# Belief change and dynamic logic
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# Overview of presentation

- dynamic epistemic logic for belief change
- *public announcement logic for belief expansion*
- successful formulas
- *dynamic doxastic logic for belief revision*
- more complex revision with action models
- dynamic versus *conditional* epistemic logic for belief change
- dynamic versus *temporal* epistemic logic for belief change
- AGM postulates in dynamic epistemic logic
- advantages of dynamic epistemic logic for belief change

# Standard belief change

A deductively closed theory $\mathcal{K}$ is changed relative to a formula $\varphi$.
Three sorts of change: expansion, contraction, and revision.
These result in changed theories: $\mathcal{K} + \varphi$, $\mathcal{K} - \varphi$, $\mathcal{K} * \varphi$.

Typically, in expansion $\varphi \notin \mathcal{K}$ and $\varphi \in \mathcal{K} + \varphi$.
Typically, in contraction $\varphi \in \mathcal{K}$ and $\varphi \notin \mathcal{K} - \varphi$.
Typically, in revision $\neg\varphi \in \mathcal{K}$, $\neg\varphi \notin \mathcal{K} * \varphi$, and $\varphi \in \mathcal{K} * \varphi$.

## Issues in standard belief change

- Multi-agent belief change
- Higher-order belief change
- Iterated belief change
- Computational complexity
- Proof tools

# Dynamic epistemic logic for belief change

- In dynamic epistemic logic *epistemic operators* describe beliefs.
- In dynamic epistemic logic *dynamic operators* describe belief change.
- Epistemic operators are interpreted on *epistemic states* (pointed Kripke models)
- Dynamic operators are interpreted as *(epistemic) state transformers*.

References:
[Segerberg 1999] (Two traditions in the logic of belief),
[Lindström & Rabinowicz 1999] (DDL unlimited), etc.

# Dynamic epistemic logic for belief change

Identify a theory $\mathcal{K}$ with the believed formulas in an epistemic state:

$$\mathcal{K} = \{\psi \mid M, s \models B\psi\}$$

For revision: $\neg\varphi \in \mathcal{K}$, and $\varphi \in \mathcal{K} * \varphi$, and $\neg\varphi \notin \mathcal{K} * \varphi$ becomes

- $M, s \models B\neg\varphi$
- $M, s \models [*\varphi]B\varphi$

For contraction: $\neg\varphi \in \mathcal{K}$, and $\varphi \in \mathcal{K} * \varphi$, and $\neg\varphi \notin \mathcal{K} * \varphi$ becomes

- $M, s \models B\varphi$
- $M, s \models [-\varphi]\neg B\varphi$

For expansion: $\varphi \notin \mathcal{K}$, and $\varphi \in \mathcal{K} + \varphi$ becomes

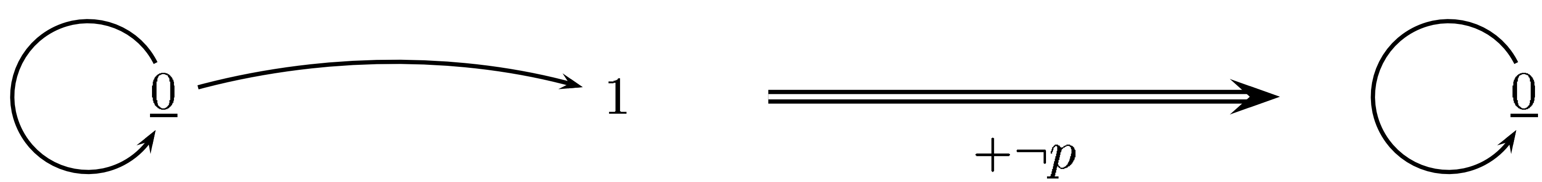
- $M, s \models \neg B\varphi$
- $M, s \models [+\varphi]B\varphi$

