

# From Onions to Broccoli: Generalizing Lewis's counterfactual logic

*Workshop on Belief Revision and Dynamic  
Logic  
ESSLLI 2005*

Patrick Girard, Stanford University

[pgirard@stanford.edu](mailto:pgirard@stanford.edu)



# Goal

- Relational belief revision. Proposal by Lindström and Rabinowicz.



# Goal

- Relational belief revision. Proposal by Lindström and Rabinowicz.
- Generalization of Segerberg's Onion semantics for belief revision (*DDL*).



# Outline

- Onion and Broccoli Logic.



# Outline

- Onion and Broccoli Logic.
- Interlude: Minimal Relational Logic ( $MRL$ ,  $[Me]$ ).



# Outline

- Onion and Broccoli Logic.
- Interlude: Minimal Relational Logic (*MRL*, [Me]).
- Broccoli Logic = Minimal conditional Logic (*MCL*, [Veltman, Burgess]).



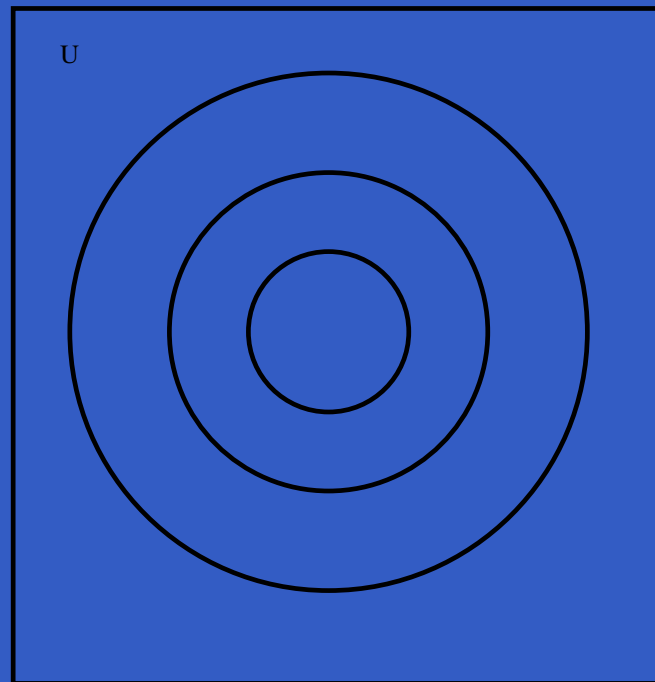
# Onions

Let  $U$  be a nonempty set. An *onion*  $\mathcal{O} \subseteq \mathcal{P}(U)$  is a linearly ordered set of subsets of  $U$  satisfying the limit condition.



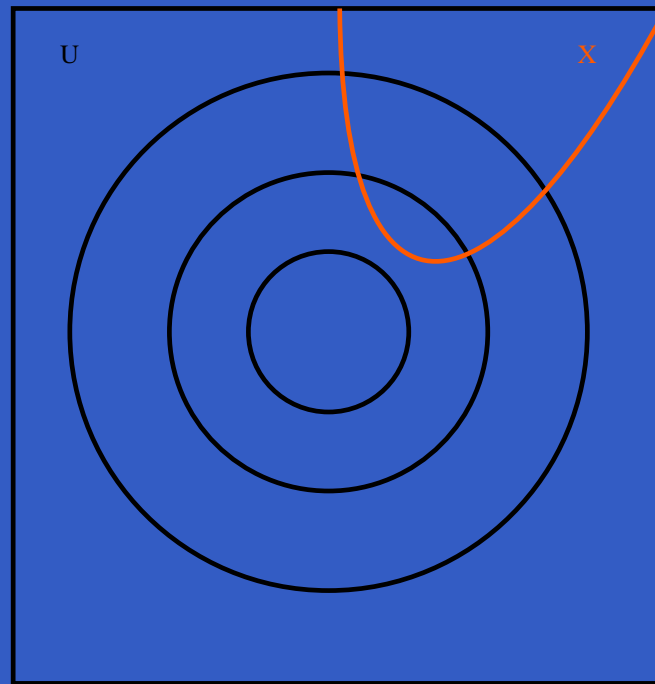
# Onions

Let  $U$  be a nonempty set. An *onion*  $\mathcal{O} \subseteq \mathcal{P}(U)$  is a linearly ordered set of subsets of  $U$  satisfying the limit condition:



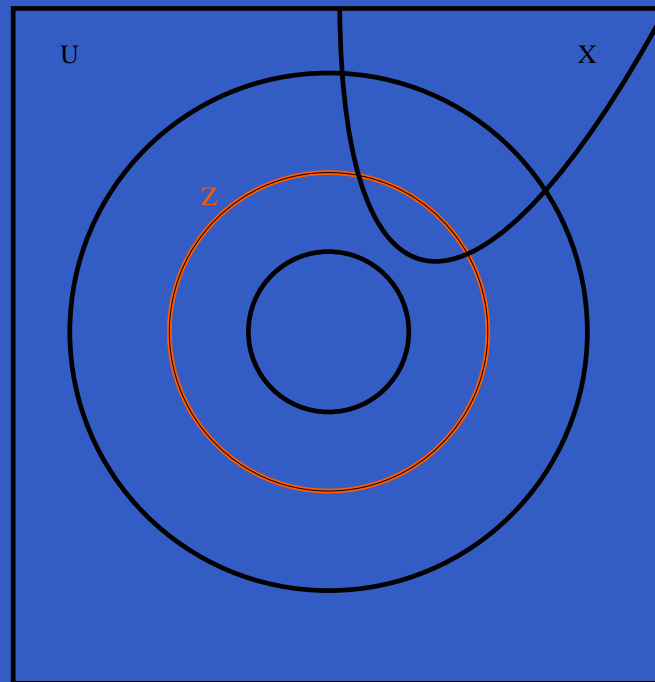
# Onions

Let  $U$  be a nonempty set. An *onion*  $\mathcal{O} \subseteq \mathcal{P}(U)$  is a linearly ordered set of subsets of  $U$  satisfying the limit condition:



# Onions

Let  $U$  be a nonempty set. An *onion*  $\mathcal{O} \subseteq \mathcal{P}(U)$  is a linearly ordered set of subsets of  $U$  satisfying the limit condition:



# Broccoli Flowers

Let  $U$  be a nonempty set. A *broccoli flower*  $\mathcal{B} \subseteq \mathcal{P}(U)$  is a set of subsets satisfying a *generalized limit condition*.



# Broccoli Flowers

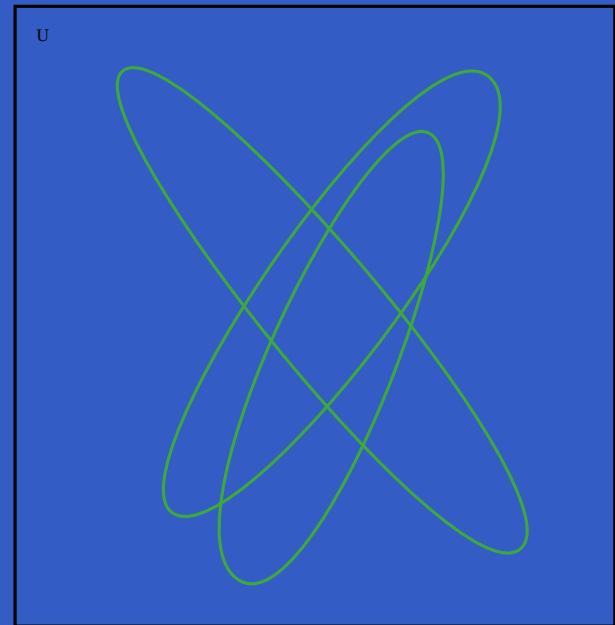
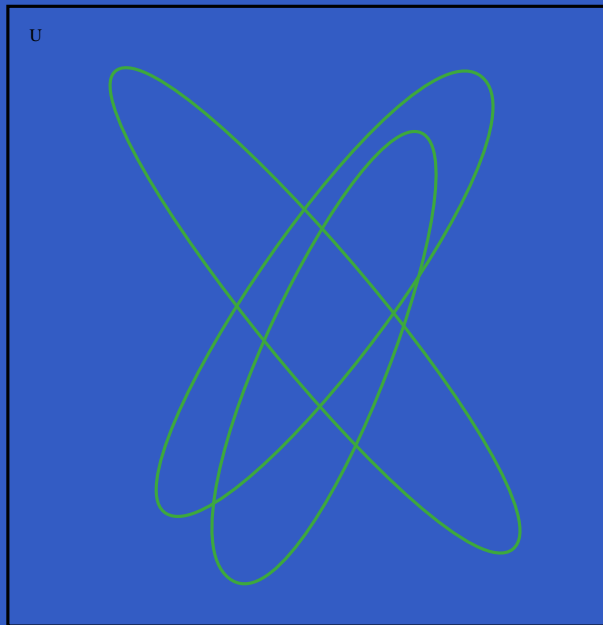
Let  $U$  be a nonempty set. A *broccoli flower*  $\mathcal{B} \subseteq \mathcal{P}(U)$  is a set of subsets satisfying a *generalized limit condition*.

