Interval-valued Fuzzy Sets, Possibility Theory and Imprecise Probability

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Abstract

Interval-valued fuzzy sets were proposed thirty years ago as a natural extension of fuzzy sets. Many variants of these mathematical objects exist, under various names. One popular variant proposed by Atanassov starts by the specification of membership and non-membership functions. This paper focuses on interpretations of such extensions of fuzzy sets, whereby the two membership functions that define them can be justified in the scope of some information representation paradigm. It particularly focuses on a recent proposal by Neumaier, who proposes to use interval-valued fuzzy sets under the name "clouds", as an efficient method to represent a family of probabilities. We show the connection between clouds, interval-valued fuzzy sets and possibility theory.

Keywords: Interval-valued fuzzy sets, possibility theory probability theory

1 Introduction

Interval-valued fuzzy sets were introduced independently by Zadeh [24], Grattan-Guiness [14], Jahn [15], Sambuc [21], in the seventies, in the same year. An interval-valued fuzzy set (IVF) is defined by an interval-valued membership function. IVFs are a special case of L-fuzzy sets in the sense of Goguen [13], so as a mathematical object, it is not of special interest. Independently, Atanassov [1] introduced the idea of defining a fuzzy set by ascribing a membership function and a non-membership function separately, in such a way that an element cannot have degrees of membership and non-membership that sum up to more than 1. Such a pair was given the misleading name of "Intuitionistic Fuzzy Sets" [6] but corresponds to an intuition that differs from IVFs, although both turned out to be mathematically equivalent notions (e.g. G. Deschrijver, E. Kerre [4]). What is somewhat debatable with all these extensions is that it is not always explained where do the two membership functions come from. This lack of foundations is embarrassing as it triggers papers that just play with the (rather derivative, as subsumed by lattice-valued constructs) induced mathematical structures, without laying bare clear motivation. Indeed, IVFs should be the by-product of some principles or some practical situations where they appear as natural constructs. Fuzzy set theory itself somewhat suffered from the same type of difficulty, even if the situation is clearer now (see the discussion in [12]). In this paper, we review motivations for fuzzy sets with two membership functions, borrowing from various applied settings or paradigms. We particularly focus on a recent proposal by Neumaier [17], who independently came up with the same mathematical object, under the name *cloud* as a tool for representing imprecise probabilities. We show the connection between the possibilistic representation of probability sets, the so-called guaranteed possibility measures [5], and the notion of clouds.

2 Fuzzy Sets with Two Membership Functions.

An IVF is defined by a mapping F from the universe U to the set of closed intervals in $[0, 1]$. Let $F(u) = [F_*(u), F^*(u)]$. The union, intersection and complementation of IVF's is obtained by canonically extending fuzzy set-theoretic operations to intervals. As such operations are monotonic, this step is mathematically obvious. For instance, the most elementary fuzzy set operations are extended as follows, for conjunction, disjunction and negation, respectively

$$F \cap G(u) = [\min(F_*(u), G_*(u)), \min(F^*(u), G^*(u))];$$
$$F \cup G(u) = [\max(F_*(u), G_*(u)), \max(F^*(u), G^*(u))];$$
$$F^c(u) = [1 - F^*(u), 1 - F_*(u)].$$

An IVF is also a special case of type 2 fuzzy set (also introduced by Zadeh [24]). See [22] for a careful study of connectives for type 2 fuzzy sets; their results apply to the special case of IVFs.

A so-called "intuitionistic fuzzy set" (IFS) in the sense of Atanassov is defined by a pair of membership functions (F^+, F^-), where $F^+(u)$ is the degree of membership of u and $F^-(u)$ is its degree of non-membership. IFSs are supposed to verify the constraint

$$F^+(u) + F^-(u) \leq 1 \quad (1)$$

The basic intuitionistic fuzzy set-theoretic (or logical) operations for such IFS's are proposed as follows for conjunction $(F^+, F^-) \cap (G^+, G^-)$, disjunction $(F^+, F^-) \cup (G^+, G^-)$, and negation of (F^+, F^-) respectively:

$$(\min(F^+(u), G^+(u)), \max(F^-(u), G^-(u)));$$
$$(\max(F^+(u), G^+(u)), \min(F^-(u), G^-(u)));$$
$$(F^-(u), F^+(u)).$$

The use of an involutive negation acting on the pair of membership and non-membership functions makes IFS theory formally collapse to IVF theory. Indeed, constraint (1) always guarantees the existence of the membership interval $[F^+(u), 1 - F^-(u)]$, which can thus be identified with $[F_*(u), F^*(u)]$, and the set theoretic operations defined for IFS agree with the standard extension of basic fuzzy set connectives to interval-valued membership recalled above. For instance, negation in IFS theory becomes the above complementation to 1 extended to intervals. The same holds for min and max connectives. This collapse was already noticed by Atanassov and Gargov [2] in the eighties, and later emphasized by several scholars.

