RELATIONAL BELIEF REVISION, BROCCOLI LOGIC AND MINIMAL CONDITIONAL LOGIC

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Abstract. We present a generalization of Segerberg's onion semantics for belief revision, in which the linearity of the spheres need not occur. The resulting logic is called broccoli logic. We provide a minimal relational logic, introducing a new neighborhood semantics operator. We then show that broccoli logic is a well-known conditional logic, the Burgess-Veltman minimal conditional logic.

Belief revision is the study of theory change in which a set of formulas is ascribed to an agent as a belief set revisable in the face of new information. A dominant paradigm in belief revision is the so-called *AGM* paradigm, which describes a functional notion of revision (cf. [1]). A natural semantics in terms of sphere systems (cf. [6]) was given by Grove in [5] and a logical axiomatization was extensively studied by Segerberg (cf. [8] and the forthcoming [9]). A generalization of the *AGM* approach in which revision is taken to be relational rather than functional was first studied by Lindström and Rabinowicz (cf. [7]). Their motivation was to formalize cases in which an agent may obtain incomparable belief sets after revision with new information. In this paper, we will pursue this generalization and propose a relational belief revision logic. We call the resulting logic "broccoli logic" (*BL*) and the type of revision it depicts "broccoli revision". As it turns out, and this will be the main result of this paper, *BL* already exists, in the guise of what we call "minimal conditional logic", studied by Burgess and Veltman (cf. Burgess [2] and Veltman [10]).

In section 1, we outline broccoli semantics. In section 2, we give a minimal relational logic (*MRL*) with its complete proof system. The semantics

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The topic of this paper arose in a seminar taught by Krister Segerberg at Stanford University in the winter quarter of 2005. I thank the participants for their enthusiasm and support, then and later. In particular, I thank Krister Segerberg for introducing me to dynamic logics for belief revision, Hannes Leitgeb and Tomasz Sadzik for their contribution in developing the abstract neighborhood models used in this paper, and Johan van Benthem for suggesting that broccoli logic is really the minimal conditional logic. Finally, I thank the anonymous referees for their helpful suggestions in improving the paper.

of this logic is in effect a neighborhood semantics (cf. [3]), but we will interpret it terms of revision instead. Finally, we will show in section 3 that *BL* is equivalent to *MCL*.[1]

1. Broccoli Semantics

This section presents the broccoli semantics. We give definitions of broccoli models and provide the semantics for the modal operators.

Definition 1.1. Let U be a nonempty set. A *broccoli flower* $\mathcal{B} \in \mathcal{P}(U)$ is a set of subsets satisfying a generalized limit condition.

There are two ways to specify the generalized limit condition of definition 1.1. Let $\mathcal{B}|X = \{Y \cap X : Y \in \mathcal{B}\}$. For all $X \subseteq U$, if $\bigcup \mathcal{B} \cap X \neq \emptyset$, either:

(1) $\exists S \subseteq \mathcal{B}$ s.t. $\forall Y \in \mathcal{B}(Y \cap X \neq \emptyset \Rightarrow \exists Z \in S\,(Z\mu(\mathcal{B} \bullet X) \wedge Z \subseteq Y))$, or
(2) $\exists S \subseteq \mathcal{B}$ s.t. $\forall Y \in \mathcal{B}(Y \cap X \neq \emptyset \Rightarrow \exists Z \in S\,((Z \cap X)\mu((\mathcal{B}|X) \bullet X) \wedge Z \subseteq Y))$.

Intuitively, a generalized limit condition states that every set intersecting a broccoli flower intersects every members of a set S of smallest elements of the flower. In the first case, the members of S are minimal sets of the broccoli that have a non-empty intersection with X. In the second case, the members of S have minimal intersection with X.

Definition 1.2. $\mathfrak{M} = (U, \{\mathcal{B}_u\}_{u \in U}, V)$ is a *broccoli model* if U is a set of worlds, $\{\mathcal{B}_u\}_{u \in U}$ is a family of broccoli flowers for each world $u \in U$ satisfying either generalized limit condition, and V is a valuation assigning sets of worlds to propositions.

In what follows, we suppress the index u whenever it is clear from context.

