QUALITATIVE OPERATORS FOR DEALING WITH UNCERTAINTY

Hamid SERIDI[*] , Florence BANNAY-DUPIN DE ST CYR[*,**], Herman AKDAG[*,***]

(*) LERI,
Université de Reims,
Rue des Crayères, BP 1035
51687 Reims Cedex 2, FRANCE
seridi@leri.univ-reims.fr

(**) LERIA,
Université d'Angers,
2, Bd Lavoisier
49045 Angers Cedex 1, FRANCE
bannay@univ-angers.fr

(***) LIP6,
Université P. & M. Curie,
4, Place Jussieu Boîte 169
75687 Paris Cedex 05, France
akdag@poleia.lip6.fr

Abstract

In Bayesian networks as well as in knowledge based systems, uncertainty in propositions can be represented by degrees of belief encoded by qualitative values. In this paper, we present an alternative approach to classic probability theory in the particular case where the set of probability degrees [0,1] is replaced by a totally ordered set of symbolic values. After having defined the four most expressive symbolic elementary operations (addition, subtraction, multiplication and division) allowing to manipulate these qualitative degrees of uncertainty, an axiomatic is proposed.

Keywords

Uncertainty, Probability Theory, Qualitative Degrees, Fuzzy Reasoning, Artificial Intelligence

I. INTRODUCTION

The aim of this paper is to present some results obtained in the framework of Qualitative Uncertainty Theory. The use of qualitative values is necessary in areas where the exact numerical value associated with a fact is unknown by experts.

Thus, in a first time, we can assert that our approach is an alternative approach to classic probability theory in the sense that we deal with qualitative uncertainty degrees instead of numerical probabilities. Beyond the use of symbolic values rather than numerical values, our approach manipulates these values with tools borrowed from the multivalued and fuzzy logic, and for these reasons, we may say that it constitutes a logico-symbolic alternative to classic probability theory. Our symbolic values can be compared with Zadeh's linguistic variables [14].

Polya emitted first the idea to conceive a probabilistic logic. Since many works in the field of Artificial Intelligence have been carried for instance by Nillson [9], with his probabilistic logic. Shafer [10] is also working on the same question. However, the main idea of these researches is that it leads to a probabilistic logic, which allocates probabilities to logical propositions and to logical deductions. More precisely, the management of knowledge is made by logical laws to which (numerical or symbolic) degrees of uncertainty are associated. Similarly, Dubois, Lang and Prade [6] have proposed a possibilistic logic in which propositions are associated with certainty degrees which are values in [0,1], called necessity measures. However, the deductions are less based on the exact numerical value of the degrees than on their relative ordering, in this way it is close to an other class of approaches described below.

A quasi-opposite approach is the management of knowledge using a classic probability theory in which numerical values and arithmetical operators are absent, so that calculations would be made symbolically with the help of logic operators. We set our framework in this second way as well as in the first, the logic and probabilities are intimately linked.

Keynes [7] and Koopmann [9] have been pioneers to propose intuitive (or symbolic) probabilities. The work of Spohn [13] can equally be quoted as precursory. A lot of similarities exist between our work and those of Aleliunas [3] and Darwiche & Ginsberg [5]. Following the works of Akdag, De Glas & Pacholczyk [1] on representation of uncertainty via multivalued logic, the first step of our work was the proposition of an axiomatic for symbolic probability theory. Khoukhi [8] and Akdag & Khoukhi [2] went further in the field of representation of uncertainty and imprecision via multivalued logic and they have proposed several solutions for the definition of each symbolic operator. Similarly, Pacholczyk & Pacholczyk [11] have proposed a new version of this theory that seems not yet satisfactory to us, in [15] the notion of linguistic operators expressivity is also presented without formalization.

The works of Aleliunas, Darwiche & Ginsberg, Akdag, De Glas & Pacholczyk, Akdag & Khoukhi and Pacholczyk &Pacholczyk have in common the following points :

1) They use non-numerical degrees of uncertainty. Indeed, it is frequent that the human expert expresses his knowledge by using qualitative approximations of numerical values.

