, \forall A \subseteq \Omega, \quad X \neq \emptyset$ and $A \neq \emptyset \Rightarrow g(X, A) \neq \emptyset$

If the representation X of the system state is consistent, and if a plausible information A characterizes its evolution, then the update of this representation by this information is always possible. It implies that every state is reachable from any other state.

U4: $\forall X \subseteq \Omega, \forall A, B \subseteq \Omega, \quad X = Y$ and $A = B \Rightarrow g(X, A) = g(Y, B)$

This axiom is useless in a set formalism.

U5: $\forall X \subseteq \Omega, \forall A, B \subseteq \Omega, \quad g(X, A) \cap B \subseteq g(X, A \cap B)$

U6: $\forall X \subseteq \Omega, \forall A, B \subseteq \Omega, \quad g(X, A) \subseteq B, g(X, B) \subseteq A \Rightarrow g(X, A) = g(X, B)$

U7: $\forall X \subseteq \Omega, \forall A, B \subseteq \Omega, \quad g(\{\omega\}, A) \cap g(\{\omega\}, B) \subseteq g(\{\omega\}, A \cup B)$

U8: $\forall X \subseteq \Omega, \forall A, B \subseteq \Omega, \quad g(X \cup Y, A) = g(X, A) \cup g(Y, A)$

U9: $\forall \omega \in \Omega, \forall A, B \subseteq \Omega, \quad g(\{\omega\}, A) \cap B \neq \emptyset \Rightarrow g(\{\omega\}, A \cap B) \subseteq g(\{\omega\}, A) \cap B$

Katsuno and Mendelzon postulates have been criticized and they can be challenged (e.g. [2]) on two points particularly. First, U2 always favors inertia: if the new information about the system change follows the representation of the current system state, then the representation must not change, so in this case the system must be considered to have remained in the same state. Thus, this postulate does not allow for the existence of transient states, i.e., states which are left immediately after being reached. But, for instance, consider the case of an object falling down from a table, the state expressing that the object is falling is transient and evolves towards a state in which the object is on the floor. This kind of situation cannot be modelled by a KM-update operator. Secondly, U3 forces every evolution from any state to any state to be possible. Meanwhile, one might need to express that a state in which an object is on the floor cannot evolve directly towards a state in which the object is on the table again, and that intermediary states are needed between those two states.

Here is the fundamental theorem which is linking those postulates with the existence of a ternary rank-ordering family (i.e. the existence of a transition graph between states):

Theorem 1 (Katsuno, Mendelzon). *g satisfies U1, U2, U3, U4, U5, U8, U9 $\Leftrightarrow$ $\forall \omega \in \Omega$, it exists a complete pre-ordering $\geq_\omega$ s.t. $\forall \omega' \in \Omega, \omega >_\omega \omega'$ and $g(X, A) = \bigcup_{\omega \in X} \{\omega' \in A$ s.t. $\forall \omega'' \in A, \omega' \geq_\omega \omega''\}$.*

The main limitation of this theorem is that the complete pre-ordering $\geq_\omega$ built on each state of the world ω must be *faithful* ($\forall \omega' \in \Omega, \omega >_\omega \omega'$), i.e., the state of the world ω on which is founded the rank-ordering is always strictly preferred to any other world. This inertia condition seems too strong and it is only due to postulate U2.

3 A new set of postulates

Considering the criticisms made above, a new set of postulates is introduced: after having removed postulate U2 which was there only to generate inertia and having modified postulate U3 in order to have a less restrict postulate, a normalization postulate U10 has been added. Here, the update operator g is a mapping from $2^{\Omega} \times 2^{\Omega \cup \{z\}}$ to 2^{Ω}, where z is an unreachable state not belonging to Ω.

U1, U5, U8 and U9 are left untouched.