There are two ways to generalize the limit condition.



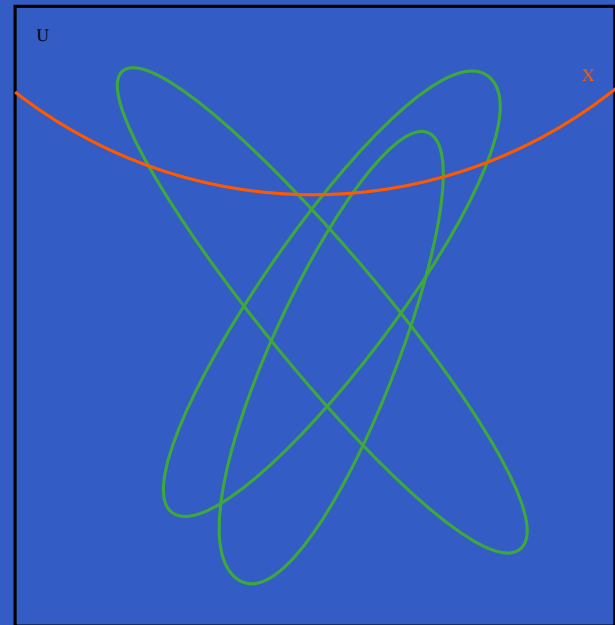
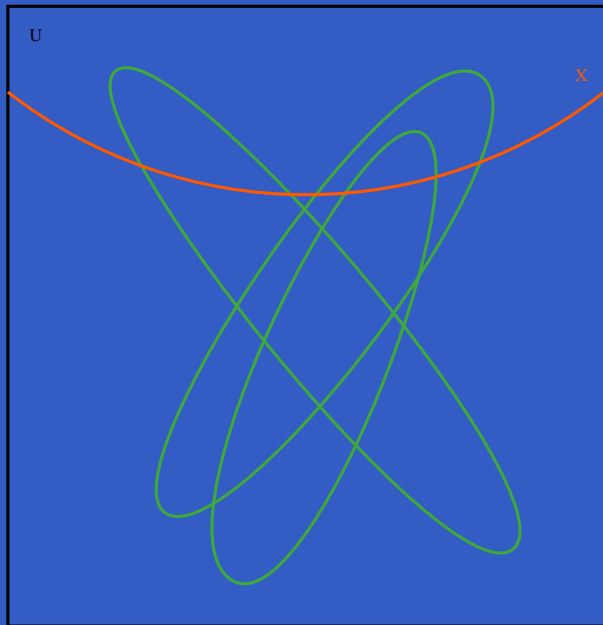
# Broccoli Flowers

Let  $U$  be a nonempty set. A *broccoli flower*  $\mathcal{B} \subseteq \mathcal{P}(U)$  is a set of subsets satisfying a *generalized limit condition*.