2) They are based on the association < probability degrees / logic > : in their work, laws of probabilities are obtained with the help of logical operators which are associated to truth tables.

3) They have both developed an axiomatic theory which allows to obtain the results either from axioms, or from theorems.

In addition, they both allow the use of uncertainty (and imprecision) expressed in a qualitative form. Darwiche & Ginsberg insisted on the necessity to have a partial ordering on symbolic values and to define a symbolic summation and a symbolic conditioning and disconditioning. Similarly, Akdag & Khoukhi and Pacholczyk & Pacholczyk have argued in favor of a symbolic graduation going from *Impossible* to *Certain*, therefore a complete ordering. This graduation represents a similar domain than in possibility theory [6], but, here, the aim is to deal with symbolic values and to compose them. In possibility theory, the numerical degrees are only composed with min and max classical operators.

Our qualitative degrees of uncertainty belong to a graduated scale L_M composed of M values:

$L_M = \{ \tau_\alpha \}_{\alpha = 0 \ldots M-1}$ with the complete ordering relationship: $\tau_\alpha \leq \tau_\beta \Leftrightarrow \alpha \leq \beta$

For sake of simplicity, in the following, all our examples are in L_7.

To each degree belonging to our scale, we can associate a linguistic value:

τ_0⇔IMPOSSIBLE		τ_4⇔RATHER -POSSIBLE
τ_1⇔NEARLY-IMPOSSIBLE	τ_3⇔POSSIBLE	τ_5⇔VERY- POSSIBLE
τ_2⇔LITTLE- POSSIBLE		τ_6⇔CERTAIN

We also use the implication of Lukaciewicz: $\tau_\alpha \rightarrow_L \tau_\beta = \tau_{M-1-\alpha+\beta}$ if $\alpha > \beta$, $\tau_\alpha \rightarrow_L \tau_\beta = \tau_{M-1}$ otherwise,.
as well as the symbolic complementation: $\neg\tau_\beta = \tau_{M-1-\beta}$.

So as to be able to rewrite (or to translate in symbolic) the different axioms and the theorems of classic probability theory[1] , and after having introduced a complete ordering on the scale of degrees, it is easy to see (as Darwiche & Ginsberg) that it is necessary to define a symbolic addition (or a symbolic t-conorm, to generalize the addition), a symbolic multiplication (or a symbolic t-norm so as to be able to translate the disconditioning and independence) and a symbolic division (to translate the conditioning).

Now, let us introduce a supplementary constraint: "it is necessary that if C is the result of the division of A by B then B times C gives A" (relationship between qualitative multiplication and qualitative division). This constraint which seams very intuitive, has not been proposed in the literature. Another originality of our work is to define a qualitative subtraction instead of using artificially the complementation operator. Thus, we are going to define a symbolic difference to generalize the subtraction.

II. MOST EXPRESSIVE QUALITATIVE OPERATORS

II.A. Qualitative Multiplication

1.Properties

As in most of the other mentioned works, our qualitative multiplication has to verify some "intuitive" properties. Thus, a qualitative multiplication is a function MUL from $L_M \times L_M$ to L_M that verifies the following properties:

M1. MUL $(\tau_\alpha, \tau_0) = \tau_0$	Absorbent element
M2 MUL $(\tau_\alpha, \tau_{M-1}) = \tau_\alpha$	Neutral element
M3. MUL $(\tau_\alpha, \tau_\beta)$ = MUL $(\tau_\beta, \tau_\alpha)$	Commutability
M4. MUL (τ_α , MUL $(\tau_\beta, \tau_\gamma)$) = MUL (MUL $(\tau_\alpha, \tau_\beta)$, τ_γ)	Associativity
M5. MUL $(\tau_\alpha, \tau_\beta) \leq$ MUL $(\tau_\gamma, \tau_\delta)$ if $\tau_\alpha \leq \tau_\gamma$ and $\tau_\beta \leq \tau_\delta$	Monotonicity
M6. MUL $(\tau_\alpha, \neg\tau_\alpha) = \tau_0$	Complementarity

[1] Especially: $P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\neg A) = 1-P(A)$, $P(A\cup B) = P(A) + P(B)$ if $A\cap B = \emptyset$,
$P(A\cup B) = P(A) + P(B) - P(A\cap B)$, $P(A\cap B) = P(B/A) * P(A)$

M2 to M5 are properties of a t-norm [4] to which we add the absorbent element and the complementarity properties.