U3bis: $\forall X \subseteq \Omega, \forall A, B \subseteq \Omega, \quad g(X, A) \neq \emptyset \Rightarrow g(X, A \cup B) \neq \emptyset$

To update every consistent representation of a system by any information is no longer necessarily possible, but, if an update of an initial representation X, with an information A is possible, then an update is necessarily still possible learning a piece of information more general than A.

U10: $\forall \omega \in \Omega, \quad g(\{\omega\}, \{z\}) = \emptyset$

Here, an unattainable state z is introduced, it constitutes the minimal bound of every rank-ordering $\geq_{\omega}$ build on Ω and relative to a given state ω (cf. Proposition 8).

Remark 1 *U1 can be deduced from U3bis, U5 and U10*

Proof. $\forall A \subseteq \Omega \cup \{z\}$, let us suppose that $\exists b \in \Omega \cup \{z\}$ such that $b \in g(X, A)$ and $b \notin A$. Using U5, $g(X, A) \cap \{b\} \subseteq g(X, \emptyset)$, i.e., $b \in g(X, \emptyset)$. Using U3bis, $g(X, \emptyset) \neq \emptyset \Rightarrow g(X, \{z\}) \neq \emptyset$, but, by U10, $g(X, \{z\}) = \emptyset$, so, the first assumption was impossible. So $\forall b \in \Omega \cup \{z\}$ such that $b \in g(X, A)$, $b \in A$. □

3.1 Completeness of the set U3bis, U5, U8, U9 and U10

This new set is more general than Katsuno and Mendelzon's one, now, a similar fundamental theorem must be established in order to show its well-foundedness. This theorem will guarantee that the existence of a transition graph between states is equivalent to the existence of an update operator satisfying our postulates (and conversely).

Lemma 2. *If g verifies U1, U5 then g also verifies:*
1. $\forall A, B \subseteq \Omega, \quad B \subseteq g(X, A) \Rightarrow g(X, B) = B$
2. $\forall A, B \subseteq \Omega, \quad A \not\subseteq g(X, A) \Rightarrow A \not\subseteq g(X, A \cup B)$
3. $\forall A, B \subseteq \Omega, \quad g(X, A \cup B) \subseteq g(X, A) \cup g(X, B)$

Proof. 1. If $B \subseteq g(X, A)$, then $g(X, A) \cap B = B$. From U1, $g(X, A) \subseteq A$, hence $B \subseteq A$, therefore $A \cap B = B$. From U5, $g(X, A) \cap B \subseteq g(X, A \cap B)$, so $B \subseteq g(X, B)$. U1 gives the converse inclusion.
2. From U5, $g(X, A \cup B) \cap A \subseteq g(X, A)$. So if $A \not\subseteq g(X, A)$ then $A \not\subseteq g(X, A \cup B) \cap A$.
3. From U5, $g(X, A \cup B) \cap A \subseteq g(X, A)$ and $g(X, A \cup B) \cap B \subseteq g(X, B)$, so $g(X, A \cup B) \cap (A \cup B) \subseteq g(X, A) \cup g(X, B)$. From U1, $g(X, A \cup B) \subseteq A \cup B$, thus $g(X, A \cup B) \cap (A \cup B) = g(X, A \cup B)$, and the result. □

Note that Lemma 2.1 entails $\forall X, A \subseteq \Omega, g(X, g(X, A)) = g(X, A)$, a property expected for an update operator.

Lemma 3. *If g verifies U5 and U9 then g also verifies:*
U6bis*:* $\emptyset \subset g(\{\omega\}, A) \subseteq B, \emptyset \subset g(\{\omega\}, B) \subseteq A \Rightarrow g(\{\omega\}, A) = g(\{\omega\}, B)$