# Dynamic modal operators for epistemic programs

(Numerical) State transformation

$$x = 2 \quad \xRightarrow[x := 5]{} \quad x = 5$$

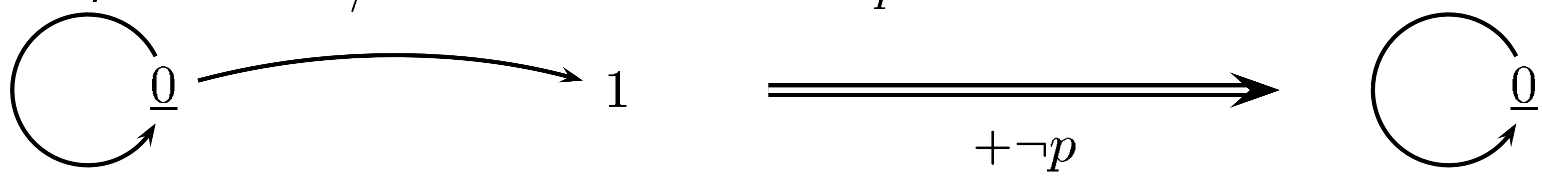
Epistemic state transformation



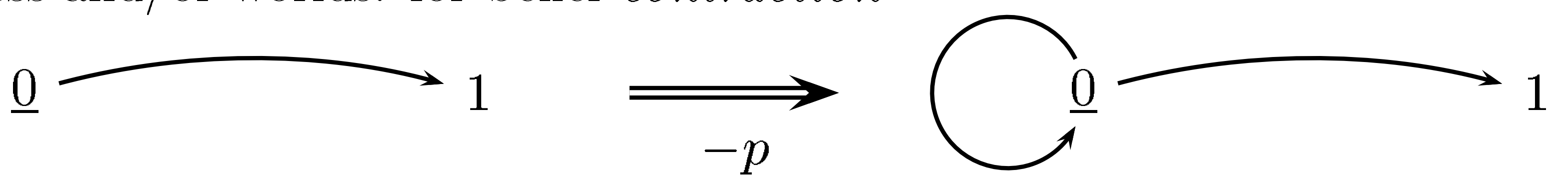

Dynamic modal operator $[+\neg p]$ is interpreted as state transformer $[\![+\neg p]\!]$.
Let $M$ be the model on the left, $M'$ the model on the right, then

$$M, s \models \neg Bp \quad M', s \models B\neg p \quad M, s \models [+\neg p]B\neg p$$

*Removing* access and/or worlds: for belief *expansion*

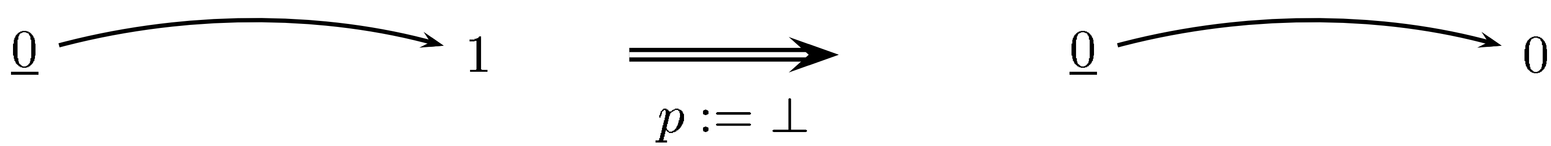



*Adding* access and/or worlds: for belief *contraction*

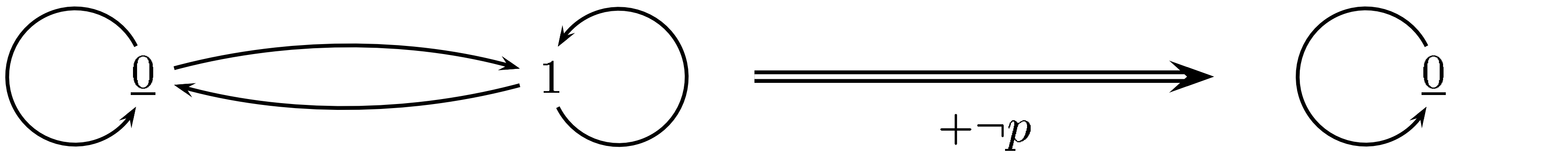



*Changing* access or domain: for belief *revision*

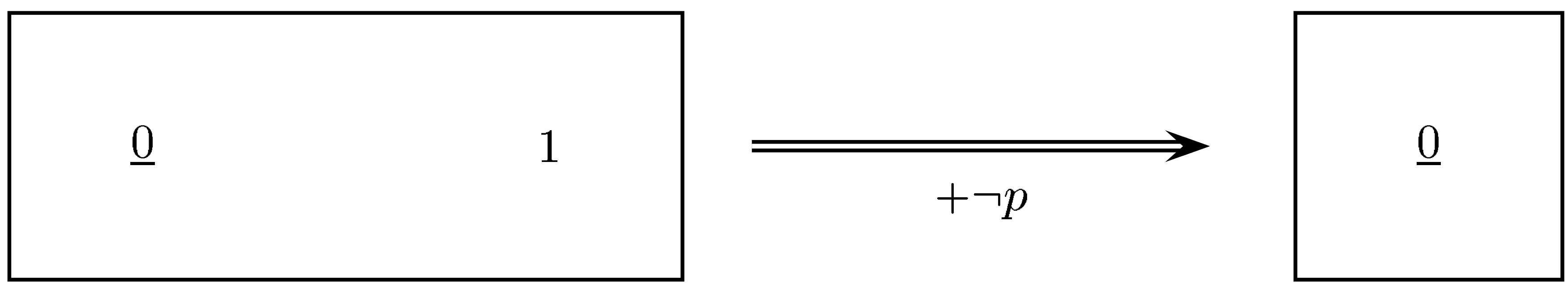

*Changing valuations*: for *update* instead of revision



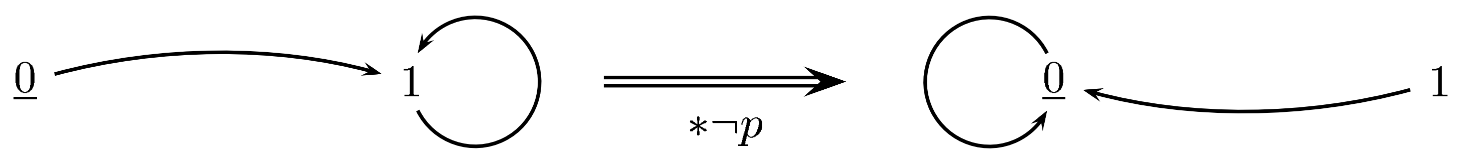
# Belief expansion for knowledge: public announcement



