Combining equilibrium logic and dynamic logic
(an introduction and a very brief overview)

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Motivation

motivation in a broad sense:

- recent studies reveals:
  - Answer Set Programming (ASP): central to various approaches in non-monotonic reasoning
  - equilibrium logic: semantical framework for ASP
    - [Pearce, Lifschitz, ...]
  - then, need for an extension of the language of ASP ...
    - (with some supportive concepts like:)
      - the representations of modalities
      - actions
      - ontologies
      - updates

main goal: beyond updates, adding other modalities to equilibrium logic (and, via that, to ASP)

motivation, in particular, for this work:

- the update of answer set programs

aim (here): manage atomic change of equi. models
Outline

1. Introduction
2. HT logic and equilibrium logic
3. A dynamic extension of HT logic and of equilibrium logic
4. DL-PA: dynamic logic of propositional assignments
5. Relating D-HT and DL-PA
6. Conclusion
Here-and-there models

**a bit history:** (idea [Gödel]) strength of “$\rightarrow$” between material implication “$\supset$” and intuitionistic implication “$\Rightarrow$”

- here-and-there model (HT model) = $(H, T)$:
  - $H, T$ sets of propositional variables from $\mathcal{P}$ with $H \subseteq T$

$$
\begin{align*}
\{p, q\} & \quad \text{‘there’} \\
\emptyset & \quad \text{‘here’}
\end{align*}
$$

- truth conditions:

  - $H, T \models p$ iff $p \in H$
  - $H, T \not\models \bot$
  - $H, T \models \varphi \land \psi$ iff $H, T \models \varphi$ and $H, T \models \psi$
  - $H, T \models \varphi \lor \psi$ iff $H, T \models \varphi$ or $H, T \models \psi$
  - $H, T \models \varphi \rightarrow \psi$ iff $H, T \models \varphi \supset \psi$ and $T, T \models \varphi \supset \psi$

- $(H, T)$ is a HT model of $\varphi$ iff $H, T \models \varphi$
Equilibrium models

Definition

$T \subseteq \mathcal{P}$ is an equilibrium model of $\varphi$ iff

1. $(T, T)$ is a HT model of $\varphi$;
2. (minimality condition) no $(H, T)$ with $H \subset T$ is a HT model of $\varphi$.

Example:

- $T = \emptyset$ is an equilibrium model of $\neg p = p \rightarrow \bot$:
  1. $\emptyset, \emptyset \models p \rightarrow \bot$
  2. min. cnd. always satisfied for $T = \emptyset$.
  3. Actually the only one: e.g. for $T = \{q\}$ we have $\emptyset, \{q\} \models p \rightarrow \bot$

- $p \lor q$ has only 2 equi. models, namely $T = \{p\}$ and $T = \{q\}$.
- $\neg\neg p$ has no equilibrium model:
  1. $\{p\}, \{p\} \models \neg\neg p$
  2. However, min. cnd. fails since $\emptyset, \{p\} \not\models \neg\neg p$. 
Equilibrium models

**Definition**

\( T \subseteq P \) is an *equilibrium model* of \( \varphi \) iff

1. \((T, T)\) is a HT model of \( \varphi \);
2. (minimality condition) no \((H, T)\) with \( H \subset T \) is a HT model of \( \varphi \).

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- \( T = \emptyset \) is an equilibrium model of \( \neg p = p \rightarrow \bot \):
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  2. however, min. cnd. fails since \( \emptyset, \{p\} \models \neg \neg p \).
Equilibrium logic

- $\chi \models \varphi$: logical consequence in HT models
- $\chi \models \varphi$: logical consequence in equilibrium models

**Definition**

$\chi \models \varphi$ iff for every equil. model $T$ of $\chi$, $(T, T)$ is HT model of $\varphi$.