Proof. Let A, B and ω be s.t. $\emptyset \subset g(\{\omega\}, A) \subseteq B$, and $\emptyset \subset g(\{\omega\}, B) \subseteq A$. Since $\emptyset \subset g(\{\omega\}, A) \subseteq B$, then $g(\{\omega\}, A) \cap B \neq \emptyset$, so we can use U9 and U5: $g(\{\omega\}, A \cap B) = g(\{\omega\}, A) \cap B$. Conversely, we also have $g(\{\omega\}, B \cap A) = g(\{\omega\}, B) \cap A$. Thus $g(\{\omega\}, A) \cap B = g(\{\omega\}, B) \cap A$. Since by assumption $g(\{\omega\}, A) \subseteq B$ and $g(\{\omega\}, B) \subseteq A$, we get $g(\{\omega\}, A) = g(\{\omega\}, B)$. □

Lemma 4. *If g verifies U1, U3bis and U9 then g also verifies:*

$$U7 : g(\{\omega\}, A) \cap g(\{\omega\}, B) \subseteq g(\{\omega\}, A \cup B)$$

Proof. Let $A, B \subseteq \Omega$ and $\omega \in \Omega$
- If $g(\{\omega\}, A \cup B) \cap A \neq \emptyset$ and $g(\{\omega\}, A \cup B) \cap B \neq \emptyset$ then using U9, $g(\{\omega\}, A) = g(\{\omega\}, (A \cup B) \cap A) \subseteq g(\{\omega\}, A \cup B) \cap A \subseteq g(\{\omega\}, A \cup B)$. Similarly $g(\{\omega\}, B) \subseteq g(\{\omega\}, A \cup B)$. Thus U7 is verified.
- If $g(\{\omega\}, A \cup B) \cap A \neq \emptyset$ and $g(\{\omega\}, A \cup B) \cap B = \emptyset$ then from U9, $g(\{\omega\}, A) \subseteq g(\{\omega\}, A \cup B)$. By U1, $g(\{\omega\}, B) \subseteq B$ therefore $g(\{\omega\}, A) \cap g(\{\omega\}, B) \subseteq g(\{\omega\}, A \cup B) \cap B = \emptyset$, so $g(\{\omega\}, A) \cap g(\{\omega\}, B) = \emptyset$ and U7 is easily verified. The proof is identical in the case where $g(\{\omega\}, A \cup B) \cap A = \emptyset$ and $g(\{\omega\}, A \cup B) \cap B \neq \emptyset$.
- If $g(\{\omega\}, A \cup B) \cap A = g(\{\omega\}, A \cup B) \cap B = \emptyset$ then $\forall x \in A \cup B, x \notin g(\{\omega\}, A \cup B)$. Using U1, $g(\{\omega\}, A \cup B) \subseteq A \cup B$. Thus $g(\{\omega\}, A \cup B) = \emptyset$. Using the converse of U3bis, $g(\{\omega\}, A) = g(\{\omega\}, B) = \emptyset$. U7 is again verified. □

Proposition 5. *Let g be an update operator verifying U1 U3bis, U5, and U9. For a given $\omega \in \Omega$, the relation on $\Omega \cup \{z\} \times \Omega \cup \{z\}$ such that $\omega' >_\omega \omega'' \Leftrightarrow \omega' \neq \omega''$ and $g(\{\omega\}, \{\omega', \omega''\}) = \{\omega'\}$ is a strict ordering.*

In the following proofs, $g(\{\omega\}, \{a, b, c\})$ will be written $g(abc)$, and so on.

Proof. Transitivity : If $a >_\omega b$ and $b >_\omega c$, we have: $g(ab) = \{a\}, g(bc) = \{b\}$ and a, b, c are distincts. Using Lemma 2.2, $b \notin g(ab)$ thus $b \notin g(abc)$, in the same way $c \notin g(bc)$ so $c \notin g(abc)$. From U1: $g(abc) \subseteq \{a, b, c\}$. By U3bis $g(abc) \neq \emptyset$ since $g(ab) \neq \emptyset$. Consequently $g(abc) = \{a\} \subseteq \{a, c\}$. From U1: $g(ac) \subseteq \{a, c\} \subseteq \{a, b, c\}$. From Lemma 2.1, $a \in g(ab)$ so $g(a) = \{a\} \neq \emptyset$. So from U3bis $g(ac) \neq \emptyset$. Thus we can use U6bis, $g(abc) = g(ac)$ and thus $g(ac) = \{a\}$ i.e., $a >_\omega c$.
Asymmetry and *Anti-reflexivity* are trivial by definition of $>_\omega$. □

