The Algebra of Multi-Agent Dynamic Belief Revision

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Multi-Agent Dynamic Belief Revision

- Belief Revision: a formal system about revising theories
- Dynamic Belief Revision: a formal system about revising theories following actions
- Multi-Agent Dynamic Belief Revision: a formal system about revising theories held by different agents following communication actions

Belief Revision is a particular form of epistemic update.

Belief Revision and Epistemic Update

- Agent's view of the world is m.
- Communication action q happens.
- Agent's updated view of the world after the action is $m \otimes q$.
- If the updated view is contradictory $m \otimes q = \perp$, then we revise agent's view according to the action $m * q$.
- The revised view no longer contradicts the action

$$(m * q) \otimes q \neq \perp.$$

Algebra of Dynamic Belief Revision

An extension of the algebra for Dynamic Epistemic Logic

- Precise mathematical structure
- A minimal language with a set of axioms
- Non-boolean: reflecting partiality of information
- Resource-sensitive
- Residuated modalities as in Moortgaat.

Syntax

Propositional Connectives:

$\top, \bot$	true and false propositions
$m \wedge m'$	conjunction
$m \vee m'$	disjunction
$f_A(m)$	agent A's view of m
$\Box_A\, m$	agent A knows that m
p	facts

Action Connectives:

$\epsilon, \bot$	skip and impossible actions
$q \vee q'$	non-deterministic choice
$q \bullet q'$	sequential composition
$f_A(q)$	agent A's view of q
$\Box_A\, q$	agent A knows that q is happening

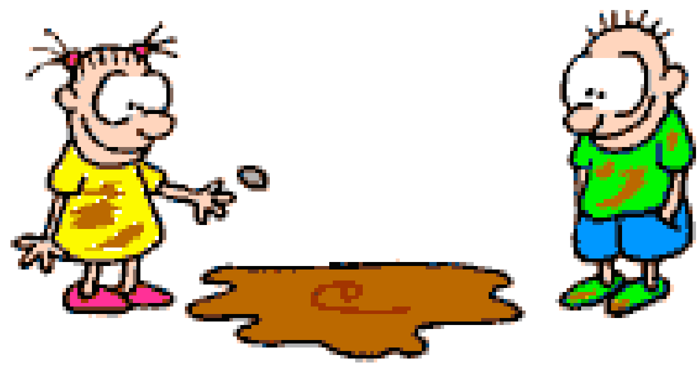
Mixed Connectives:

$m \otimes q$	update proposition m by action q
$[q]m$	after action q, proposition m holds

What do we have to specify?

Algebra		**Kripke Semantics**
Agents	A, B	agents
Facts	p	facts
Propositions	m, m'	sets of states
Partial order on props	$m \leq m'$	entailment
Views of agents about props	$f_A(m)$	accessibility relation
Actions	q	sets of states
Views of agents about actions	$f_A(q)$	accessibility relation
Kernel of each action	$ker(q)$	precondition

Example



Encoding the muddy children puzzle

Agents: four children, three of them dirty $\{1, 2, 3, 4\}$

Facts: D_i : child i is dirty, $\overline{D}_i$: child i is clean

Propositions: all possible situations $s_{2,3}$ $s_{1,2,3}$

Partial order: $s_{1,2,3} \leq D_1$

Views: in each situation each child thinks either he is dirty or clean

$$f_i(s_\beta) = s_{\beta \cup i} \vee s_{\beta \setminus i}$$
$$f_1(s_{1,2,3}) = s_{1,2,3}, \vee s_{2,3}$$

Father's announcement: q_0

$$f_i(q_0) = q_0, \qquad ker(q_0) = D_0$$

Children's replies: q

$$f_i(q) = q, \qquad ker(q) = \wedge_i \Box_i D_i$$

Axioms of the Algebra

Propositions

$f_A(\bot) = \bot$	contradiction
$f_A(m \vee m') = f_A(m) \vee f_A(m')$	disjunction-preserving

Actions

$q \bullet \epsilon = \epsilon \bullet q = q$	skip action
$f_A(\epsilon) = \epsilon$	skip is public
$f_A(q \vee q') = f_A(q) \vee f_A(q')$	disjunction-preserving
$f_A(q \bullet q') = f_A(q) \bullet f_A(q')$	composition-preserving

Mixed

$p \otimes q \leq p$	facts are update-stable
$m \otimes (q \vee q') = (m \otimes q) \vee (m \otimes q')$	disjunction-preserving
$(m \vee m') \otimes q = (m \otimes q) \vee (m' \otimes q)$	disjunction-preserving
$m \otimes (q \bullet q') = (m \otimes q) \otimes q'$	associativity of update
$f_A(m \otimes q) \leq f_A(m) \otimes f_A(q)$	update inequality

$f_A(m) \leq m'$	$\Leftrightarrow$	$m \leq \Box_A m'$	epistemic adjunction
$m \otimes q \leq m'$	$\Leftrightarrow$	$m \leq [q]m'$	dynamic adjunction

Solving the muddy children puzzle

Proposition

$$s_{1,2,3} \leq [q_0 \bullet q \bullet q] \,\Box_1\, D_1$$

(I) Induction

$$s_{2,3} \otimes q_0 \otimes q \in ker(q) \quad \Rightarrow \quad s_{2,3} \otimes q_0 \otimes q \otimes q = \bot \leq D_1$$