3 Naturally Generated Fuzzy Sets with Double Membership Functions

By *generated* IVF or IFS, we mean that the two membership functions can be interpreted in the framework of some meaningful paradigm of information representation. In other words, there is a process, related to a well-defined problem, for generating them, instead of considering them as primitive objects. This approach makes IVF or IFS closer to applications, and provides guidelines for the choice of connectives.

3.1 Examples of generated IVFs

The idea behind IVFs is that membership grades can hardly be precise. As fuzzy sets are supposed to model ill-defined concepts, some scientists have argued that requiring precision in membership grades may sound paradoxical. Although this view could be challenged, it naturally leads to IVFs in a first step of departure away from standard fuzzy sets. Indeed it is a long tradition in economics, engineering, etc., that intervals were used to represent values of quantities in case of uncertainty.

For instance, dealing with uncertain possibilistic information about the potential elements of an ill-known set, gives birth to a special kind of IVF named twofold fuzzy set [9]. Suppose that there is a set of objects Ω, and an attribute $\mathcal{A}$ on a scale U such that generally the value of attribute $\mathcal{A}$ is ill-known, say $\mathcal{A}(\omega) \in [u^-(\omega), u^+(\omega)]$ for $\omega \in \Omega$. The fuzzy set F of objects that verify a (standard) fuzzy property defined by a regular fuzzy subset Φ of U is then interval-valued : $\Phi(\omega) \in [\inf_{u \in [u^-(\omega), u^+(\omega)]} \Phi(u), \sup_{u \in [u^-(\omega), u^+(\omega)]} \Phi(u)]$.

A rough set [19] is a kind of IVF induced by indiscernibility of elements, modelled by an equivalence relation R on U. Subsets of U are approximated by their upper approximation $S^* = SoR$ and their lower approximation $S_* = (S^c oR)^c$. Fuzzy sets can be approximated likewise by an IVF, even if R is a fuzzy similarity relation (See Dubois and Prade [10]).

Interestingly these notions are not truth-functional in general: the upper and lower approximations of unions and intersection of fuzzy sets in these settings cannot be completely defined by the union and intersection defined for IVFs. So, while extensions of fuzzy connectives to IVFs can be straightforwardly defined, they do not always make sense. But this is related to the lack of compositionality of uncertainty since upper and lower degrees $F^*(u), F_*(u)$ quantify uncertainty about membership and may not express degrees of membership. For instance, in the twofold fuzzy sets paradigm, if the property F is Boolean and the $\mathcal{A}(\omega)$'s lie in fuzzy sets, F is approximated by two membership functions expressing degrees of possibility and necessity of Boolean membership.

3.2 Examples of generated membership and nonmembership functions

However, the membership and non-membership degrees in the IFSs may represent something else, namely the idea that concepts are more naturally approached by separately envisaging positive and negative instances. It is worth pointing out that the ideas developed by Atanassov may be seen as a fuzzification of sub-definite sets, introduced some years before by Narin'yani [16] who separately handles the (ordinary) set F^+ of elements known as belonging to the sub-definite set and the (ordinary) set F^- of elements known as not belonging to it, with the condition $F^+ \cap F^- = \emptyset$ (together with some additional information expressed by bounds on the cardinalities of F^+ and F^-). Condition (1) extends this requirement to the two membership functions F^+ and F^-. In other words, $F^+(u)$ is viewed as a lower bound on actual membership degree, and $F^-(u)$ is a lower bound on non-membership. It leads to the idea of bipolarity in information representation.