<u>Proposition:</u> The following function verifies the properties M1 ÷M6 :

$$\mathbf{MUL_L(\tau_\alpha, \tau_\beta) = \neg(\tau_\alpha \rightarrow_L \neg\tau_\beta) = \tau_\gamma \quad \text{with } \tau_\gamma = \max(\tau_{\alpha+\beta-(M-1)}, \tau_0).}$$

It corresponds in L_7 to the following table:

τ_α \ τ_β	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_0	τ_0	τ_0	τ_0	τ_0	τ_0	τ_0	τ_0
τ_1	τ_0	τ_0	τ_0	τ_0	τ_0	τ_0	τ_1
τ_2	τ_0	τ_0	τ_0	τ_0	τ_0	τ_1	τ_2
τ_3	τ_0	τ_0	τ_0	τ_0	τ_1	τ_2	τ_3
τ_4	τ_0	τ_0	τ_0	τ_1	τ_2	τ_3	τ_4
τ_5	τ_0	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5
τ_6	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6

Table 1 : Multiplication in L_7

It is the complementarity property that imposes the numerous occurrences of τ_0 in the table. Because of the complementarity, the product of two values different from τ_0 can be equal to τ_0; which is normal and prudent, given that we manipulate qualitative values that represent " intervals ". This function or this table is not the unique choice to define a multiplication satisfying the above properties, but as we shall see, it is the most expressive choice.

<u>2.The most expressive operator</u>

Let us examine the operators satisfying M1-M6 in some particular cases :

<u>a) MUL in L_1</u>

If there is only one probability degree in L_M (M = 1) then there is only one corresponding operator MUL :

MUL ↗	τ_0
τ_0	τ_0

<u>b) MUL in L_2</u>

If there are two probability degrees (M = 2) then there is also only one operator MUL satisfying M1-M6.

- using M1, we obtain the first column : MUL(τ_0, τ_0) = MUL(τ_1, τ_0) = τ_0 ;
- using M1 and M3, we obtain the first line : MUL(τ_0, τ_1) = τ_0 ;
- using M2, the last case : MUL(τ_1, τ_1) = τ_1.

This is shown in the following table :

MUL ↗	τ_0	τ_1
τ_0	τ_0	τ_0
τ_1	τ_0	τ_1

Properties leading to this result :

M1	M1, M3
	M2

<u>c) MUL in L_3</u>

If there are three probability degrees (M = 3) then there is also only one operator satisfying M1-M6 :

MUL ↗	τ_0	τ_1	τ_2
τ_0	τ_0	τ_0	τ_0
τ_1	τ_0	τ_0	τ_1
τ_2	τ_0	τ_1	τ_2

Properties leading to this result :

M1	M1, M3	
	M6	M2
	M2,M3	

<u>d) MUL in L_4</u>

If there are four probability degrees (M =4) then there are several possibilities for MUL according to the properties M1-M6 :

MUL ↗	τ_0	τ_1	τ_2	τ_3
τ_0	τ_0	τ_0	τ_0	τ_0
τ_1	τ_0	τ_0	τ_0	τ_1
τ_2	τ_0	τ_0	?	τ_2
τ_3	τ_0	τ_1	τ_2	τ_3

M1	M1, M3		
	M5	M6	M2
	M6		
	M2, M3		

For the cell MUL(τ_1, τ_1), we can see that the neighboring cells have the value τ_0, using M5 we get that

$$MUL(\tau_1, \tau_1) \leq MUL(\tau_1, \tau_2) = \tau_0,$$

hence the result.