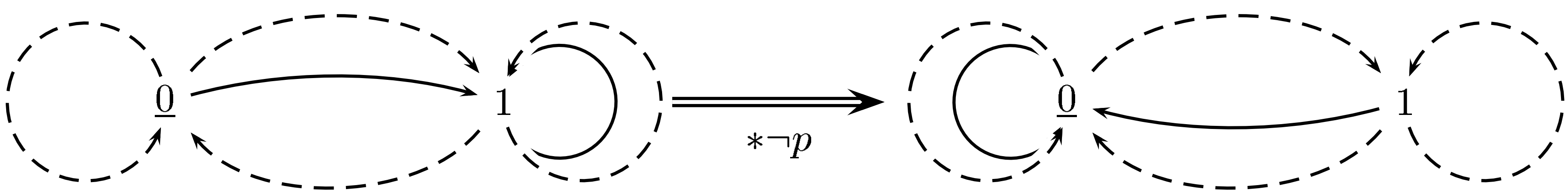

Simpler visualization



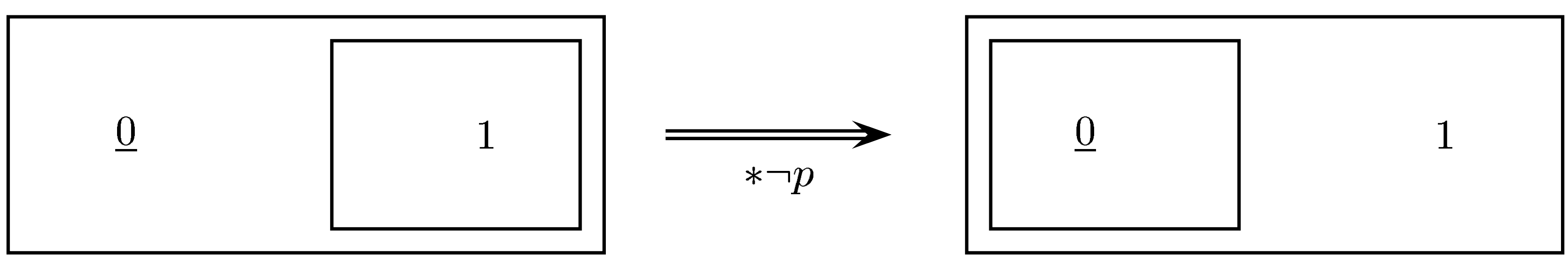
# Belief revision for degrees of introspective belief




Add another degree of belief




Simpler visualization



# Public Announcement Logic

**Language** $\varphi \; ::= \; p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_n\varphi \mid C_G\varphi \mid [\varphi]\psi$

**Structures** *epistemic model*: $M = \langle S, \sim, V \rangle$; *epistemic state*: $(M, s)$

**Semantics**

$$\begin{array}{ll} M, s \models p & \text{iff } s \in V_p \\ M, s \models K_n\varphi & \text{iff for all } s' \sim_n s : M, s' \models \varphi \quad \text{etc, etc, ...} \end{array}$$

$$M, s \models [\varphi]\psi \;\text{ iff }\; M, s \models \varphi \text{ implies } M|\varphi, s \models \psi$$

$M|\varphi \equiv \langle S', \sim', V' \rangle$ is defined as

$$\begin{array}{lcl} S' & = & [\![\varphi]\!]_M \\ \sim'_n & = & \sim_n \cap ([\![\varphi]\!]_M \times [\![\varphi]\!]_M) \\ V'_p & = & V_p \cap [\![\varphi]\!]_M \end{array}$$

$M|\varphi$ is the restriction of $M$ to the $\varphi$-states/worlds.

# Public Announcement Logic

Frame correspondence

- $K_n\varphi \to \varphi$
- $K_n\varphi \to K_nK_n\varphi$
- $\neg K_n\varphi \to K_n\neg K_n\varphi$

References: [Plaza 1989], [Gerbrandy & Groeneveld 1997],
[Baltag, Moss & Solecki 1998], [vBenthem 2000], [vDitmarsch, vdHoek, Kooi 2006].