The idea of handling information in terms of separate positive vs. negative chunks is actually confirmed by psychological investigations [3]. It is currently studied in various domains of information engineering including preference modelling, learning, and reasoning (see for instance Dubois et al. [5], and papers from two special sessions at the IPMU 2004 conference in Perugia [7]) under the term "bipolarity". Dubois et al. [5] consider that information coming from the background knowledge of people is not of the same nature as information contained in observed data. The former is negative information because it rules out situations considered impossible a priori. They define a possibility distribution π on the universe of discourse as $\pi(u) = 1 - \iota(u)$, where $\iota(u)$ expresses a degree of impossibility reflecting the amount of surprise felt by an agent hearing that situation u occurred. Observed data are positive information summarised by a distribution δ. $\delta(u)$ measures the extent to which the possibility of u is supported by evidence; it is a degree of guaranteed possibility (in the sense that it focuses on situations that were actually observed). Clearly the pair of functions (δ, ι) is close to Atanassov intuitions. Especially, the IFS condition $\delta \leq \pi$ reflects the necessary consistency between positive and negative knowledge. The stronger condition requiring that δ and ι have disjoint supports can be enforced but may be challenged, as one may observe to a limited extent situations that are not fully ruled out by our background knowledge (for instance it is known that birds fly but one can see a few species that don't). $\delta = \pi$ correspond to the situation where our knowledge is fully supported by experience.

These ideas of bipolar information also make sense in various other contexts like preference modelling (desires vs. constraints), deontic logic (forbidden vs. explicitly permitted by jurisprudence), learning (counter-examples vs. examples), and voting. In Szmidt and Kacprzyk [20] the IFSs were illustrated on a voting example when "yes", "no" and "abstain" votes are possible.

4 Clouds and Imprecise Probabilities

This section recalls basic definitions and results due to Neumaier [17], cast in the terminology of fuzzy sets and possibility theory. A *cloud* is an IVF F such that $(0, 1) \subseteq \cup_{u \in U} F(u) \subseteq [0, 1]$. In the following it is defined on a finite set U or it is an interval-valued fuzzy interval (IVFI) on the real line (then called a cloudy number). In the latter case each fuzzy set has cuts that are intervals. When the upper membership function coincides with the lower one, ($F_* = F^*$) the cloud is called *thin.* When the lower membership function is identically 0, the cloud is said to be *fuzzy.*

Note that in both cases an IVF is characterised by a standard fuzzy set. However, while most fuzzy set specialists would call a thin cloud a fuzzy set, Neumaier considers a fuzzy set to be interpreted as a fuzzy cloud. A subcloud $G \sqsubseteq F$ of a cloud F is such that $\forall u \in U, [G_*(u), G^*(u)] \subseteq [F_*(u), F^*(u)]$. A minimal cloud has no other subcloud but itself [18]. This notion does NOT generalize fuzzy set inclusion, but expresses the idea that the membership grades of G are more precise than those of F.

Let $\overline{F}_\alpha = \{u, F^*(u) > \alpha\}$ be the upper cut of F and $\underline{F}_\alpha = \{u, F_*(u) \geq \alpha\}$ be the lower cut of F. A random variable x with values in U is said to belong to a cloud F if and only if $\forall \alpha \in [0,1]$:

$$P(x \in \underline{F}_\alpha) \leq 1 - \alpha \leq P(x \in \overline{F}_\alpha) \qquad (2)$$

under all suitable measurability assumptions.

Moreover $x \in F$ if and only if $x \in F^c$, the latter being called the *mirror* cloud. When the cloud is thin, it reduces to

$$P(x \in \underline{F}_\alpha) = P(x \in \overline{F}_\alpha) = 1 - \alpha. \qquad (3)$$

A distribution function (DF) defines a thin cloud containing all random variables having this distribution. So a thin cloud (or precise fuzzy set in our terms) is in this sense a generalized DF.

5 Casting Clouds in Possibility theory

A possibility distribution π is a mapping from U to the unit interval (hence a fuzzy set) such that $\pi(u) = 1$ for some $u \in U$. Several set-functions can be defined from them [5]:

- Possibility measures: $\Pi(A) = \sup_{u \in A} \pi(u)$
- Necessity measures: $N(A) = 1 - \Pi(A^c)$
- Guaranteed possibility measures: $\Delta(A) = \inf_{u \in A} \pi(u)$

Possibility degrees express the extent to which an event is plausible, i.e., consistent with a possible state of the world, necessity degrees express the certainty of events and Δ-measures the extent to which all states of the world where A occurs are plausible, and generally apply to guaranteed possibility distributions denoted by δ.

A possibility degree can be viewed as an upper bound of a probability degree [11]. Let $\mathcal{P}(\pi) = \{P, \forall A \subseteq U \text{ measurable}, P(A) \leq \Pi(A)\}$ be the set of probability measures encoded by π. Viewed in the scope of bipolarity, a possibility distribution describes negative information in the sense that all values u such that $\pi(u) = 0$ are ruled out. But in the absence of positive information, the guaranteed possibility distribution is $\delta = 0$. Hence, as expressing upper bound on probabilities, or on the actual possibility of occurrence of outcomes, π is better equated to the IVF with $F_* = 0$, $F^* = \pi$ (a fuzzy cloud, for Neumaier).