The cell MUL(τ_2, τ_2), is not determined by the properties M1-M6, we can only say, using M5, that its value is between τ_0 and τ_2 : MUL(τ_2, τ_1) ≤ MUL(τ_2, τ_2) ≤ MUL(τ_2, τ_3). A choice can be done between the three values considering that it is more interesting to have the maximum of levels represented. With this point of view, *the value* τ_1 *is the relevant choice*. We call this choice of the operator MUL : *the most expressive operator*.

e) MUL in L_7

If there are seven probability degrees (M = 7) then there are also several possibilities for MUL according to the properties M1-M6, but there is only one most expressive operator satisfying them :

MUL ↗	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_0	τ_0	τ_0	τ_0	τ_0	τ_0	τ_0	τ_0
τ_1	τ_0	τ_0	τ_0	τ_0	τ_0	τ_0	τ_1
τ_2	τ_0	τ_0	τ_0	τ_0	τ_0	τ_1	τ_2
τ_3	τ_0	τ_0	τ_0	τ_0	τ_1	τ_2	τ_3
τ_4	τ_0	τ_0	τ_0	τ_1	τ_2	τ_3	τ_4
τ_5	τ_0	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5
τ_6	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6

The grey cells correspond to the choice of the most expressive operator.

f) Generalization :

Let MUL be an operator verifying the properties M1-M6.

- The value of every cell above the diagonal is τ_0, more formally, we have :

$$\forall \mathbf{i,j} \in \mathbf{[0,M\text{-}1]} \quad \textbf{if } \mathbf{i + j \leq M - 1} \textbf{ then } \mathbf{MUL(\tau_i, \tau_j) = \tau_0}.$$

Proof : From M1, $\forall i \in [0,M\text{-}1]$ $\quad MUL(\tau_0, \tau_i) = \tau_0$.

From M1 and M3, $\forall i \in [0,M\text{-}1]$ $\quad MUL(\tau_i, \tau_0) = \tau_0$.

From M6, $\forall i \in [0,M\text{-}1]$ $\quad MUL(\tau_i, \tau_{M-1-i}) = \tau_0$.

Hence, using M5, $\forall i,j \in [0,M\text{-}1]$ $\quad 0 \leq j \leq M\text{-}1\text{-}i$ $\quad MUL(\tau_i, \tau_j) = \tau_0$.

- The values of every bordering cell is also given by the properties of MUL :

$$\forall \mathbf{i} \in \mathbf{[0,M\text{-}1]} \qquad \mathbf{MUL(\tau_i, \tau_{M-1}) = \tau_i} \qquad \textbf{and} \qquad \mathbf{MUL(\tau_{M-1}, \tau_i) = \tau_i}$$

Proof : The result is given directly by M2 and M3.

- The values of the remaining cells (the cells under the diagonal) are constrained by M5 and M3, if we use the most expressive operator, we obtain :

$$\forall \mathbf{i,j} \in \mathbf{[0, M\text{-}1]}, \textbf{ if } \mathbf{i + j \geq M\text{-}1}, \qquad \mathbf{MUL(\tau_i, \tau_j) = \tau_{i+j-(M-1)}}$$

Proof : Using M6, $\forall i \in [0,M\text{-}1]$ $\quad MUL(\tau_i, \tau_{M-1-i}) = \tau_0 = \tau_{i+(M-1-i)-(M-1)}$.

Using M2, $\forall i \in [0,M\text{-}1]$ $\quad MUL(\tau_i, \tau_{M-1}) = \tau_i = \tau_{i+(M-1)-(M-1)}$

Hence, using M5, $\forall i,j \in [0, M\text{-}1]$, if $j \geq M\text{-}1\text{-}i$, then $\quad MUL(\tau_i, \tau_{M-1-i}) \leq MUL(\tau_i, \tau_j) \leq MUL(\tau_i, \tau_{M-1})$.

So, $\quad \tau_0 \leq MUL(\tau_i, \tau_j) \leq \tau_i$.