# Public announcement logic for belief expansion: growth of knowledge

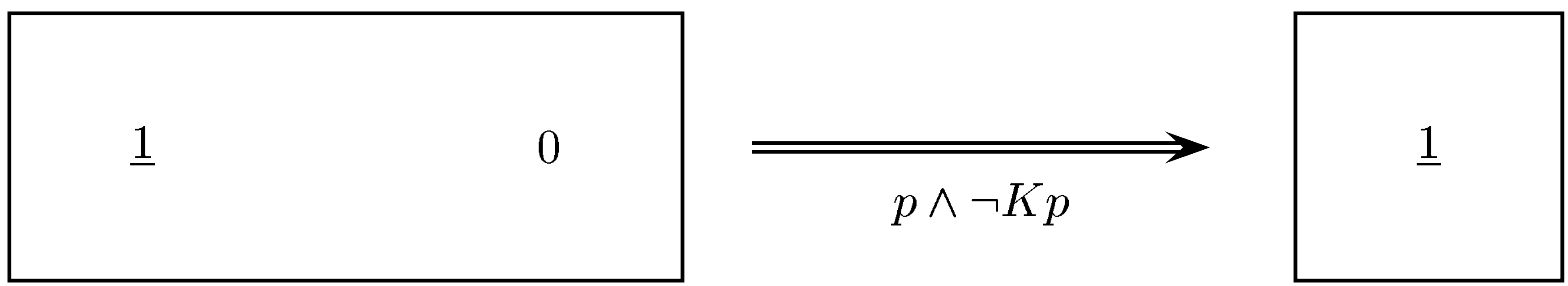
Identify $\mathcal{K}$ as before with $\{\psi \mid M, s \models K\psi\}$.
Let $M, s \models \varphi$. Suppose $\mathcal{K} \subset \mathcal{K} + \varphi$: let $\psi \in \mathcal{K} + \varphi$ but $\psi \notin \mathcal{K}$.

From $\psi \in \mathcal{K} + \varphi$ follows by positive introspection that $K\psi \in \mathcal{K} + \varphi$.
From $\psi \notin \mathcal{K}$ follows by negative introspection that $\neg K\psi \in \mathcal{K}$.
From $\neg K\psi \in \mathcal{K} \subset \mathcal{K} + \varphi$ and $K\psi \in \mathcal{K} + \varphi$ follows a contradiction.

Knowledge *growth* is contradictory.
(Because ignorance of the expansion formula cannot be preserved.)
Fortunately, knowledge *change* is possible.
Also, knowledge growth is possible for a *fragment* of the language.
[vDitmarsch, vdHoek & Kooi 2005] (Public announcements and belief expansion)

# Public announcement logic for belief expansion: unsuccessful updates

After saying "($p$ is true and) You don't know that $p$ is true" this is no longer true.




Formally, $M, 1 \models \langle p \wedge \neg K p \rangle \neg (p \wedge \neg K p)$.

Related to Moore-sentences: $K(p \wedge \neg K p)$ is inconsistent. [Hintikka 1962]

[Gerbrandy 1999] (Bisimulations on Planet Kripke) introduces unsuccessful updates.
See also [vDitmarsch & Kooi 2005] (The Secret of my Success)

# Unsuccessful updates

- $\varphi$ is a successful formula iff $[\varphi]\varphi$ is valid
- $\varphi$ is an unsuccessful formula iff $\varphi$ is not successful
- $\varphi$ is a successful update in $(M, s)$ iff $(M, s) \models \langle\varphi\rangle\varphi$
- $\varphi$ is an unsuccessful update in $(M, s)$ iff $(M, s) \models \langle\varphi\rangle\neg\varphi$

The definition of successful formula captures the notion of success in belief revision.
Note that:

- $[\varphi]\psi$ is valid iff $[\varphi]C_N\psi$ is valid
- $[\varphi]\psi$ is logically equivalent to $\varphi \to [\varphi]\psi$

From these two follows that $[\varphi]\varphi$ is equivalent to $\varphi \to [\varphi]C_N\varphi$.
The latter says: if $\varphi$ is true, then announcing $\varphi$ makes it common knowledge.

# Unsuccessful updates

What formulas are successful? Unclear!
Obvious inductive definitions fail. Even when both $\varphi$ and $\psi$ are successful:

- $\neg\varphi$ may be unsuccessful (for $\varphi = p \wedge \neg Kp$)
- $\varphi \wedge \psi$ may be unsuccessful (for $\varphi = p$ and $\psi = \neg Kp$)
- $[\varphi]\psi$ may be unsuccessful
- $\varphi \rightarrow \psi$ may be unsuccessful

There are relevant successful fragments of the language:

- *common knowledge formulas* are successful: $[C_N\varphi]C_N\varphi$ is valid
in other words: announcing common knowledge is a waste of time!
- *preserved formulas* are successful: $\varphi ::= p \mid \neg p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid K_n\varphi \mid C\varphi \mid [\neg\varphi]\psi$
(submodel) preservation is a special case of success:
from $\varphi \rightarrow [\psi]\varphi$ for arbitrary $\psi$ follows $\varphi \rightarrow [\varphi]\varphi$.

# Unsuccessful updates

Preservation is indeed closed under $[\neg\varphi]\psi$!
Let $M, s \models [\neg\varphi]\psi$, and $M' \subseteq M$ such that $s \in M'$.

Assume $M', s \models \neg\varphi$.
Then $M, s \models \neg\varphi$ by *contraposition* of the inductive hypothesis for $\varphi$.
From that and $M, s \models [\neg\varphi]\psi$ follows $M|\neg\varphi, s \models \psi$.
From the inductive hypothesis for $\psi$ follows $M'|\neg\varphi, s \models \psi$.
Therefore $M', s \models [\neg\varphi]\psi$ by definition.