Intuitively, $\omega' >_\omega \omega''$ means that starting from ω, ω' is a strictly more plausible next state than ω''. Therefore updating a knowledge base, whose extension is reduced to only one initial state ω, with a piece of information saying that the world has evolved either in ω' or in ω'' will give a resulting knowledge base which extension is the state ω'.

Proposition 6. *Let g be an update operator verifying U1, U3bis, U5, and U9. For a given $\omega \in \Omega$, the relation on $\Omega \cup \{z\} \times \Omega \cup \{z\}$ such that $\omega' \geq_\omega \omega'' \Leftrightarrow \omega' \in g(\{\omega\}, \{\omega', \omega''\})$ or $g(\{\omega\}, \{\omega', \omega''\}) = \emptyset$ is a complete pre-ordering.*

Remark 2 *g satisfies U1 $\Rightarrow \geq_\omega$ and $>_\omega$ defined from g are such that $a \geq_\omega b \Leftrightarrow b \not>_\omega a$.*

Proof. NOT ($b \neq a$ and $g(ba) = \{b\}$) $\equiv b = a$ or $g(ba) \neq \{b\}$. So, by U1, $b \not>_\omega a \equiv a \in g(ab)$ or $g(ab) = \emptyset$. □

Proof of Proposition 6. Transitivity : If $a \geq_\omega b$ and $b \geq_\omega c$, there are four cases:
- $a \in g(ab)$ and $b \in g(bc)$. Note that then $g(abc) = \emptyset$ is impossible (because of U3bis), as well as $g(abc) = \{b\}$ or $g(abc) = \{c\}$ or $g(abc) = \{b, c\}$. Indeed using U9 it is easy to see that any of these hypotheses leads to a contradiction, since $g(ab) \subseteq g(abc) \cap \{a, b\}$ leads to $g(ab) = \{b\}$, for $g(abc) = \{b\}$ or $\{b, c\}$ and thus to $a \notin g(ab)$; similarly U9 leads to $g(bc) = \{c\}$ if $g(abc) = \{c\}$. Due to U1, the only remaining possible case is $a \in g(abc)$. Now using U5, $a \in g(abc) \cap \{a, c\} \subseteq g(ac)$.
- $a \in g(ab)$ and $g(bc) = \emptyset$ then using the converse of U3bis, $g(b) = g(c) = \emptyset$. Now, using Lemma 2.2: $b \notin g(abc)$ and $c \notin g(abc)$. By U3bis, $g(abc) \neq \emptyset$. Using U1, we have $g(abc) = \{a\}$, and, as above, we get $a \in g(ac)$.
- The case $g(ab) = \emptyset$ and $b \in g(bc)$ is impossible (by the converse of U3bis, $g(b) = \emptyset$ and by Lemma 2.2 $b \notin g(b) \Rightarrow b \notin g(bc)$).
- $g(ab) = \emptyset$ and $g(bc) = \emptyset$, we get $g(ac) = \emptyset$ (using Lemma 2.3 and $g(a) = g(c) = \emptyset$)

In all cases, $a \geq_\omega c$.

Reflexivity : Since $>_\omega$ is anti-reflexive then $\geq_\omega$ is reflexive.

Completeness : If $a \not\geq_\omega b$, then from U1, $b \in g(ab)$ i.e., $b \geq_\omega a$. □

Intuitively, $\omega' \geq_\omega \omega''$ means that ω' is at least as accessible than ω'' from ω. So, for sure, if it is possible, updating a knowledge base whose extension is reduced to one initial state ω with the information that, now, the state of the system is ω' or ω'' gives a resulting knowledge base whose extension must contain the state ω'.