It means that i+1 values can be affected to i+1 cells (since j varies from M-1-i to M-1). So the most expressive way to affect a value to each product is to give to each one a different value:

$MUL(\tau_i, \tau_{M-1-i}) = \tau_0 \quad MUL(\tau_i, \tau_{M-1-i+1}) = \tau_1 \quad \ldots \quad MUL(\tau_i, \tau_j) = \tau_{i+j-(M-1)}$

$\ldots \quad MUL(\tau_i, \tau_{M-2}) = \tau_{i-1} \quad MUL(\tau_i, \tau_{M-1}) = \tau_i$

These results can be summarized by the following proposition :

<u>Proposition :</u> The most expressive operator satisfying the properties M1-M6 is such that :

$$\forall \mathbf{i,j} \in \mathbf{[0,M\text{-}1]}, \quad \mathbf{MUL(\tau_i, \tau_j) = max(\tau_{i+j-(M-1)}, \tau_0)}$$

II.B. Qualitative Addition:

Similarly, a qualitative addition is a function ADD from $L_M \times L_M$ to L_M that verifies the following properties :

A1. $ADD(\tau_\alpha, \tau_{M-1}) = \tau_{M-1}$	Absorbent Element
A2. $ADD(\tau_\alpha, \tau_0) = \tau_\alpha$	Neutral element
A3. $ADD(\tau_\alpha, \tau_\beta) = ADD(\tau_\beta, \tau_\alpha)$	Commutability
A4. $ADD(\tau_\alpha, ADD(\tau_\beta, \tau_\gamma)) = ADD(ADD(\tau_\alpha, \tau_\beta), \tau_\gamma)$	Associativity
A5. $ADD(\tau_\alpha, \tau_\beta) \leq ADD(\tau_\gamma, \tau_\delta)$ if $\tau_\alpha \leq \tau_\gamma$ and $\tau_\beta \leq \tau_\delta$	Monotonicity
A6. $ADD(\tau_\alpha, \neg\tau_\alpha) = \tau_{M-1}$	Complementarity

A2 to A5 are properties of a t-conorm [4] to which we add the absorbent element and the complementarity properties.

This second form of complementarity allows to have the sum of two complementary values equal to τ_{M-1}.

Proposition: **The following function verifies the properties A1÷A6 :**

$$\mathbf{ADD_L(\tau_\alpha , \tau_\beta) = (\neg\tau_\alpha \rightarrow_L \tau_\beta) = \tau_\delta \quad with\ \tau_\delta = min\ (\tau_{\alpha+\beta} , \tau_{M-1})}$$

It corresponds in L_7 to the following table:

τ_β / τ_α	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_0	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_1	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_6
τ_2	τ_2	τ_3	τ_4	τ_5	τ_6	τ_6	τ_6
τ_3	τ_3	τ_4	τ_5	τ_6	τ_6	τ_6	τ_6
τ_4	τ_4	τ_5	τ_6	τ_6	τ_6	τ_6	τ_6
τ_5	τ_5	τ_6	τ_6	τ_6	τ_6	τ_6	τ_6
τ_6	τ_6	τ_6	τ_6	τ_6	τ_6	τ_6	τ_6

Table 2 : Addition in L_7

It is the symbolic conorm associated with the implication of Lukaciewicz, also proposed by Pacholczyk & Pacholczyk.

The following proposition can be shown using the same approach as for multiplication:

Proposition : **The most expressive operator satisfying the properties A1-A6 is such that :**

$$\forall \mathbf{i,j} \in \mathbf{[0,M-1]}, \quad \mathbf{ADD(\tau_i, \tau_j) = min\ (\tau_{i+j}, \tau_{M-1})}$$

II.C Subtraction Qualitative:

A symbolic subtraction of two degrees τ_α and τ_β such that $\tau_\beta \leq \tau_\alpha$ is defined by the function SOUS from $L_M \times L_M$ to L_M that verifies the following properties:

S1. SOUS $(\tau_\alpha , \tau_0) = \tau_\alpha$ — Neutral element

S2. SOUS $(\tau_\alpha , \tau_\gamma) \leq$ SOUS $(\tau_\beta , \tau_\gamma)$ if $\tau_\alpha \leq \tau_\beta$ — Increasing relatively to the first argument

S3. SOUS $(\tau_\alpha , \tau_\beta) \geq$ SOUS $(\tau_\alpha , \tau_\gamma)$ if $\tau_\beta \leq \tau_\gamma$ — Decreasing relatively to the second argument