Definition 1.3. We say that φ is true at world u in a broccoli model $\mathfrak{M}$, written $\mathfrak{M}, u \vDash \varphi$ iff (taking standard truth definition for the propositional and the Boolean cases):

(1) $\mathfrak{M}, u \vDash^{\mathfrak{M}} [\varphi]\psi$ iff $\forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|)$, and
(2) $\mathfrak{M}, u \vDash^{\mathfrak{M}} [\varphi\rangle\psi$ iff $\forall Z\mu(\mathcal{B} \bullet |\varphi|)(Z \cap |\varphi| \cap |\psi|) \neq \emptyset$.

Here, as usual, $|\varphi| = \{u : \mathfrak{M}, u \vDash \varphi\}$. We call $|\varphi|$ the *associated proposition* to φ.

These two modalities are meaningful with either generalized limit condition proposed above. Figure 1 illustrates the semantics of both operators.

[1]The completeness of the *MRL* proof system is provided in an extended version of this paper, along a with a proposed extension to *BL* via generalized selection function and the inherent difficulties of this approach (cf. [4])

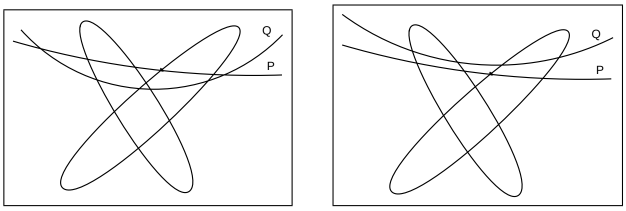

FIGURE 1. Broccoli semantics of the counterfactual operators $[\varphi]\psi$ and $[\varphi\rangle\psi$, with $|\varphi| = P$ and $|\psi| = Q$.

2. Minimal relational logic

Since our goal is to get a logic that captures a notion of belief revision in which revision is relational rather than functional, we need a language that can express notions like "all sets obtained under revision by φ are ψ-sets" and "ψ is consistent with all sets obtained under revision by φ". In counterfactual terminology, the same claims would be read as "all minimal φ-sets are ψ-sets" and "all minimal φ-sets intersect ψ-sets". In this section, we introduce a minimal relational logic that captures the core of these ideas. Section 2.1 introduces the language and the semantics of this minimal logic. We give the axiomatization of the minimal logic in section 2.2.

2.1. Language and Semantics.

We use a standard propositional language whose primitive Boolean connectives are negation $\neg$ and disjunction $\vee$, augmented with two modalities $[\varphi]\psi$ and $[\varphi\rangle\psi$.

Definition 2.1. Given a finite set of propositional variables P, a *minimal relational model* is a triple (U, R, V), where:

- U is a nonempty set, the universe;
- $R = \{R_{|\varphi|} : \varphi \text{ is a formula}, R_{|\varphi|} \subseteq U \times \mathcal{P}(U)\}$; and
- $V : P \longrightarrow \mathcal{P}(U)$.

Definition 2.2. Let $\mathfrak{M}$ be a model and let $w \in U$. The truth-definition for atomic propositions, negations and disjunction is standard. We say that the formula φ is true at point u in a minimal relational model $\mathfrak{M}$, written $\mathfrak{M}, u \vDash \varphi$ if :

$$\mathfrak{M}, u \vDash [\varphi]\psi \quad \text{iff} \quad \forall X((u, X) \in R_{|\varphi|} \Rightarrow \forall v \in X, \mathfrak{M}, v \vDash \psi)$$
$$\mathfrak{M}, u \vDash [\varphi\rangle\psi \quad \text{iff} \quad \forall X((u, X) \in R_{|\varphi|} \Rightarrow \exists v \in X, \mathfrak{M}, v \vDash \psi))$$