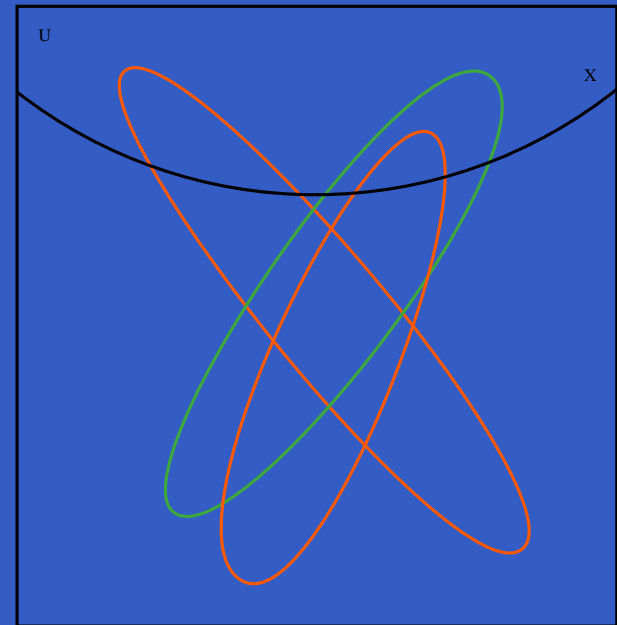
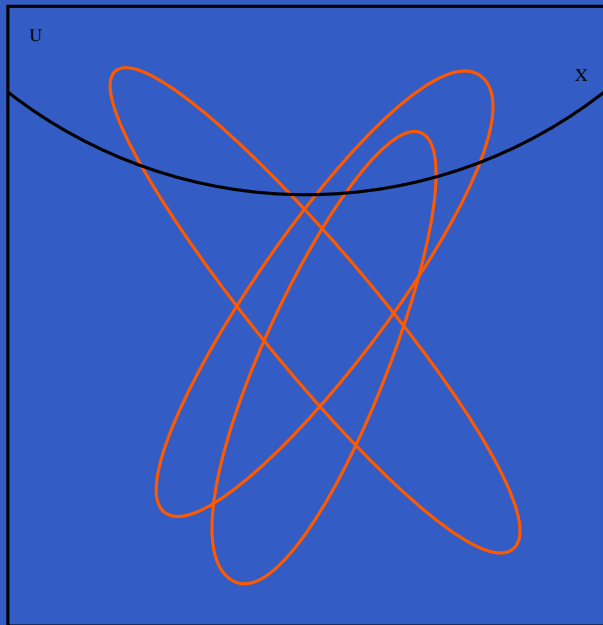
# Broccoli Flowers

Let  $U$  be a nonempty set. A *broccoli flower*  $\mathcal{B} \subseteq \mathcal{P}(U)$  is a set of subsets satisfying a *generalized limit condition*.



# Broccoli Flowers

Let  $U$  be a nonempty set. A *broccoli flower*  $\mathcal{B} \subseteq \mathcal{P}(U)$  is a set of subsets satisfying a *generalized limit condition*.



# Notation

- Let  $\mathcal{B} \bullet X = \{Y \in \mathcal{B} : Y \cap X \neq \emptyset\}$ .



# Notation

- Let  $\mathcal{B} \bullet X = \{Y \in \mathcal{B} : Y \cap X \neq \emptyset\}$ .
- We use the notation  $Z_{\mu}(\mathcal{B} \bullet X)$  to express that  $Z$  is a minimal element of the broccoli flower  $\mathcal{B}$  intersecting  $X$ .



# Notation

- Let  $\mathcal{B} \bullet X = \{Y \in \mathcal{B} : Y \cap X \neq \emptyset\}$ .
- We use the notation  $Z_{\mu}(\mathcal{B} \bullet X)$  to express that  $Z$  is a minimal element of the broccoli flower  $\mathcal{B}$  intersecting  $X$ .
- Let  $|\varphi| = \{u : \mathfrak{M}, u \models \varphi\}$ . We call  $|\varphi|$  the *associated proposition* to  $\varphi$ .



# Language

$$\mathcal{L} = p \mid \neg\varphi \mid \varphi \vee \psi \mid [\varphi]\psi \mid [\varphi]\psi$$



# Broccoli model

A *broccoli model*  $\mathfrak{M}$  is a triple  $\langle U, \{\mathcal{B}_u\}_{u \in U}, V \rangle$ ,  
where:



# Broccoli model

A *broccoli model*  $\mathfrak{M}$  is a triple  $\langle U, \{\mathcal{B}_u\}_{u \in U}, V \rangle$ ,  
where:

- $U$  is a set of worlds,



# Broccoli model

A *broccoli model*  $\mathfrak{M}$  is a triple  $\langle U, \{\mathcal{B}_u\}_{u \in U}, V \rangle$ , where:

- $U$  is a set of worlds,
- $\{\mathcal{B}_u\}_{u \in U}$  is a family of broccoli flowers for each world  $u \in U$  satisfying either generalized limit condition, and



# Broccoli model

A *broccoli model*  $\mathfrak{M}$  is a triple  $\langle U, \{\mathcal{B}_u\}_{u \in U}, V \rangle$ , where:

- $U$  is a set of worlds,
- $\{\mathcal{B}_u\}_{u \in U}$  is a family of broccoli flowers for each world  $u \in U$  satisfying either generalized limit condition, and
- $V$  is a valuation assigning sets of worlds to propositions.