**Example:** $\top \models \neg p$ and $\neg p \rightarrow q \models q$
Equilibrium logic

- \( \chi \models \varphi \): logical consequence in HT models
- \( \chi \models \varphi \): logical consequence in equilibrium models

**Definition**

\[ \chi \models \varphi \text{ iff for every equil. model } T \text{ of } \chi, (T, T) \text{ is HT model of } \varphi. \]

**example:** \( \top \models \neg p \) and \( \neg p \rightarrow q \models q \)
The language $\mathcal{L}_{D-HT}$

- extension of $\mathcal{L}_{HT}$ with dynamic modalities:

  (common to D-HT and dynamic equilibrium logic)

$$
\mathcal{L}_{D-HT} : \quad \varphi ::= p \mid \bot \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid [\pi] \varphi \mid \langle \pi \rangle \varphi \\
\pi ::= +p \mid -p \mid \pi; \pi \mid \pi \cup \pi \mid \pi^* \mid \varphi?
$$

where $p$ ranges over $\mathbb{P}$.

- atomic programs: $+p$ and $-p$
  - minimally update an HT model $(H, T)$

- abbreviations:

  $$
  \neg \varphi = \varphi \rightarrow \bot \\
  T = \bot \rightarrow \bot
  $$
Dynamic here-and-there logic (1)

notation:
\[ \text{HT} = \{(H, T) : H \subseteq T \subseteq \mathcal{P}\} : \text{the set of all HT models} \]

- interpretation of formulas and programs in D-HT:
  - for a formula \( \varphi \), \( \parallel \varphi \parallel_{D-HT} \subseteq \text{HT} \).

examples:

1. \( \parallel \neg p \parallel_{D-HT} = \{(H, T) : p \notin T\} \)
   (and so \( p \notin H \) by the heredity constraint in HT logic, i.e., \( H \subseteq T \))

2. \( \parallel p \lor \neg p \parallel_{D-HT} = \{(H, T) : p \in H \text{ or } p \notin T\} \)

3. (trivial, but important) \( \parallel \neg \neg p \parallel_{D-HT} = \{(H, T) : p \in T\} \)
   (and therefore, upshot: \( \parallel p \parallel_{D-HT} \subset \parallel \neg \neg p \parallel_{D-HT} \))

- for a program \( \pi \), \( \parallel \pi \parallel_{D-HT} \) is a relation on \( \text{HT} \).
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- for a program \( \pi \), \( \| \pi \|_{\text{D-HT}} \) is a relation on HT.
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$$HT = \{(H, T) : H \subseteq T \subseteq P\}$$: the set of all HT models

- interpretation of formulas and programs in D-HT:
  - for a formula $\varphi$, $\|\varphi\|_{D-HT} \subseteq HT$.

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- interpretation of formulas and programs in D-HT:
  - for a formula \( \varphi \), \( \| \varphi \|_{D-HT} \subseteq \text{HT} \).
  
  examples:

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     (and therefore, upshot: \( \| p \|_{D-HT} \subseteq \| \neg \neg p \|_{D-HT} \))

- for a program \( \pi \), \( \| \pi \|_{D-HT} \) is a relation on \( \text{HT} \).
Dynamic here-and-there logic (2)

interpretation of atomic update operations:

- **upgrade p:** $+p$ executable, viz. when $p \notin H$ (‘here’)
  (ex: all, but black below)
- **downgrade p:** same for $−p$, viz. when $p \in T$ (‘there’)
  (ex: all, but blue below)

$\langle \emptyset, \{p, q\} \rangle$

\begin{align*}
\|−p\|_{D-HT} & \quad \|+r\|_{D-HT} \\
\|−p\|_{D-HT} & \quad \|+r\|_{D-HT}
\end{align*}

$\langle \{p\}, \{p, q\} \rangle$

$\langle \emptyset, \{p, q, r\} \rangle$

$\langle \emptyset, \{q\} \rangle$

$\langle \{r\}, \{p, q, r\} \rangle$

remark:

- not allowed to apply $−p$ to blue again, neither $+r$ to green...
- Similarly, neither $+p$ to black, nor $−r$ to orange...
Dynamic here-and-there logic (3)

**Mixed and more sophisticated examples:**

1. $\ll (\langle +p \rangle T) \ll_{D-HT} = \{(H, T) : p \notin H\}$
2. $\ll (\langle +p \rangle T \land \langle -p \rangle T) \ll_{D-HT} = \{(H, T) : p \notin H \text{ and } p \in T\}$
3. $\ll \neg p? \ll_{D-HT} = \{((H, T), (H, T)) : p \notin T\}$
4. $\ll (\neg p \lor q)? \cup (\neg p; +q) \ll_{D-HT}$
   
   $$ = \{((H_1, T_1), (H_2, T_2)) : (H_2, T_2) \in \ll p \rightarrow q \ll_{D-HT}\}$$