The semantics of the modalities $[\varphi]$ and $[\varphi\rangle$ contains two levels of quantification and should be read in two stages: 1) the left bracket picks out a set of φ-subsets of the universe and 2) the right bracket evaluates where ψ is true in these subsets. Notice that the semantics given by minimal relational models is a neighborhood semantics (cf. [3]). Indeed, the relation R is a relation between worlds and subsets of the universe. The modality $[\varphi]$ is the usual neighborhood universal modality, but indexed with associated propositions $|\varphi|$. It comes with its dual modality $\langle\varphi\rangle$ with the obvious semantics. The interesting addition of our language is the modality $[\varphi\rangle$, which expresses that every set $R_{|\varphi|}$-related to u satisfies ψ in at least one point. In neighborhood terminology, this modality expresses that every φ-neighborhood contains at least one ψ-point. This latter modality also come with its natural dual $\langle\varphi]$. In the remainder of this paper, we shall no longer appeal to neighborhood semantics. We will instead provide an interpretation in terms of revision.

2.2. **Proof system.** The following set of axioms and rules is complete with respect to onion selection models:

Axioms:

(1) Classical tautologies
(2) $\langle\varphi\rangle\psi \equiv \neg[\varphi]\neg\psi$
(3) $\langle\varphi]\psi \equiv \neg[\varphi\rangle\neg\psi$
(4) $[\varphi](\psi \to \theta) \to ([\varphi]\psi \to [\varphi]\theta)$
(5) $\langle\varphi]\psi \to \langle\varphi](\psi \vee \theta)$
(6) $[\varphi]\psi \wedge \langle\varphi]\theta \to \langle\varphi](\psi \wedge \theta)$
(7) $\neg\langle\varphi]\top \to [\varphi]\psi$

Rules:

(1) Modus Ponens.
(2) Necessitation for $[\varphi]$ and $[\varphi\rangle$.
(3) If φ and φ' are formulas differing only in φ having an occurrence of θ in one place where φ' has an occurrence of θ', and if $\theta \equiv \theta'$ is a theorem, then $\varphi \equiv \varphi'$ is also a theorem.

Rule 3, *substitution of equivalents*, is applied indiscriminately inside or outside the modal operators. We count the presence of 'φ' inside $[\varphi]$ and $[\varphi\rangle$ as occurrences of φ. For example, if $\psi \equiv \theta$, then both $[\varphi]\psi \equiv [\varphi]\theta$ and $[\psi]\alpha \equiv [\theta]\alpha$ are instances of rule 3.

Axioms 2 and 3 provide the dual modalities of $[\varphi]$ and $[\varphi\rangle$ respectively. Axiom 4 is a typical K axiom for the modality $[\varphi]$ and yields modus ponens under the scope of $[\varphi]$.[2] Axioms 5 is a monotonicity axiom for the modality $\langle\varphi]$. Intuitively, if ψ is consistent with some revision by φ, then anything weaker than ψ is also consistent with some revision by φ. Finally, axiom 6 provides a minimal interaction between the modalities: If ψ is consistent

[2]There is no corresponding K axiom for the $[\varphi\rangle$. Consider a model M such that the set $X \subseteq U$ is the only subset of U that is φ-related to the world $u \in U$, i.e, $R_{|\varphi|} = \{(u, X)\}$. Suppose that both $|\psi| \cap X \neq \emptyset$ and $|\neg\psi| \cap X \neq \emptyset$, but that $|\theta| \cap X = \emptyset$. Then $u \vDash [\varphi\rangle(\psi \to \theta)$ (since $|\neg\psi| \cap X \neq \emptyset$) and $u \vDash [\varphi\rangle\psi$, but $u \nvDash [\varphi\rangle\theta$. Hence, $[\varphi\rangle\psi(\psi \to \theta) \to ([\varphi\rangle\psi \to [\varphi\rangle\theta)$ is not valid.

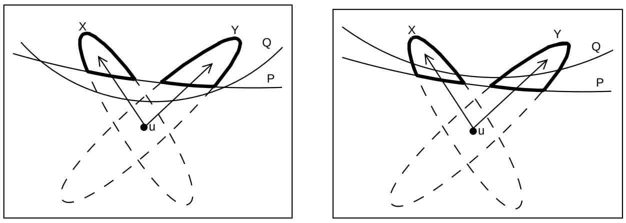

FIGURE 2. Intended semantics of the Broccoli revision operators $[\varphi]\psi$ and $[\varphi\rangle\psi$, with $|\varphi| = P$ and $|\psi| = Q$.

with every revision by φ and there is a revision by φ such that θ is consistent, then there is a revision by φ such that both ψ and θ are consistent. Finally, axiom 7 says that if there is no revision by φ, then every $[\varphi]$ formula holds vacuously. This is akin to saying that every necessary formula holds at en end-point in a Kripke model.