S4. SOUS $(\tau_\alpha , \tau_\alpha) = \tau_0$

Proposition: **The following function verifies the properties S1 ÷ S4 :**

$$\mathbf{SOUS_L (\tau_\alpha , \tau_\beta) = \neg\ (\tau_\alpha \rightarrow_L \tau_\beta) = \tau_S \qquad with\ \tau_S = max\ (\tau_{\alpha-\beta} , \tau_0)}$$

It corresponds in L_7 to the following table:

τ_β / τ_α	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_0	τ_0						
τ_1	τ_1	τ_0					
τ_2	τ_2	τ_1	τ_0				
τ_3	τ_3	τ_2	τ_1	τ_0			
τ_4	τ_4	τ_3	τ_2	τ_1	τ_0		
τ_5	τ_5	τ_4	τ_3	τ_2	τ_1	τ_0	
τ_6	τ_6	τ_5	τ_4	τ_3	τ_2	τ_1	τ_0

Table 3 : Subtraction in L_7

It is the first time that a qualitative subtraction is introduced in this type of theory. It will allow to define an important axiom linking the probability of the union to the probability of the intersection (see ***A.6***). The table that we propose corresponds in fact to the bounded difference of Zadeh, defined in the framework of fuzzy logic[4].

As in the previous cases, **the most expressive operator satisfying the properties S1-S4 is $\mathbf{SOUS_L (\tau_\alpha ,\tau_\beta)}$**

II.D. Qualitative Division:

A qualitative division of two degrees τ_α and τ_β such that $\tau_\alpha \leq \tau_\beta$ with $\tau_\beta \neq \tau_0$ is defined by the function DIV from $L_M \times L_M$ to L_M that verifies the following properties :

D1. $DIV(\tau_0 , \tau_\alpha) = \tau_0$ — Absorbtive element

D2. DIV $(\tau_\alpha , \tau_{M-1}) = \tau_\alpha$ — Neutral element

D3. DIV $(\tau_\alpha , \tau_\beta) \leq$ DIV $(\tau_\gamma , \tau_\beta)$ if $\tau_\alpha \leq \tau_\gamma$ — Increasing relatively to the first argument

D4. DIV $(\tau_\alpha , \tau_\beta) \geq$ DIV $(\tau_\alpha , \tau_\gamma)$ if $\tau_\beta \leq \tau_\gamma$ — Decreasing relatively to the second argument

D5. If $\tau_\alpha \neq \tau_0$ then $DIV(\tau_\alpha , \tau_\alpha) = \tau_{M-1}$, (where $DIV(\tau_0 ,\tau_0)$ is undetermined) — Boundary conditions

<u>Proposition:</u> The following function verifies the properties D1 ÷ D5 :

$$\mathbf{DIV}_L(\tau_\alpha , \tau_\beta) = \mathbf{MUL}_L(\tau_\alpha , \tau_\beta) \quad \textbf{if } \tau_\alpha = \tau_0 \quad \textbf{and } \tau_\beta \neq \tau_0$$

$$\mathbf{DIV}_L(\tau_\alpha , \tau_\beta) = \mathbf{ADD}_L(\tau_\alpha , \neg\tau_\beta) \textbf{ if } \tau_\alpha \leq \tau_\beta \quad \textbf{and } \tau_\beta \neq \tau_0$$

It corresponds in L_7 to the following table:

τ_β / τ_α	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_0							
τ_1	τ_0	τ_6					
τ_2	τ_0	τ_5	τ_6				
τ_3	τ_0	τ_4	τ_5	τ_6			
τ_4	τ_0	τ_3	τ_4	τ_5	τ_6		
τ_5	τ_0	τ_2	τ_3	τ_4	τ_5	τ_6	
τ_6	τ_0	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6

Table 4 : Division in L_7

Compared to the table proposed by Pacholczyk & Pacholczyk, we notice that all ambiguities are removed[2], and that the link between division and multiplication exists effectively, in other words :

If $\tau_C = \mathrm{DIV}(\tau_\alpha , \tau_\beta)$ Then $\tau_\alpha = \mathrm{MUL}(\tau_\beta , \tau_C)$.