**$\varphi$ is D-HT valid if and only if $\ll \varphi \ll_{D-HT} = \text{HT}$.**

**Examples:**

- neither $\langle +p \rangle T$ nor $\langle -p \rangle T$ is valid, but $\langle +p \cup -p \rangle T$ is.
- $[+p][+p]p$ and $[-p][-p] \neg p$ are valid.
- $[p? \cup -p?](p \lor \neg p)$ is valid too.
- finally, $[\pi] \varphi \rightarrow \neg \langle \pi \rangle \neg \varphi$ is valid, but the converse is not.
Dynamic here-and-there logic (3)

- **Mixed and more sophisticated examples:**
  
  1. \(\|\langle +p \rangle_T \|_{D-HT} = \{(H, T) : p \notin H\}\)
  
  2. \(\|\langle +p \rangle_T \land \langle -p \rangle_T \|_{D-HT} = \{(H, T) : p \notin H \text{ and } p \in T\}\)
  
  3. \(\|\neg p ? \|_{D-HT} = \{\left(\left(\left(H, T\right), \left(H, T\right)\right) : p \notin T\}\}
  
  4. \(\|\langle -p \lor q \rangle \lor \langle -p ; +q \rangle \|_{D-HT} = \left\{\left(\left(H_1, T_1\right), \left(H_2, T_2\right)\right) : \left(H_2, T_2\right) \in \|p \to q \|_{D-HT}\right\}\)

- \(\varphi\) is **D-HT valid** if and only if \(\|\varphi\|_{D-HT} = HT\).

**Examples:**

- neither \(\langle +p \rangle_T\) nor \(\langle -p \rangle_T\) is valid, but \(\langle +p \lor -p \rangle_T\) is.
- \([+p][+p]p\) and \([-p][-p]\neg p\) are valid.
- \([p? \lor \neg p?](p \lor \neg p)\) is valid too.
- finally, \([\pi]\varphi \to \neg \langle \pi \rangle \neg \varphi\) is valid, but the converse is not.
D-HT is more expressive than HT itself.

and even more interesting examples, but why?:

\[ [-p] \bot \leftrightarrow \neg p, \]
\[ \langle -p \rangle \top \leftrightarrow \neg \neg p, \text{ and} \]
\[ [+p] \bot \leftrightarrow p \]

are all valid.

1. \( \langle +p \rangle \top \) cannot be expressed in \( \mathcal{L}_{HT} \), but why?
   - no formula \( \mathcal{L}_{HT} \) that conveys \( p \in T \setminus H \).

2. 'heredity property of intuitionistic logic' does not hold in D-HT (in general), but always in HT.

**Definition**

if \((H, T) \in \| \varphi \|_{D-HT} \) then \((T, T) \in \| \varphi \|_{D-HT} \).

**counterex:** consider \((H, T) = (\emptyset, \{p\})\) and \(\varphi = \langle +p \rangle \top\)
D-HT is more expressive than HT itself.

and even more interesting examples, but why?:

- $[-p] \bot \leftrightarrow \neg p$,
- $\langle -p \rangle \top \leftrightarrow \neg \neg p$, and
- $[+p] \bot \leftrightarrow p$

are all valid.

1. $\langle +p \rangle \top$ cannot be expressed in $\mathcal{L}_{HT}$, but why?
   - no formula $\mathcal{L}_{HT}$ that conveys $p \in T \setminus H$.

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if $(H, T) \in \|\varphi\|_{D-HT}$ then $(T, T) \in \|\varphi\|_{D-HT}$.

**counterex:** consider $(H, T) = (\emptyset, \{p\})$ and $\varphi = \langle +p \rangle \top$
D-HT is more expressive than HT itself.

and even more interesting examples, but why?:

\[-p] \perp \leftrightarrow \neg p,\]
\[\langle -p \rangle \top \leftrightarrow \neg \neg p, \text{ and}\]
\[+[p] \perp \leftrightarrow p\]

are all valid.