Preservation (excluding updates), see [v Benthem 2002] (One is a lonely number)

# Doxastic epistemic logic for belief revision

**Language** $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid B_n^x\varphi \mid K_n\varphi \mid [*\varphi]\psi$

**Structures** *doxastic epistemic model*: $M = \langle S, <, V \rangle$;
*doxastic epistemic state*: $(M, s)$

*Belief function* $<$ partitions the domain into disjoint total orders $\langle \mathcal{X}, < \rangle$.
For each agent and state, $<_n^s$ totally orders the plausible states in the domain.
Write $<_n^s(s')$ for the degree/level of state $s'$ in order $<_n^s$.
Accessibility $\rightarrow_n^x$ is induced by: $s \rightarrow_n^x s'$ iff $<_n^s(s') \leq x$.
Accessibility $\rightarrow_n^{\mathcal{X}}$ is defined as $\bigcup_{x \in \mathcal{X}} \rightarrow_n^x$.

**Semantics**
$M, s \models B_n^x\varphi$ iff for all $s'$ with $s \rightarrow_n^x s'$: $M, s' \models \varphi$.
$M, s \models K_n\varphi$ iff for all all $s'$ with $s \rightarrow_n^{\mathcal{X}} s'$: $M, s' \models \varphi$.

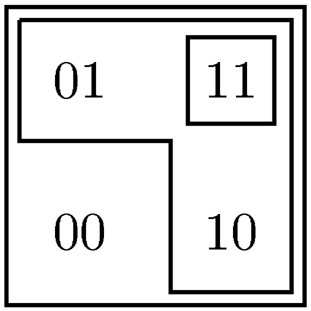
# Doxastic epistemic logic

References

Similar logics (static part) have been investigated in [Kraus & Lehmann 1988], [Stalnaker 1996], [vdHoek 1980s], [Ferguson & Labuschagne 2002], [vDitmarsch & Labuschagne 2003], [Aucher 2003], [Liu 2004], [Asheim & Søvik 2005], [Board 2005].

This setup is based on
[vDitmarsch 2005] (Prolegomena to Dynamic Logic for Belief Revision).

**Example** Doxastic epistemic model $M = \langle \{00, 01, 10, 11\}, <, V \rangle$



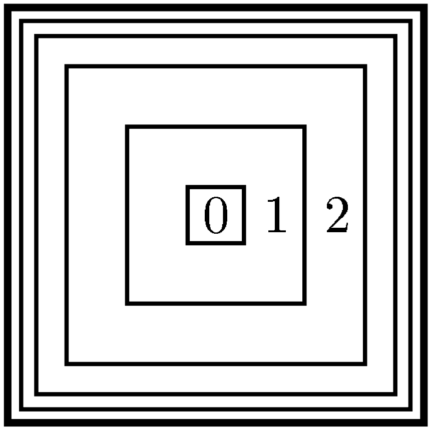

$11 <^{00} 01 =^{00} 10 <^{00} 00$, and $11 <^{01} 01 =^{01} 10 <^{01} 00$, and (all the same);

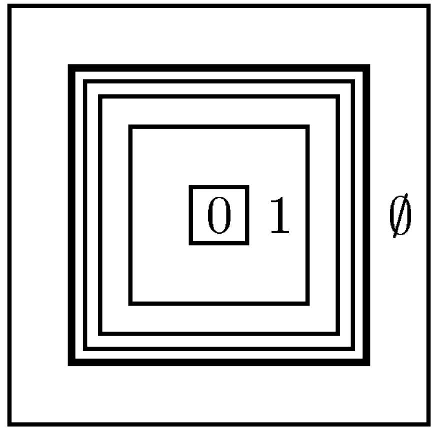
$V_p = \{10, 11\}$, and $V_q = \{01, 11\}$;

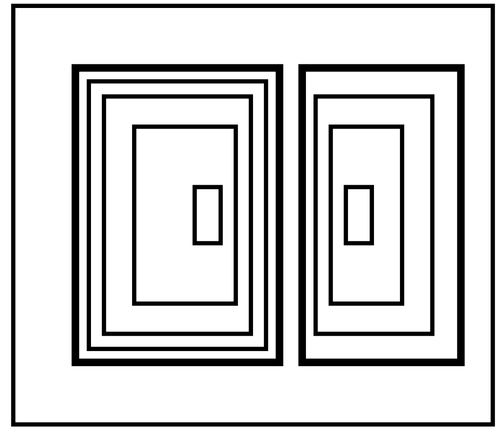
$M \models B(p \wedge q) \quad (= B^0(p \wedge q))$
$M \models B^1(p \vee q)$
$M \models K(p \vee \neg p) \quad (= B^2(p \vee \neg p))$

$M, 00 \models B^1(p \vee q)$, because $00 \rightarrow^1 10$, and $00 \rightarrow^1 01$, and $00 \rightarrow^1 11$, and $M, 10 \models p \vee q$, and $M, 01 \models p \vee q$, and $M, 11 \models p \vee q$

# Preference



- Revision is relative to an order / system of spheres / preference relation [Lewis 1973], [Grove 1988], [Spohn 1988].
- For each state in the domain, there is such a preference order.
- The innermost sphere corresponds to what is normally believed.
- Assume a total order with a least element.
- With additional constraints, degrees of belief and knowledge result.

# Frame correspondence for doxastic epistemic logic

Let $x, y \in \mathcal{X}$ be arbitrary.