Proposition 7. *Let g be an update operator verifying U1, U3bis, U5 and U6bis, for a given $\omega \in \Omega$, the relation on $\Omega \cup \{z\} \times \Omega \cup \{z\}$ such that $\omega' =_\omega \omega'' \Leftrightarrow g(\{\omega\}, \{\omega', \omega''\}) = \{\omega', \omega''\}$ or $g(\{\omega\}, \{\omega', \omega''\}) = \emptyset$ is an equivalence relation.*

Remark 3 *If g satisfies U1 then $\geq_\omega$ and $=_\omega$ defined from g are such that $a =_\omega b \Leftrightarrow (a \geq_\omega b)$ and $(b \geq_\omega a)$*

Proof. $(a \geq_\omega b)$ and $(b \geq_\omega a) \equiv (a \in g(ab)$ or $g(ab) = \emptyset)$ and $(b \in g(ab)$ or $g(ab) = \emptyset) \equiv \{a, b\} \subseteq g(ab)$ or $g(ab) = \emptyset$. By U1, $\{a, b\} \subseteq g(ab) \equiv \{a, b\} = g(ab)$. Hence the result. □

Proof of Proposition 7. Since the relation $\geq_\omega$ is a pre-ordering, the relation $=_\omega$ defined by $x =_\omega y$ iff $(x \geq_\omega y$ and $y \geq_\omega x)$ is trivially transitive, reflexive and symmetric . □

Intuitively, $\omega' =_\omega \omega''$ means that from the initial state ω, it is as plausible to evolve towards ω' than towards ω''. Therefore, if it is possible, updating a knowledge base whose extension is the initial state ω with an information that the next state now is either ω' or ω'', gives the result that the world has evolved either in ω' or in ω'', (no more precision about this evolution is available).

Proposition 8. *If g verifies U1, U5 and U10 then $\geq_\omega$, defined from g, is such that $\forall a, \omega \in \Omega$, $a \geq_\omega z$*

Proof. Let us assume that $\exists a \in \Omega, z >_\omega a$, then since $a \neq z$, $g(\{\omega\}, \{a, z\}) = \{z\}$. But by U10, $g(\{\omega\}, \{z\}) = \emptyset$, so using Lemma 2.2, $z \notin g(\{\omega\}, \{a, z\})$. $\square$

This proposition points out that given an update function g, z, the unattainable state from any initial state ω, is the lowest bound of all pre-ordering $\geq_\omega$ associated to g.

Theorem 9. $\exists g : 2^\Omega \times 2^{\Omega \cup \{z\}} \to 2^\Omega$ *satisfying U3bis, U5, U8, U9 and U10* $\Rightarrow$ $\forall \omega \in \Omega$, *there exists a complete pre-ordering* $\geq_\omega$ *s. t.* $\forall S, X \subseteq \Omega$, $g(X, S) = \bigcup_{\omega \in X} \{\omega' \in S^\star$ *s.t.* $\forall \omega'' \in S, \omega' \geq_\omega \omega''\}$ *with* $S^\star = \{\omega' \in S,\ \omega' >_\omega z\}$.

Proof. If $\exists g : 2^\Omega \times 2^{\Omega \cup \{z\}} \to 2^\Omega$ satisfying U1, U3bis, U5, U8, U9 and U10.
- by Proposition 6, for a given $\omega \in \Omega$, the relation on $\Omega \cup \{z\} \times \Omega \cup \{z\}$ s.t.

$$\omega' \geq_\omega \omega'' \Leftrightarrow \begin{cases} \omega' \in g(\{\omega\}, \{\omega', \omega''\}) \\ or \\ g(\{\omega\}, \{\omega', \omega''\}) = \emptyset \end{cases} \quad \text{is a complete pre-ordering.}$$