1. \(\langle +p \rangle \top\) cannot be expressed in \(\mathcal{L}_{HT}\), but why?
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D-HT is more expressive than HT itself. and even more interesting examples, but why?:

\[-p] \bot \leftrightarrow \neg p, \\
\langle -p \rangle \top \leftrightarrow \neg \neg p, \text{ and} \\
[+p] \bot \leftrightarrow p
\]

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**Definition**

if \((H, T) \in \|\varphi\|_{D-HT} \) then \((T, T) \in \|\varphi\|_{D-HT}\).

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Dynamic equilibrium logic

- same definition as in HT for equilibrium models of D-HT...

  examples:

  1. $T = \emptyset$ is the only equi. model for all valid formulas of D-HT.
  2. $\neg \neg p$ has no equi. model.
  3. $T = \{q\}$ is the only equi. model for $\neg p \rightarrow q$.
  4. $T = \{p\}$ is the only equi. model for $\langle \neg p \rangle (\neg p \rightarrow q)$ and $\langle +q; +q \rangle (p \land q)$.
  5. however, $\langle \neg q \rangle (p \land q)$ does not have any, because neither even does a D-HT model.

- again same definition for logical consequence...

  examples:

  1. $\neg \neg p \models \varphi$, for every $\varphi$
  2. $p \lor q \models [\neg p?]q$
  3. $p \lor q \models [\neg p?][+p; +p](p \land q)$
A bit DlPa...

- recently studied in [BalbianiHerzigTroquard-Lics13]
- (here) expansion of assignments to arbitrary formulas as ‘atomic programs’, and without converse operator, still same expresivity and complexity results...
- same abbreviations as before... moreover,
  - $[\pi]\varphi := \neg\langle\pi\rangle\neg\varphi$
  - $\text{skip} := \top?$ (“nothing happens”)

**language:**

$$
\begin{align*}
\pi & := \ p := \varphi \mid \pi ; \pi \mid \pi \cup \pi \mid \pi^* \mid \varphi? \\
\varphi & := \ p \mid \bot \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \supset \varphi \mid \langle\pi\rangle\varphi
\end{align*}
$$

where $p$ ranges over a fixed set of propositional variables $\mathbb{P}$. 
DLPa continues...

**semantics:**
- a ‘valuation’ \( V \) is a subset of \( \mathbb{P} \).
- \( \|\varphi\|_{DL-PA} \in 2^\mathbb{P} \)
- \( \|\pi\|_{DL-PA} \): relation between valuations

**Definition**
- \( \varphi \) is DL-PA **valid** if and only if \( \|\varphi\|_{DL-PA} = 2^\mathbb{P} \).
- \( \varphi \) is DL-PA **satisfiable** if \( \|\varphi\|_{DL-PA} \neq \emptyset \).

**examples:**
1. \( \langle p:=\top \rangle \top, \langle p:=\top \rangle p \) and \( \langle p:=\bot \rangle \neg p \) are all valid.
2. as well as, \( \psi \land [\psi?] \varphi \supset \varphi \) and \( [p:=\top \cup q:=\top] (p \lor q) \)
Complexity of the full language (including the Kleene star)

- proved in [BalbianiHerzigTroquard-Lics13] that:
  - model and satisfiability checking are both EXPTIME complete.

⇒ we expect: both problems for our DL-PA are also EXPTIME complete because
  - lower bounds for both problems clearly transfer.
  - upper bounds for both problems can be established as in [BalbianiHerzigTroquard-Lics13].
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What we do hereafter: a quick summary

1. define a polynomial translation, $tr_1$ from D-HT to DL-PA
2. embed some notions of D-HT into DL-PA such as:
   - D-HT satisfiability, consequence in equilibrium models, etc.
3. check the problems of D-HT validity and consequence in equilibrium models
4. establish an EXPTIME upper bound for the complexity of latter problem
5. then define another translation, $tr_2$, the other way around and establish the EXPTIME hardness of these problems
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Key points of the talk

- propose a neat (sound and complete), but also simple logic: D-HT

- notice: strong relation between DL-PA and D-HT
- construct the correspondence via polynomial translations
- even prove mathematical properties of extensions

Further goals:
- first, epistemic extension
- then go on with reexamining the logical foundations of equilibrium logic, and ASP....