- $B^x\varphi \to B^y\varphi$ iff $x \leq y$ (*strong belief implies weak belief*)
- $B^x\varphi \to \neg B^x\neg\varphi$ (*seriality*)
- $B^x\varphi \to B^y B^x\varphi$ (*arbitrary positive introspection*)
- $\neg B^x\varphi \to B^y\neg B^x\varphi$ (*arbitrary negative introspection*)
- $K\varphi \to \varphi$ (*truth axiom*)

Properties of knowledge ($K\varphi \to KK\varphi$, $\neg K\varphi \to K\neg K\varphi$) are derivable.
Properties relating belief to knowledge ($B^x\varphi \to KB^x\varphi$, $K\varphi \to B^x\varphi$) are derivable.

All states occupy a unique level in $\langle \mathcal{X}, < \rangle$:
if $<^s(u) = x$ and $<^t(u) = y$, then $x = y$.

# Dynamic logic for belief revision

The formula $[*\varphi]\psi$ reads '$\psi$ holds after belief revision with $\varphi$'.
Dynamic operator $[*\varphi]$ is interpreted as (doxastic epistemic) state transformer $[\![\varphi]\!]$.

$$(M, s) \models [*\varphi]\psi \quad \text{iff} \quad \text{for all} \ \ (M^*, s^*) : (M, s)[\![*\varphi]\!](M^*, s^*) \text{ implies } (M^*, s^*) \models \psi$$

Change what is plausible (epistemic class) or change preferences (within a class).
If *only* preferences, revision $*\varphi$ is *tentative* public announcement of $\varphi$:
both $\varphi$ and $\neg\varphi$ may be true, but it is considered more likely that $\varphi$ is true.
In that case, dynamic belief revision becomes:

$$(\langle S, <, V\rangle, s) \models [*\varphi]\psi \quad \text{iff} \quad (\langle S, <^*, V\rangle, s) \models \psi$$

where $<^*$ is a *revised* belief function.

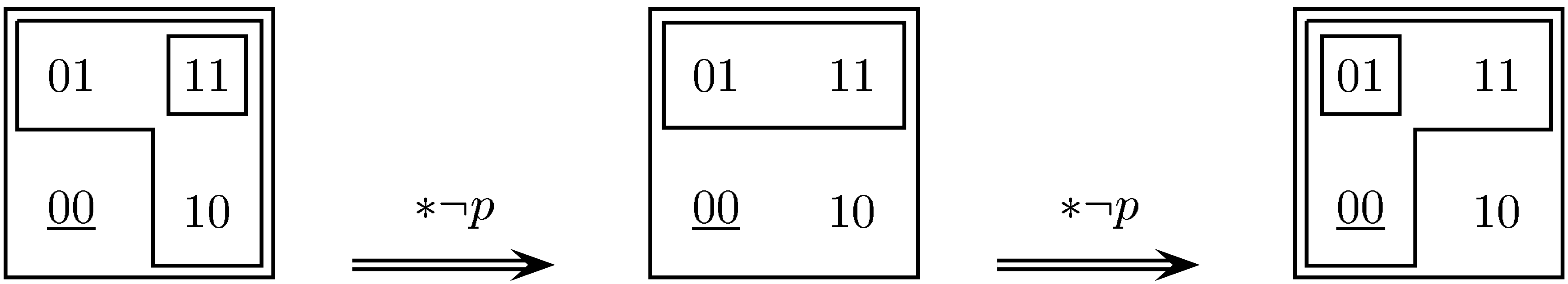
# Dynamic logic for belief revision: success

- Revision is *successful* for $\varphi$ iff $\neg K \neg \varphi \to [*\varphi] B\varphi$ is valid.
- Revision is *propositionally successful* iff successful for all propositional $\varphi$.
- Revision is *eventually successful* for $\varphi$ iff $\neg K \neg \varphi \to [*\varphi]^* B\varphi$ is valid.
  (In other words: iff it is successful for $\varphi$ after (finitely) iterated revision.)
- Revision is revocable iff $\psi \to [*\varphi][*\neg\varphi]\psi$ is valid (for arbitrary $\varphi, \psi$).

Similarly for 'successful for agent $n$'.

# Example: Minimal belief revision $\mathcal{X} = \mathbb{N}$

$<^*(s) = \ <(s)$ if $M, s \models \varphi$, and else $<^*(s) = \ <(s) + 1$ (plus 'normalization')

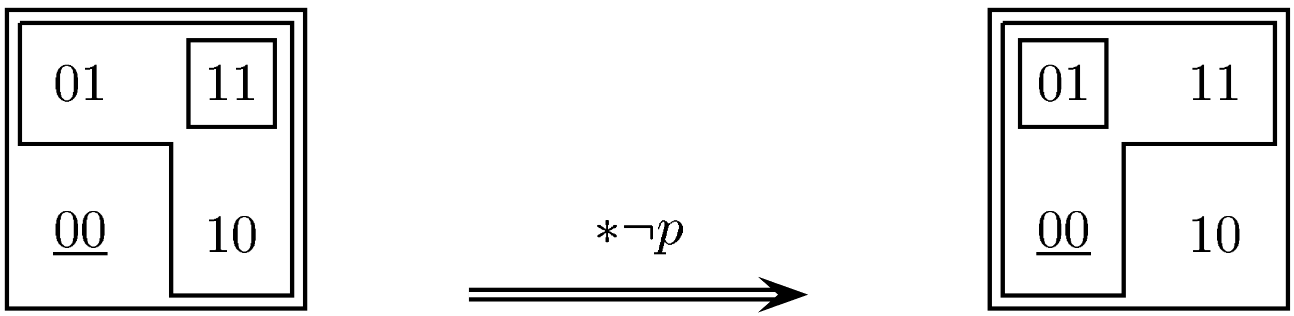



This belief revision is *eventually* propositionally successful.
It is also *revocable*.