Similarly, **the most expressive operator satisfying the properties D1-D5 is** $\mathrm{DIV}_L(\tau_\alpha , \tau_\beta)$

III. AN AXIOMATIC FOR A QUALITATIVE THEORY OF UNCERTAINTY

Considering the symbolic probability concept introduced by the aforementioned authors, we can propose a new axiomatic by using the predicate Uncert of a multivalued logic. The proposition "A is τ_α-certain" is translated into Uncert(A) = τ_α. The first axioms are those proposed by Akdag, De Glas & Pacholczyk, and that we find in aforementioned works[3].

U1 If $|=\tau_{M-1}$ A then Uncert(A) = τ_{M-1} i.e. if A is a tautology then A is certain
U2 If $|=\tau_0$ A then Uncert(A) = τ_0 i.e. if A is false then A is impossible
U3 If $|=\tau_\alpha$ A then Uncert($\neg$A) = $\neg\tau_\alpha$ i.e. if A is τ_α -certain then $\neg$A is $\neg\tau_\alpha$ -certain
U4 If A ≡ B then Uncert(A) = Uncert(B) i.e. if A is equivalent to B then Uncert(A) = Uncert(B)

This allows us to have trivial theorems:
T.1: If Uncert(A) = τ_{M-1} then Uncert($\neg$A) = τ_0 **T.2:** If Uncert(A) = τ_0 then Uncert($\neg$ A) = τ_{M-1}

The following axioms are more original:
<u>Definition</u>: A and B are quasi - independent events if with Uncert(A) = τ_α and Uncert(B) = τ_β we have: Uncert(A∩B) = τ_γ where $\tau_\gamma = \mathbf{MUL}_L(\tau_\alpha , \tau_\beta) = \max(\tau_{\alpha+\beta-(M-1)}, \tau_0)$

U5 If Uncert(A) =τ_α and Uncert(B) = τ_β and Uncert(A∩B) = τ_0 then Uncert(A∪B) = τ_δ
with $\tau_\delta = \mathrm{ADD}_L(\tau_\alpha , \tau_\beta)$.
It allows us to generalize: P(A∪B) = P(A) + P(B) if A∩B = ∅.
U6 If Uncert(A) =τ_α and Uncert(B) = τ_β and B⊂ A then Uncert(A∩¬B) = τ_S with $\tau_S = \mathrm{SOUS}_L(\tau_\alpha , \tau_\beta)$.
It allows us to generalize: P(A∩¬B) = P(A) - P(B) if B⊂ A.
U7 If Uncert(A) = τ_α and Uncert(B) = τ_β and Uncert(A∩B) = τ_γ then Uncert(A∪B) = τ_θ
with $\tau_\theta = \theta_L(\tau_\alpha , \tau_\beta) = \mathrm{ADD}_L[\ \mathrm{SOUS}_L(\tau_\alpha , \tau_\gamma) , \tau_\beta\ ]$.
It allows us to generalize: P(A∪B) = P(A) + P(B) - P(A∩B).
U8 If Uncert(A∩B) = τ_γ and Uncert(A) = τ_α then Uncert(B/A) = τ_D with $\tau_D = \mathrm{DIV}_L(\tau_\gamma , \tau_\alpha)$.
It allows us to generalize: P(B/A) = P(A∩B)/P(A).

The reader must notice the originality of axiom U7 where it is necessary first to undertake the subtraction, then the addition. One should not perform $\mathrm{SOUS}_L[\ \mathrm{ADD}_L(\tau_\alpha , \tau_\beta) , \tau_\gamma]$ as in the numerical case.
<u>Remark</u>: $\mathrm{ADD}_L[\ \mathrm{SOUS}_L(\tau_\alpha , \tau_\gamma), \tau_\beta\ ]$ is equivalent to $\mathrm{ADD}_L[\ \mathrm{SOUS}_L(\tau_\beta , \tau_\gamma) , \tau_\alpha\ ]$
As all axiomatics, the latter allows to demonstrate theorems. In our case, what is desirable is that these theorems are acceptable generalizations of classic probability theory. The goal of this paragraph is to show that the

[2] the result of a division is a unique value, and never an interval as Pacholczyk & Pacholczyk propose it.
[3] They allow us to generalize other properties: P(∅) = 0, P(Ω) = 1, P(¬A) = 1 - P(A).

symbolic operators that we have chosen and especially our axiomatic allow us to demonstrate intuitive properties (that will be exposed in a future paper).