Now suppose two possibility distributions π_1 and π_2 are known to dominate a certain family of probabilities. The largest such family is clearly $\mathcal{P}_{12} = \mathcal{P}(\pi_1) \cap \mathcal{P}(\pi_2)$. Note that the probability of any event induced by $\mathcal{P}_{12}$ is such that

$$\max(N_1(A), N_2(A)) \leq P(A) \leq \min(\Pi_1(A), \Pi_2(A)) \qquad (4)$$

It is important to point out that $\mathcal{P}$ is not the set of probability measures generated by $\min(\pi_1, \pi_2)$. The latter may be empty without the former being empty. For instance, if $U = \{u, v\}$, $\pi_1(u) = 1, \pi_1(v) = 0.5, \pi_2(v) = 1, \pi_2(u) = 0.5$. Then the uniform probability on U lies in $\mathcal{P}(\pi_1)$ and $\mathcal{P}(\pi_2)$, but $\mathcal{P}$ is empty as $\min(\pi_1, \pi_2)$ is subnormalized. Let us recall the following result (see Dubois et al.[8]):

Proposition 1 *$P \in \mathcal{P}(\pi)$ if and only if $1 - \alpha \leq P(\pi(x) > \alpha), \forall \alpha \in (0,1]$*

Besides note that $P(x \in \underline{F}_\alpha) \leq 1 - \alpha$ is equivalent to $P(1 - F_*(x) > \beta) \geq 1 - \beta$, letting $\beta = 1 - \alpha$. So if we let $\pi_1 = F^*, \pi_2 = 1 - F_*$, it is clear from equation (2) that a random variable with probability measure P is in the cloud F if and only if it is in $\mathcal{P}(\pi_1) \cap \mathcal{P}(\pi_2)$. Hence eq.(4) is the possibilistic version of eq. (2). It also confirms that a possibility distribution is what Neumaier calls a fuzzy cloud (letting $F_* = 0$).

The following results indicate conditions under which the family $\mathcal{P}_{12} = \mathcal{P}(\pi_1) \cap \mathcal{P}(\pi_2)$ is empty or not. First consider a finite partition $\{A_1, \ldots, A_m\}$ of U. Since $\sum_{i=1}^m P(A_i) = 1$, it is clear that a necessary condition for $\mathcal{P}_{12} \neq \emptyset$ is due to eq.(4)

$$\sum_{i=1}^{m} \min(\Pi_1(A_i), \Pi_2(A_i)) \geq 1 \qquad (5)$$

In particular, $\sum_{i=1}^{n} \min(\pi_1(u_i), \pi_2(u_i)) \geq 1$ must hold if U is a finite set with n elements. Based on two-partitions another condition can be derived:

Theorem 1 *If* $c_{12} = \sup_{u\in U} \min(\pi_1(u), \pi_2(u)) < 0.5$ *then* $\mathcal{P}_{12} = \emptyset$.

Proof Indeed if the premise holds then $\exists c_{12} < \alpha < 0.5$ and a subset A such that $(\underline{\pi}_1)_\alpha \subseteq A$ and $(\underline{\pi}_2)_\alpha \subseteq A^c$. Then $\Pi_1(A) = \Pi_2(A^c) = 1$, so $\Pi_1(A^c) + \Pi_2(A) \leq \Pi_1(((\underline{\pi}_1)_\alpha)^c) + \Pi_2(((\underline{\pi}_2)_\alpha)^c) \leq 2\alpha < 1$, violating (5).

An important case where this proposition applies is when $\forall u \in U, \pi_1(u)+\pi_2(u) < 1$, i.e. a case when $(\pi_1, 1-\pi_2)$ is not a cloud. Recalling that a possibility distribution stands for an upper membership function, this condition does contradict the one linking (lower) membership function and non-membership function as requested by Atanassov.