(II) Stability of Fact

$$s_{1,2,3} \leq D_1 \quad \Rightarrow \quad s_{1,2,3} \otimes q_0 \otimes q \otimes q \leq D_1$$

(I) and (II): $(s_{1,2,3} \vee s_{2,3}) \otimes q_0 \otimes q \otimes q \leq D_1$

Apply assumptions and axioms (taste of heaven!)

$$\begin{aligned}
f_1(s_{1,2,3}) \otimes f_1(q_0) \otimes f_1(q) \otimes f_1(q) &\leq D_1 \\
f_1(s_{1,2,3} \otimes q_0 \otimes q \otimes q) &\leq D_1 \\
s_{1,2,3} \otimes q_0 \otimes q \otimes q &\leq \Box_1\, D_1 \\
s_{1,2,3} &\leq [q_0 \bullet q \bullet q] \,\Box_1\, D_1
\end{aligned}$$

Cheating Muddy Children

After father's announcement: action q_0 , in the first round all 4 reply 'No!': action q

Then, children 2 and 3 cheat by secretly communicating to each other that they are dirty: action π. As a result, 1 and 4 reply 'No!', and 2 and 3 reply 'Yes!' in the second round: mixed action q'. This makes children 1 and 4 confused!

Child 1 wrongly concludes that he is clean

$$s_{1,2,3,4} \leq [q_0 \bullet q \bullet \pi \bullet q']\Box_1 \overline{D_1}\,.$$

Child 4 goes crazy

$$s_{1,2,3,4} \leq [q_0 \bullet q \bullet \pi \bullet q']\Box_4 \bot\,.$$

Help Child 4: Belief Revision

Goal: We do not want agents to get contradictory views from actions $f_A(m \otimes q) = \bot$ when there is no contradiction in the real world $m \otimes q \neq \bot$.

$$f_A(m \otimes q) = \bot \quad \Rightarrow \quad f_A(m) \otimes f_A(q) = \bot$$

Method:

Use agent's Revised Belief instead of his updated belief

replace $f_A(m) \otimes f_A(q) = \bot$ with $f_A(m) * f_A(q) \neq \bot$

Axioms of Belief Revision

1. $\exists m' \in M, m * q = m' \otimes q$

   Consistency of revised view with action.

2. If $m \otimes q \neq \perp$ then $m * q = m \otimes q$

3. If $m * q = \perp$ then $q = \perp$

4. $f_A(m \otimes q) \leq f_A(m) * f_A(q)$

   Replace the updated belief with revised belief.

Axioms of Belief Revision

Lemma:

The revised belief is the weakest belief consistent with the action

$$m * q = \vee\{m'' \mid m'' \otimes q \neq \bot\} .$$

Note: actions can also be revised $q * q'$, details in the proceedings.

Revise Belief of Child 4

In the cheating muddy children, child 4 is fine until the last round of answers:

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \neq \bot$$

But after the last round, child 4 gets a contradiction

$$\begin{aligned} f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \otimes f_4(q') &= \bot \\ f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \otimes q' &= \bot \end{aligned}$$

In order to avoid this, we revise his beliefs

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) * q' \neq \bot$$

So child 4 does not get a contradiction any more

$$s_{1,2,3} \otimes q_0 \otimes q \otimes \pi \otimes q' \not\leq \Box_4 \bot \,.$$

Calculate the Revised Belief of Child 4

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) * q' = m \otimes q'$$

Make child 4 suspect the cheating:

change $\quad f_4(\pi) = \epsilon \quad$ to $\quad f_4(\pi) = \pi \vee \epsilon$

Contradiction gets eliminated

$$f_4(s_{1,2,3} \otimes q_0 \otimes q \otimes \pi) \otimes q' = f_4(s_{1,2,3}) \otimes q_0 \otimes q \otimes (\pi \vee \epsilon) \otimes q' \neq \bot$$

The revised belief of child 4 is

$$m = f_4(s_{1,2,3}) \otimes q_0 \otimes q \otimes (\pi \vee \epsilon) .$$

instead of the original

$$f_4(s_{1,2,3}) \otimes q_0 \otimes q \otimes \epsilon .$$

The Mathematical Structure behind all this

The mathematical structure that governs all this is a

Belief Revision Epistemic System

- A system, i.e. a pair module-quantale (M, Q) that encodes dynamics,
- with an update action $- \otimes - : M \times Q \to M$ satisfying some axioms,
- and a belief revision operator $- * - : M \times Q \to M$ also satisfying some axioms.

- The system is endowed with pairs of homomorphisms

$$f_A = (f_A^M : M \to M, f_A^Q : Q \to Q)$$

  that encode epistemics.

- It satisfies the revision inequality:

$$f_A(m \otimes q) \leq f_A(m) * f_A(q)\,.$$

Conclusion

- Belief Revision is a particular form of epistemic update
- Epistemic update is usually done using Kripke structures
- I have presented an algebraic way to do it
  - Strong mathematical structure as its basis
  - Explicit structure on epistemic actions
  - Uniform way of dealing with modalities: residuation
  - Multi-agent belief revision

Future Work

- Connections with AGM axioms
- Sequent calculus

References

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2- A. Baltag, M. Sadrzadeh, ‘The algebra of multi-agent dynamic belief revision’.

3- A. Baltag, B. Coecke and M. Sadrzadeh, ’Algebra and sequent calculus for epistemic actions’, in *ENTCS*, Vol. 126, March 2005, pp. 27-52.