IV. APPLICATION IN L_7

Let us consider an example of a gambler who proposes to a public to bet on three distinct cards in order to find a given card chosen beforehand. When someone wants to bet, he chooses a card among these three cards then with the help of rapid gestures, the gambler displace the cards and put them on-line ahead of the bettor.
If the player is not an accomplice, he found himself in front of two possibilities : either he avoids to follow the manipulations of the gambler in order not to be influenced or he paids attention to its gestures in order to follow the good card.
Considering the three types of bettor (accomplice (B), avoiding to follow manipulations (C), following manipulations (D)), it's very hard to deal with this problem numerically because there is no statistical information available about the type of the bettor. To avoid this kind of difficulty, we prefer to approach this problem qualitatively rather than quantitavely.
Let us consider four propositions about the player :
A = "the player wins"
B = "the player is an accomplice"
C = "the player avoids to follow manipulations"
D = "the player follows the gestures"
We can assign uncertainty degrees to each proposition according to our beliefs :

- we set Uncert(B)=τ_2 because it is "little possible" that the player is an accomplice since in this example, the accomplice must play rarely; Uncert(A/B)=τ_6 because it is "certain" that the accomplice wins each time he plays.
- Uncert(C)=τ_1 because it is "nearly impossible" that a player does not follow manipulations; Uncert(A/C)=τ_2 because it is "little possible" that a player who follows manipulations can win;
- Uncert(D)=τ_3 because it is "possible" that a player follows manipulations (usually people do so); Uncert(A/D)=τ_1 because it is "nearly impossible" for a player to win when he is influenced by manipulations.

Let us compute the uncertainty degree that a given player wins:
Uncert(A)=Uncert(($A \cap B$) $\cup$ ($A \cap C$) $\cup$ ($A \cap D$)) because the propositions B, C and D are mutually exclusives and cover the set of all the players.
Using axioms A4 and U5, we get Uncert(A)=ADD_L[Uncert($A \cap B$), ADD_L [Uncert($A \cap C$), Uncert($A \cap D$)]].
Since A and B are not quasi-independent, we can not use the definition Uncert($A \cap B$)=MUL_L(Uncert(A), Uncert(B)). This is why it is necessary to use conditionnal uncertainty U8 and the link between division and multiplication (cf. II.D) : Uncert($A \cap B$)=MUL_L(Uncert(A/B),Uncert(B)) .
We obtain Uncert(A) = ADD_L [MUL_L (Uncert(A/B),Uncert(B)),
ADD_L [MUL_L (Uncert(A/C),Uncert(C)), MUL_L (Uncert(A/D),Uncert(D))]]
so Uncert(A)=ADD_L (MUL_L (τ_6, τ_2), ADD_L [MUL_L (τ_2, τ_1), MUL_L (τ_1, τ_3)]) = τ_2.
It shows that it is little possible that a given person can win.

Computing the uncertainty degree that a winner is an accomplice, we find Uncert(B/A)=τ_6. It means that it is certain that a winner must be an accomplice.

The obtained results are coherent with the intuition, this example is also treated by Pacholczyk [ ], but, as far as we are concerned, his results are less intuitive than ours.

V. CONCLUSION

We have proposed a coherent and rigorous theoretical framework to deal with uncertainty values described in a qualitative manner, in the absence of reliable quantitative estimation. The four elementary operators that we propose are based unambiguously on well-defined functions. These operators are the most expressive ones which verify the intuitive properties that we impose. For shortness reasons, the intuitive character of these properties is left to be discussed in a next paper. In some next paper, we shall see how they allow us to propose new theorems (deduced from our new axiomatic) that will extend the classic probability theory.

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