- *let us show that* $g(\{\omega\}, S) \subseteq \{\omega' \in S^\star, \forall \omega'' \in S, \omega' \geq_\omega \omega''\}$. If $g(\{\omega\}, S) = \emptyset$, it's trivial. Let $a \in g(\{\omega\}, S)$. From U1,$a \in S$. From Lemma 2.1, $g(\{\omega\}, \{a\}) = \{a\}$. Using Lemma 2.3 $g(\omega, az) \subseteq g(\omega, a) \cup g(\omega, z)$. Using U10, $g(\omega, az) \subseteq \{a\}$. By U3bis, $g(\omega, az) \neq \emptyset$, so $g(\omega, az) = \{a\}$, i.e., $a >_\omega z$, therefore $a \in S^\star$. Let us assume that it exists $b \in S$ such that $b >_\omega a$, in this case $a \notin g(\omega, ab)$. From U5, $g(\{\omega\}, S) \cap \{a, b\} \subseteq g(\{\omega\}, ab)$, but $a \in g(\{\omega\}, S) \cap \{a, b\}$, so it's impossible. Therefore $a \in \{\omega' \in S^\star, \forall b \in S, \omega' \geq_\omega b\}$.
- *let us show that* $\{\omega' \in S^\star, \forall \omega'' \in S, \omega' \geq_\omega \omega''\} \subseteq g(\{\omega\}, S)$. Let $a \in \{\omega' \in S^\star$ s.t. $\forall \omega'' \in S, \omega' \geq_\omega \omega''\}$ then $a \in S^\star$. Let us note $\omega_1, \ldots \omega_n$ the elements of S. Since $a \in S, S = \{a, \omega_1\} \cup \cdots \cup \{a, \omega_n\}$. Since $a \in S^\star$, $g(az) = \{a\}$. By Lemma 2.1, $g(a) = \{a\}$. Using U3bis, $\forall \omega_i, g(a\omega_i) \neq \emptyset$. Moreover, $\forall \omega_i \in S, a \geq_\omega \omega_i$, so $a \in g(\{\omega\}, \{a, \omega_i\})$. Using U7, we get $\forall \omega_{1\ldots n} \in S, a \in g(\{\omega\}, \{a, \omega_1\} \cup \cdots \cup \{a, \omega_n\})$.
- by U8, $g(X, S) = \bigcup_{\omega \in X} g(\{\omega\}, S)$. $\square$

Here is the converse of the previous theorem:

Theorem 10. *There exists a function mapping each $\omega \in \Omega$ to a complete pre-order $\geq_\omega$ on $\Omega \cup \{z\} \times \Omega \cup \{z\}$ s.t. $\forall \omega, \omega' \in \Omega, \omega' \geq_\omega z \Rightarrow \exists g : 2^\Omega \times 2^{\Omega \cup \{z\}} \to 2^\Omega$ s. t. $\forall S, X \subseteq \Omega$, $g(X, S) = \bigcup_{\omega \in X} \{\omega' \in S^\star$ s.t. $\forall \omega'' \in S, \omega' \geq_\omega \omega''\}$ with $S^\star = \{\omega' \in S,\ \omega' >_\omega z\}$, which satisfies U3bis, U5, U8, U9 and U10*

Proof. g satisfies *U8* and *U10* by construction.

- *U3bis:* If $g(\{\omega\},A) \neq \emptyset$, let $a \in g(\{\omega\},A)$, i.e, $\forall \omega'' \in A, a \geq_\omega \omega''$, and $a \in A^\star$. Since $\geq_\omega$ is complete, $\forall B \neq \emptyset, \exists b \in B$ s.t. $\forall \omega'' \in B, b \geq_\omega \omega''$. If $b \geq_\omega a$ then, since $\geq_\omega$ is transitive, $\forall \omega'' \in A, b \geq_\omega \omega''$, so, $\forall \omega'' \in A \cup B, b \geq_\omega \omega''$. Moreover $b >_\omega z$ (else $z \geq_\omega a$ which is impossible), so $b \in (A \cup B)^\star$, thus, $b \in g(\{\omega\}, A \cup B)$. Otherwise $a \geq_\omega b$ then, similarly, $\forall \omega'' \in A \cup B, a \geq_\omega \omega''$, moreover, $a \in A^\star$, so, $a \in (A \cup B)^\star$ and $a \in g(\{\omega\}, A \cup B)$.