# Truth-definition

- The truth definition for the propositional and the Boolean cases is standard.



# Truth-definition

- The truth definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  $\forall Z \mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$ .



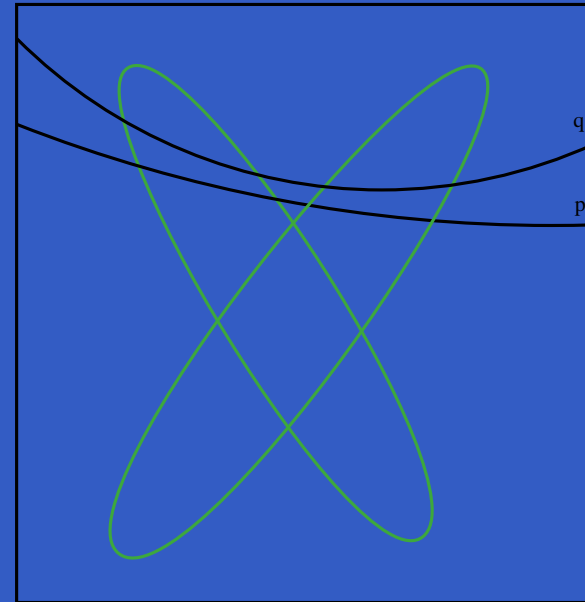
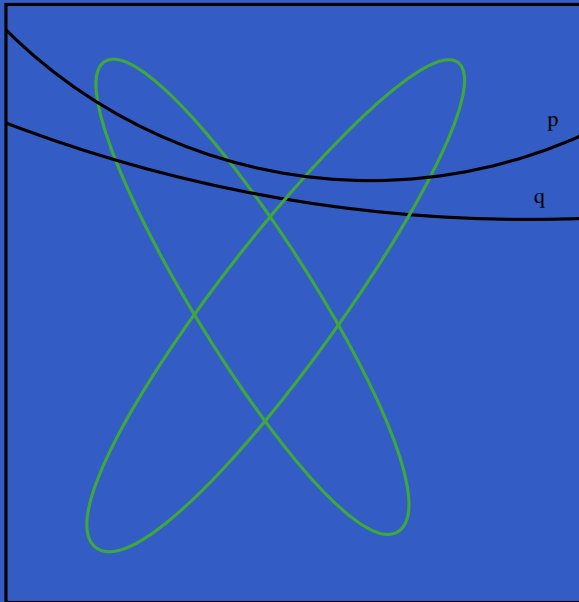
# Truth-definition

- The truth definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  $\forall Z \mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$ , and
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  $\forall Z \mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi|) \neq \emptyset$ .



# In pictures

Broccoli semantics of the counterfactual operators  $[p]q$  and  $[p \rangle q$ .



# *MRL Models*

*A minimal relational model is a triple  $(U, R, V)$ , where:*



# *MRL Models*

A *minimal relational model* is a triple  $(U, R, V)$ , where:

- $U$  and  $V$  are as above.



# MRL Models

A *minimal relational model* is a triple  $(U, R, V)$ , where:

- $U$  and  $V$  are as above;
- $R = \{R_{|\varphi|} : \varphi \text{ is a formula, } R_{|\varphi|} \subseteq U \times \mathcal{P}(U)\}$ .



# *MRL* Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.



# *MRL* Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \models \psi)$ , and



# MRL Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \models \psi)$ , and
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \models \psi)$ .



# MRL Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \models \psi)$ , and
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \models \psi)$ .
- **NB. 2 Quantifiers!**



# MRL Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \models \psi)$ , and
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \models \psi)$ .
- NB. 2 Quantifiers! **Read in two stages.**



# MRL Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \models \psi)$ , and
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \models \psi)$ .
- NB. 2 Quantifiers! Read in two stages.