See [vDitmarsch 2005] (Prolegomena).

# Example: Successful minimal belief revision $\mathcal{X} = \mathbb{N}$

$<^*(s) = \ <(s) - \mathsf{Min}\{<(t) \mid M, t \models \varphi\}$ if $M, s \models \varphi$ and else
$<^*(s) = \ <(s) + 1 - \mathsf{Min}\{<(t) \mid M, t \models \neg\varphi\}$

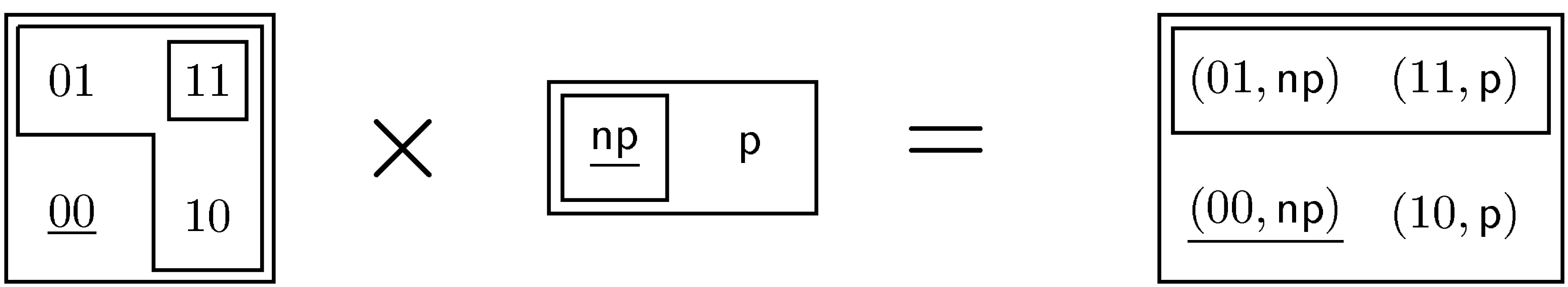



This belief revision is propositionally successful.

See [Aucher 2003] (Belief and update logic, MSc thesis), motivated by [Spohn 1988].

# Doxastic epistemic actions

Belief revision is (in a multi-agent setting) a *public* doxastic epistemic action.
A doxastic epistemic action can be represented semantically as an action model.
Action models were introduced in [Baltag, Moss & Solecki 1998].
Executing an action can be seen as computing a restricted modal product.
Revised degree $<^*(s,a)$ of pair $(s,a)$ is a function of degree $<(s)$ and degree $<(a)$.
For example (minimal belief revision): $<^*(s,a) = <(s) + <(a)$ (+ normalization).




For more information, see [Aucher 2003]. See also [Liu 2004], [vDitmarsch 2005], [Herzig, Lang & Marquis 2005] (Revision and update in multi-agent belief structures)

# Dynamic versus conditional epistemic logic

Success problematic in dynamic epistemic logic, not in *conditional* epistemic logic:
$[p \wedge \neg Bp](p \wedge \neg Bp)$ is invalid but
$B^{p \wedge \neg Bp}(p \wedge \neg Bp)$ is valid.

$B_n^{\varphi}\psi$ stands for '$\psi$ is true in the *minimal* $\varphi$-states (for agent $n$)'.
'Minimal for $n$' ($B_n = B_n^0$) corresponds to access $\rightarrow_n$ ($\rightarrow_n = \rightarrow_n^0$):
$M, s \models B_n^{\varphi}\psi$ iff for all $s'$: if $s \rightarrow_n s'$ and $M, s' \models \varphi$, then $M, s' \models \psi$.
Abbreviate interpretation of $B_n^{\varphi}$ as $\rightarrow_n^{\varphi}$: $s \rightarrow_n^{\varphi} s'$ iff $s \rightarrow_n s'$ and $M, s' \models \varphi$.
$C_G$ is interpreted as trans./refl. closure of the union of all $\rightarrow_n$:

$$(\bigcup_{n \in G} \rightarrow_n)^*$$

Conditional common knowledge $C_G^{\varphi}$ is similarly interpreted as the transitive and reflexive closure of the union of all $\rightarrow_n^{\varphi}$:

$$(\bigcup_{n \in G} \rightarrow_n^{\varphi})^*$$

# Dynamic versus conditional epistemic logic

Conditional belief/kn. $B_n^{\varphi}\psi$ is logically equivalent to $B_n(\varphi \to \psi)$.
Conditional common belief/kn. $C_G^{\varphi}\psi$ is *not* logically equivalent to $C_G(\varphi \to \psi)$;
consider the dual version $\neg C_G^{\varphi}\neg\psi$:
$\varphi$ has to hold in every state of the path to the $\psi$-state, *not just* in the final state.