Now, if $g(X,A) \neq \emptyset$, i.e, $\bigcup_{\omega \in X}\{\omega' \in A^\star$ s.t. $\forall \omega'' \in A, \omega' \geq_\omega \omega''\} \neq \emptyset$ then $\exists \omega \in \Omega$ s.t. $\{\omega' \in A^\star$ s.t. $\forall \omega'' \in A, \omega' \geq_\omega \omega''\} \neq \emptyset$. For this ω, we have just seen that $g(\{\omega\}, A \cup B) \neq \emptyset$, so $g(X, A \cup B) \neq \emptyset$.

- *U5:* If $g(\{\omega\},A) \cap B = \emptyset$, U5 is trivial. Let $a \in g(\{\omega\},A) \cap B$, thus, $\forall \omega' \in A \cap B, a \geq_\omega \omega'$. Moreover, $a \in A^\star$ hence $a \in (A \cap B)^\star$. So $a \in g(\{\omega\}, A \cap B)$. Thus, $g(\{\omega\},A) \cap B \subseteq g(\{\omega\}, A \cap B)$ and therefore $\bigcup_{\omega \in X} g(\{\omega\},A) \cap B \subseteq \bigcup_{\omega \in X} g(\{\omega\}, A \cap B)$, then, by U8, $g(X,A) \cap B \subseteq g(X, A \cap B)$.

- *U9:* If $g(\{\omega\},A) \cap B \neq \emptyset$, let $a \in g(\{\omega\},A)$ and $a \in B$ therefore $a \in A^\star \cap B \subseteq A \cap B$ and $\forall \omega' \in A, a \geq_\omega \omega'$. Let $b \in g(\{\omega\}, A \cap B)$, so $b \in (A \cap B)^\star \subseteq B$ and $\forall \omega' \in A \cap B, b \geq_\omega \omega'$. Since $a \in A \cap B$, then $b \geq_\omega a$. By transitivity: $\forall \omega' \in A, b \geq_\omega \omega'$. Since $b \in (A \cap B)^\star, b \in A^\star$ and $b \in B$. Consequently $b \in g(\{\omega\},A) \cap B$. □

These two theorems show that the existence of an update operator satisfying U3bis, U5, U8, U9 and U10 is equivalent to the existence of rank-orderings on each set of transitions starting from a given state of the system. Let us notice that, here, the update operation must always lead to *reachable* states, this condition is added because U3 has been modified, this allows us to deal with unattainable states. Moreover, the inertia condition is suppressed by removing postulate U2 and therefore the complete pre-ordering mapped to each state of the world, this time, has not to be *faithful* [7] i.e. it is not necessarily inert. These theorems point also out that the update of a knowledge base whose extension is a set of state, can be computed by updating independently the knowledge bases corresponding to each state of the set, and then by doing the union of the results. $\{\geq_\omega, \omega \in \Omega\}$ describe a transition graph between states. *Every update operator satisfying U3bis, U5, U8, U9, U10 always admit an underlying ranked transition graph.*