# MRL Truth-definition

- Again, the truth-definition for the propositional and the Boolean cases is standard.
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \models \psi)$ , and
- $\mathfrak{M}, u \models [\varphi]\psi$  iff  
 $\forall X ((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \models \psi)$ .
- NB. 2 Quantifiers! Read in two stages.



# MRL Proof System

## RULES

1. Modus Ponens.
2. Necessitation for  $[\varphi]$  and  $[\varphi]$ .
3. If  $\varphi$  and  $\varphi'$  are formulas differing only in  $\varphi$  having an occurrence of  $\theta$  in one place where  $\varphi'$  has an occurrence of  $\theta'$ , and if  $\theta \equiv \theta'$  is a theorem, then  $\varphi \equiv \varphi'$  is also a theorem.



# MRL Proof System

## AXIOMS

1. Classical tautologies
2.  $\langle \varphi \rangle \psi \equiv \neg [\varphi] \neg \psi$
3.  $\langle \varphi ] \psi \equiv \neg [\varphi \rangle \neg \psi$
4.  $[\varphi](\psi \rightarrow \theta) \rightarrow ([\varphi]\psi \rightarrow [\varphi]\theta)$
5.  $\langle \varphi ] \psi \rightarrow \langle \varphi ](\psi \vee \theta)$
6.  $[\varphi]\psi \wedge \langle \varphi ] \theta \rightarrow \langle \varphi ](\psi \wedge \theta)$
7.  $\neg \langle \varphi ] \top \rightarrow [\varphi ] \psi$



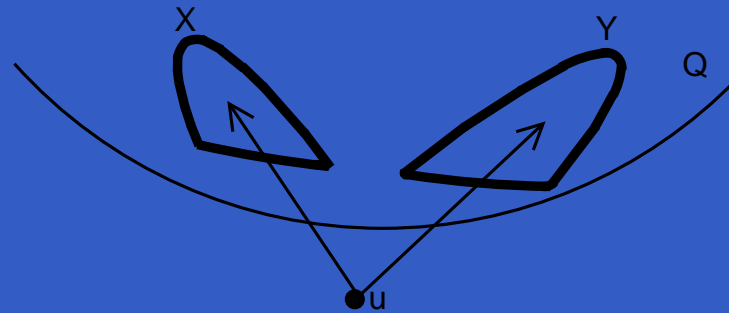
# Two ways to proceed

1. Add restrictions on the relation  $R_{|\varphi|}$ , or introduce so-called selection functions, in order to get minimal sets returned under revision by  $\varphi$  from broccoli flowers:



# Two ways to proceed

1. Add restrictions on the relation  $R_{|\varphi|}$ , or introduce so-called selection functions, in order to get minimal sets returned under revision by  $\varphi$  from broccoli flowers:



# Two ways to proceed

1. Add restrictions on the relation  $R_{|\varphi|}$ , or introduce so-called selection functions, in order to get minimal sets returned under revision by  $\varphi$  from broccoli flowers:



# Two ways to proceed

1. Add restrictions on the relation  $R_{|\varphi|}$ , or introduce so-called selection functions, in order to get minimal sets returned under revision by  $\varphi$  from broccoli flowers.

DIFFICULT!



# Two ways to proceed

1. Add restrictions on the relation  $R_{|\varphi|}$ , or introduce so-called selection functions, in order to get minimal sets returned under revision by  $\varphi$  from broccoli flowers.
2. Show that broccoli logic is equivalent to a well-know conditional logic, viz., the *minimal conditional logic* of Veltman and Burgess.



# *MCL* Language

$$\mathcal{L} = p \mid \neg\varphi \mid \varphi \vee \psi \mid [\varphi]\psi$$



# Minimal Conditional Logic

A *Minimal conditional logic model* is a triple  $(U, \leq_x, V)$ , where



# Minimal Conditional Logic

A *Minimal conditional logic model* is a triple  $(U, \leq_x, V)$ , where

- $U$  and  $V$  are as above;



# Minimal Conditional Logic

A *Minimal conditional logic model* is a triple  $(U, \leq_x, V)$ , where

- $U$  and  $V$  are as above;
- $\leq_x$  is a reflexive and transitive ternary relation.



# Minimal Conditional Logic

A *Minimal conditional logic model* is a triple  $(U, \leq_x, V)$ , where

- $U$  and  $V$  are as above;
- $\leq_x$  is a reflexive and transitive ternary relation.