References: [Kooi & vBenthem 2004], [vBenthem, vEijck & Kooi 2005] (Logics of communication and change)

This development is relevant because

- Conditional epistemic logics have pleasing properties: successful revision [Stalnaker 1996], [Asheim & Søvik 2005], [Board 2004], [Bonanno 2005]
- Conditional common knowledge is more expressive than public announcement!
  $C_G\varphi \leftrightarrow C_G^{\top}\varphi \qquad [\varphi]C_G\psi \leftrightarrow C_G^{\varphi}[\varphi]\psi$

# Dynamic versus temporal epistemic logic

Expression $[\varphi]\psi$ is replaced by $X\psi$ ('next, $\psi$') for some proper transition.
We relate dynamic epistemic logics to temporal epistemic logics (LTL, CTL, ...).
Temporal epist. logics successful in automated verification (tableaux, model checking).
For model checking, $M, s \models [\varphi]\psi$ is replaced by $(M + M'), (s, 0) \models X\psi$,
where $M'$ is a copy of $M$ representing the next state of the system.
We then have that $(M + M'), (s, 0) \models X\psi$ iff $(M + M'), (s, 1) \models \psi$.
For more transitions, we need more copies...

Assume an *interpreted system* $\mathcal{I}$, consisting of global states and runs.
There is a natural correspondence to Kripke models / epistemic models, and to epistemic state transitions: an environmental variable performs the part of a revision formula.

# Dynamic versus temporal epistemic logic

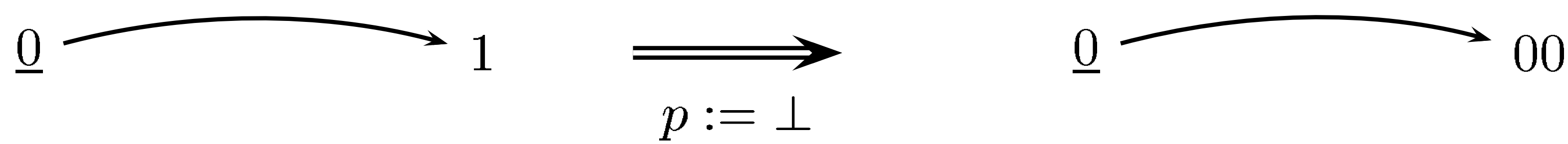
*References*

Temporal epistemic logic: [Halpern, vdMeyden & Vardi 2004], [Pnueli 1980s], [Vardi 80s 90s], ...
Epistemic model checking: [Lomuscio & Raimondi 2003] (MCMAS), [Gammie & vd Meyden 2004] (MCK), [vEijck 2005] (DEMO)
Dynamic versus temporal epistemics: [vDitmarsch, vdHoek, vdMeyden, Ruan 2005] (Model Checking Russian Cards)

# Revision and update

The distinction revision/update in belief revision corresponds to the distinction epistemic/factual change in dynamic epistemics.

Factual change can be modelled by assignments.
Assignments are modelled as dynamic modal operators interpreted as epistemic state transformers.




In the above, we have $M, 0 \models \neg Kp \wedge [p := \bot]Kp$.
Assignments are currently under much investigation in dynamic epistemic logic.
For example: [vBenthem, vEijck, Kooi 2005] (Logics of communication and change)

# Belief change and dynamic epistemic logic

Winding up! There are problems with bulleted postulates for belief revision:

| | | | |
|---|---|---|---|
| | $*_{\mathsf{agm}}1$ | $\mathcal{K} * \varphi$ is a theory | type |
| • | $*_{\mathsf{agm}}2$ | $\varphi \in \mathcal{K} * \varphi$ | success |
| • | $*_{\mathsf{agm}}3$ | $\mathcal{K} * \varphi \subseteq \mathcal{K} + \varphi$ | upper bound |
| • | $*_{\mathsf{agm}}4$ | if $\neg\varphi \notin \mathcal{K}$, then $\mathcal{K} + \varphi \subseteq \mathcal{K} * \varphi$ | lower bound |
| | $*_{\mathsf{agm}}5$ | $\mathcal{K} * \varphi = \mathcal{K}_\perp$ iff $\varphi$ is inconsistent | triviality |
| | $*_{\mathsf{agm}}6$ | if $\varphi$ is equivalent to $\psi$ then $\mathcal{K} * \varphi = \mathcal{K} * \psi$ | extensionality |
| • | $*_{\mathsf{agm}}7$ | $\mathcal{K} * (\varphi \wedge \psi) \subseteq (\mathcal{K} * \varphi) + \psi$ | iteration upper bound |
| • | $*_{\mathsf{agm}}8$ | if $\neg\psi \notin \mathcal{K} * \varphi$, then $(\mathcal{K} * \varphi) + \psi \subseteq \mathcal{K} * (\varphi \wedge \psi)$ | iteration lower bound |

# Belief change and dynamic epistemic logic

Winding up! Advantages of dynamic logic for belief change:

- higher order belief revision
- multi-agent belief revision
- iterated belief revision
- alternatives to success and growth
- clear distinction between revision and update
- computational properties
- tools for automated verification