Proposition 11. *If there exists a mapping associating to each interpretation $\omega \in \Omega$ a complete pre-ordering $\geq_\omega$ on $\Omega \cup \{z\} \times \Omega \cup \{z\}$, s.t. $\forall \omega, \omega' \in \Omega$, $\omega' \geq_\omega z$ then $\forall \omega, \omega' \in \Omega$, $z \geq_\omega \omega' \Leftrightarrow g(\{\omega\},\{\omega'\}) = \emptyset$ (ω' is not reachable from ω) where g from $2^\Omega \times 2^{\Omega \cup \{z\}}$ to 2^Ω is defined by: $g(X,S) = \bigcup_{\omega \in X}\{\omega' \in S^\star$, s.t. $\forall \omega'' \in S$, $\omega' \geq_\omega \omega''\}$ with $S^\star = \{\omega' \in S, \omega' >_\omega z\}$.*

Proof. If $\omega' \in \Omega$ is such that $z \geq_\omega \omega'$ then $\{\omega'\}^\star = \emptyset$ thus $g(\{\omega\},\{\omega'\}) = \emptyset$. □

The above property shows the possible existence of unattainable states from a given state. From a family of pre-orderings $\geq_\omega$ on $\Omega \cup \{z\} \times \Omega \cup \{z\}$, where z is the lowest bound of every $\geq_\omega$, the unattainable states from ω for g (the update function g associated to this family) are the states under z with respect to $\geq_\omega$.

3.2 Minimality of postulates U3bis, U5, U8, U9 and U10

Proposition 12. *U3bis, U5, U8, U9 and U10 are constituting a minimal set.*

Proof. It must be shown that no postulate is deducible from the others, i.e, for each postulate, one must find a function g violating it and satisfying the five remaining ones. For instance, the independence of U9 can be proven using the function g such that $\forall X \subseteq \Omega, \exists a \in \Omega, g(X, \Omega) = \{a\}$ and $\forall A \subseteq \Omega \cup \{z\}, A \neq \Omega, g(X, A) = A^{\star}$. U3bis, U5, U8 and U10 are true, U9 is not verified because $g(X, \Omega) \cap \{a, b\} \neq \emptyset$ and $g(X, \Omega \cap \{a, b\}) = \{a, b\} \not\subset g(X, \Omega) \cap \{a, b\} = \{a\}$. □

4 Conclusion

In relation with Katsuno and Mendelzon postulates, Boutilier [2] has discussed the update of a knowledge base by an observation. He introduces events which are functions that maps each world to a set of worlds (result of the event). To compute the update of a world ω by an observation O, one has to find the most plausible event e which transforms ω into a set of worlds where O is true. Boutilier's analysis agrees with our view in the sense that his class of update operators is also more general than Katsuno and Mendelzon's one (it doesn't verify neither U2 nor U3). He shows, however, that his framework can be restrained in order to satisfy every KM-postulate from U1 to U9.

Let us notice that a set of update postulates is not enough to update a knowledge base practically, because it does not define the update operator precisely but only its properties. In [3] it is pointed out that these postulates can be verified by some operators that are not intuitively satisfying as update operators, and thus concludes that they are not constraining enough.

The ternary relation of the form $\omega' >_{\omega} \omega''$ which expresses "closeness" between states and whose existence is guaranteed by the new set of updating postulates we propose, can be seen as a justification of the use of possibilistic Markov chains introduced in [4, 5]. Indeed the ternary relation $\omega' >_{\omega} \omega''$ reads in possibilistic terms $\pi(\omega'|\omega) > \pi(\omega''|\omega)$ where $\pi(.|\omega)$ is a conditional possibility distribution. It expresses that ω' is strictly more possible than ω'' as being the next state after ω. It should be clear that the use of the possibilistic view leads to commensurate $>_{\omega_1}$ and $>_{\omega_2}$, an hypothesis which is not required by Katsuno and Mendelzon. The construction of the function g from the ternary relation agrees with the extension principle in possibility theory which enables us to extend a function to (fuzzy) subsets of its domain; here, the function under consideration is the one which associates to ω its closest neighbours in A, and which is then extended to X. The coherence of Katsuno and Mendelzon postulates with possibility theory has been pointed out and established in [6].

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