$y \leq_x z$  should be read as:

“According to world  $x$ , world  $y$  is no farther away than world  $z$ .”



# Some more notation

$W_u = \{y : \exists z, y \leq_u z\}$  is the *zone of entertainability* for world  $u \in U$ .

Intuitively, worlds outside the zone of entertainability for  $u$  are worlds so far away that their distance from the real world is not appreciable.



# MCL Truth-definition

- $\mathfrak{M}, u \models [\varphi]\psi$ , iff:

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi) [z \leq_u y \& \forall w \in W_u \cap V(\varphi) (w \leq_u z \rightarrow w \in V(\psi))]$$



# MCL Truth-definition

- $\mathfrak{M}, u \models [\varphi]\psi$ , iff:

$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi) [z \leq_u y \& \forall w \in W_u \cap V(\varphi) (w \leq_u z \rightarrow w \in V(\psi))]$

- **No minimality condition!**



# MCL Truth-definition

- $\mathfrak{M}, u \models [\varphi]\psi$ , iff:

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi) [z \leq_u y \& \forall w \in W_u \cap V(\varphi) (w \leq_u z \rightarrow w \in V(\psi))]$$

- No minimality condition! But we only need **finite models**.



# MCL Truth-definition

- $\mathfrak{M}, u \models [\varphi]\psi$ , iff:

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi) [z \leq_u y \& \forall w \in W_u \cap V(\varphi) (w \leq_u z \rightarrow w \in V(\psi))]$$

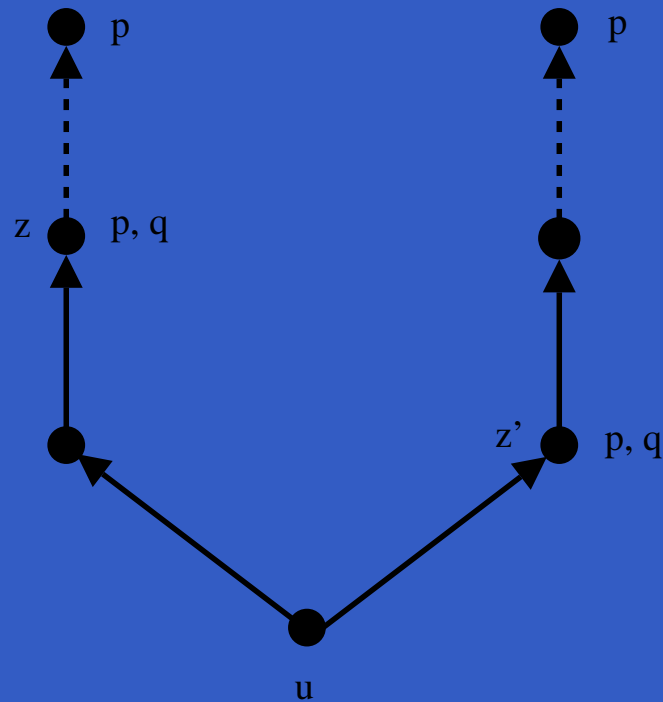
- No minimality condition! But we only need finite models. The semantic condition reduces to:

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi) [z \leq_u y \& \forall w <_u z, w \notin V(\varphi) \& z \in V(\psi)].$$



# In picture

$\mathfrak{M}, u \models [p]q$ :



# *MCL* Proof System

## AXIOMS

1. Classical tautologies
2.  $[\varphi]\varphi$
3.  $[\varphi]\psi \wedge [\varphi]\theta \rightarrow [\varphi](\psi \wedge \theta)$
4.  $[\varphi](\psi \wedge \theta) \rightarrow [\varphi]\psi$
5.  $[\varphi]\psi \wedge [\varphi]\theta \rightarrow [\varphi \wedge \psi]\theta$
6.  $[\varphi]\psi \wedge [\theta]\psi \rightarrow [\varphi \vee \theta]\psi$