Now let us find natural sufficient conditions for $\mathcal{P}_{12} \neq \emptyset$, and the natural candidate is $\forall u \in U, \pi_1(u) + \pi_2(u) \geq 1$, i.e. the pair $(\pi_1, 1-\pi_2)$ defines an IVF or a regular cloud. The most interesting case is the one of thin clouds (IVFs reduced to fuzzy sets) $\pi_1 = 1 - \pi_2$. Some general results are proved by Neumaier [18]. Here we basically give some examples that have practical importance:

Theorem 2 *If* U *is finite, then* $\mathcal{P}(\pi) \cap \mathcal{P}(1-\pi)$ *is empty.*

Proof Choose A to be the 0.5-cut of π. Then $\Pi(A) = 1, \Delta(A) = 0.5$, $\Delta(A^c) \leq \Pi(A^c) < 0.5$. It is easy to see that if $\pi = \pi_1 = 1 - \pi_2$, then eq. (4) now comes down to $1 - \Pi(A^c) \leq P(A) \leq 1 - \Delta(A))$. Hence eq. (4) is violated.

In fact, this is due to finiteness, and a simple shift of indices solves the difficulty. Let $\pi_1(u_i) = a_i$ such that $a_1 = 1 > \ldots > a_n > a_{n+1} = 0$. Consider $\pi_2(u_i) = 1 - a_{i+1} > 1 - \pi_1(u_i)$. Then $\mathcal{P}(\pi_1) \cap \mathcal{P}(\pi_2)$ contains the unique probability measure P such that the probability weight attached to u_i is $p_i = a_i - a_{i+1}, \forall i = 1 \ldots n$. To see it let $A_i = \{u_i \ldots u_n\}$. Then, $N_1(A_i) = 0$, $\Pi_1(A_i) = a_i$, $N_2(A_i) = 0$, $\Pi_2(A_i) = 1$, so, eq.(4) reads $a_i \leq P(A_i) \leq a_i$. So the values $P(A_1), \ldots, P(A_n)$ are completely determined, and enforce $p_i = a_i - a_{i+1}, \forall i = 1 \ldots n$. This result is formulated by Neumaier in terms of clouds [17], and he concludes that clouds generalise histograms. Note that possibility function π is viewed a kind of distribution function with respect to the ordering induced by the probability values p_i.

Theorem 3 *If* π *is a continuous fuzzy interval then* $\mathcal{P}(\pi) \cap \mathcal{P}(1-\pi) \neq \emptyset$.

Proof. Let $DF(x) = \Pi((-\infty, x])$. This is the distribution function of a probability measure $P \leq \Pi$. It is such that $\forall \alpha \in [0,1], P_\pi(x \in \overline{\pi}_\alpha) = 1 - \alpha$ according to eq.(3). $\overline{\pi}_\alpha$ forms a continuous sequence of nested sets and its definition involves the differentiability assumption (see [8] p. 285). Such a probability lies in $\mathcal{P}(\pi)$. Moreover, $P_\pi(x \in \underline{\pi}_\alpha) = P_\pi(x \in \overline{\pi}_\alpha)$ (due to continuity) $= 1 - \Pi((\overline{\pi}_\alpha)^c) = 1 - \Delta(\overline{\pi}_\alpha)$ (due to continuity) $= \sup_{u\in\overline{\pi}_\alpha} 1 - \pi(u)$. Hence $P_\pi \in \mathcal{P}(1-\pi)$.

Another probability measure in the thin cloud induced by a unimodal differentiable possibility distribution π has density of the form $p(u) = \frac{\pi'(u)\pi'(f(u))}{\pi'(f(u))-\pi'(u)}$ where π' is the derivative of π whose cuts are of the form $[u, f(u)]$ (see [8] p. 285). If π is convex on each side of the mode, then p is unimodal and is ordinally equivalent to π, with the same support. If π is concave on each side of the mode then p has two modes located on each end of the support of π and has its minimum on the mode of π.

In general, the probability measure whose density has the same support as a unimodal continuous π, and is ordinally equivalent to it, is unique. It lies in the corresponding thin cloud. For instance consider a symmetric π. Let M be the mode of π. By symmetry, $\overline{\pi}_\alpha$ is the open interval $(a_\alpha, 2M - a_\alpha)$. Then $P_\pi(x \in \overline{\pi}_\alpha) = DF(2M-a_\alpha)-DF(a_\alpha) = 1-2\cdot DF(a_\alpha) = 1-\alpha$. So for $u = a_\alpha \leq M, DF(u) = \pi(u)/2$, and $u = 2M - a_\alpha \geq M, DF(u) = \frac{1-\pi(u)}{2}$. This DF uniquely defines P_π. If π is a triangular fuzzy interval, then P_π is the uniform distribution on the corresponding support (see [8]). The determination of all P's in a continuous thin cloud is a matter of further research. They are clearly infinitely many.

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