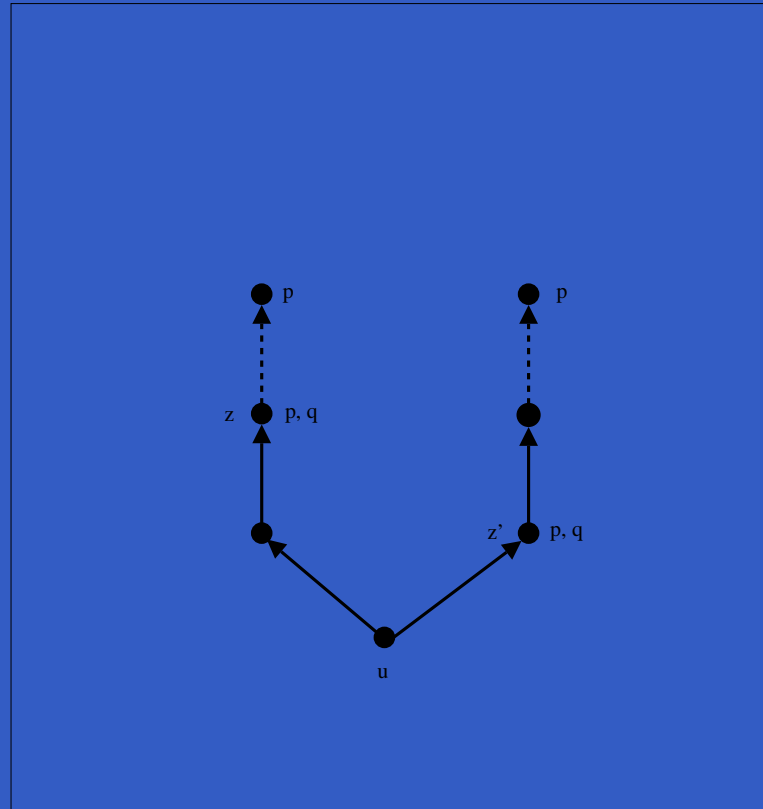


# Main Theorem

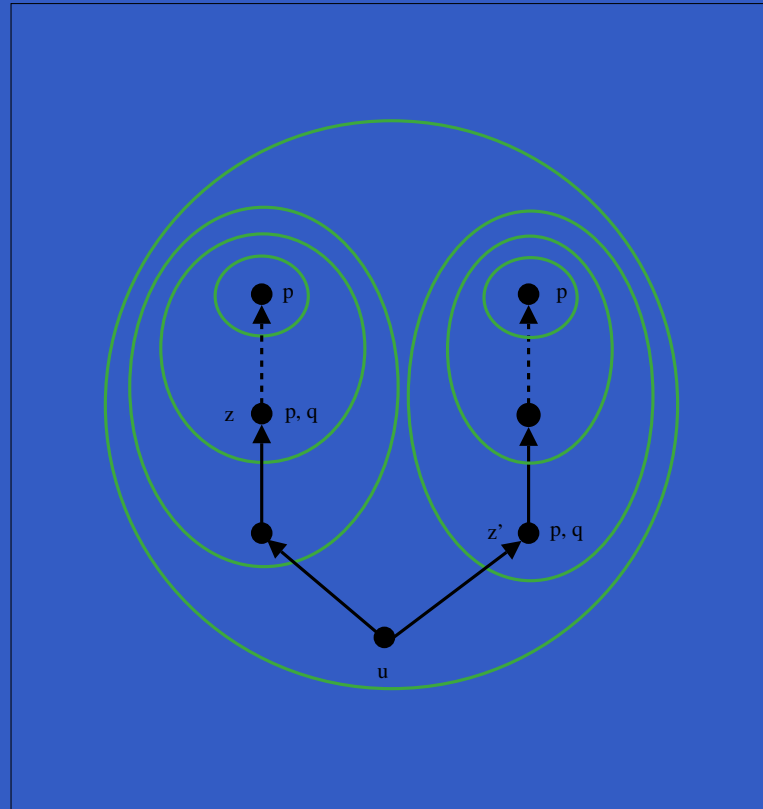
BROCCOLI LOGIC = MINIMAL CONDITIONAL LOGIC



# Key Idea – Induced Broccoli Models



# Key Idea – Induced Broccoli Models



# Conclusion

- Broccoli Logic = Minimal Conditional Logic.



# Conclusion

- Broccoli Logic = Minimal Conditional Logic.
- Open question: What is the complete minimal logic of  $[\varphi]\psi$  and  $[\varphi]\psi$  over the Burgess-Veltman models?



# Lesson

As so often over the past years, we see that logics of belief revision are largely conditional logics.