MRL provides the core of broccoli logic and there are two ways to proceed. One approach to get a complete logic for *BL* is to add restrictions on the relation $R_{|\varphi|}$, or to introduce so-called selection functions, in order to get the sets X and Y as two minimal sets returned under revision by φ. Once these sets are selected, the minimal relational logic of the present section will provide the logic to evaluate what holds in these sets. This is illustrated in picture 2, where $|\varphi| = P$ and $|\psi| = Q$. In the picture on the left-hand-side, $[\varphi]\psi$ is true at world u, since every set obtained under revision by φ is a ψ-sets. Similarly, $[\varphi\rangle\psi$ is true at u in the right-hand-side picture, since ψ is consistent with every revision by φ.

The second approach is to find a logic that satisfies the properties of broccoli logic outlined in section 1. This is what we do in section 3, by showing that broccoli logic is equivalent to a well-know conditional logic, the minimal conditional logic of Veltman and Burgess.

3. Broccoli Logic and Minimal Conditional Logic

Minimal conditional logic (*MCL*) was studied by Stalnaker, Pollock, Burgess and Veltman to capture the idea that a conditional $\varphi \Rightarrow \psi$ is true if an only if the conjunction $\varphi \wedge \neg\psi$ is less possible than the conjunction $\varphi \wedge \psi$, and no more. Their modeling comes with a reflexive and transitive $\leq$-order for each world x and no spheres need occur. In a sphere system, two worlds lying on the same sphere agree on which worlds are farther away and which are closer. This assumption is dropped in *MCL*. Hence, if two worlds x and y are equally far away in the underlying order from the real world u and if the world z is farther away than the world y, no conclusions may be drawn as to whether world z is farther from the real world than the world x, or vice versa. Instead of changing the onion picture by allowing non-linearly ordered sphere system as we wish to do in *BL*, *MCL* ignores spheres altogether. It has been a difficult task to find completeness for *MCL*, and we refer the reader to Burgess [2] for a detailed proof. This section will show how to avoid similar completeness difficulties with *BL* by showing that it is actually equivalent to *MCL*.

Section 3.1 provides the *MCL* semantics, section 3.2 gives the complete proof system and section 3.3 shows the equivalence of *MCL* and *BL*.

3.1. **Minimal Conditional Logic.** A *Minimal conditional logic model* is a triple (U, R^3, V), where U and V are as above, and R^3 is a ternary relation on U that respects reflexivity and transitivity (cf. [2]). The relation $Rxyz$ should be read as "according to world x, world y is no farther away than world z". We shall write the more suggestive $y \leq_x z$ instead of $Rxyz$. We let $W_u = \{y : \exists z, y \leq_x z\}$ be the zone of entertainability for world $u \in U$. Intuitively, worlds outside the zone of entertainability for u are worlds so far away that their distance from the real world is not appreciable. The *minimal conditional logic language* contains a set of propositional variables, together with negation $\neg$, disjunction $\vee$ and a counterfactual modality $[\varphi]$ for every formula φ.

Definition 3.1. We say that the formula $[\varphi]\psi$ is true at world u in the model $\mathfrak{M}$, and we write $\mathfrak{M}, u \vDash [\varphi]\psi$, iff:

$$\forall y \in W_u \cap V(\varphi), \exists z \in W_u \cap V(\varphi)[z \leq_u y \& \forall w \in W_u \cap V(\varphi)(w \leq_u z \rightarrow w \in V(\psi)]$$

Notice that the semantic definition of $[\varphi]\psi$ does not contain a minimality condition. However, if the model is finite and $\mathfrak{M}, u \vDash [\varphi]\psi$, then there is a minimal world $z \in U$ such that $z \in V(\varphi) \cap V(\psi)$. Since we will only use finite models for our equivalence result, we use the minimality formulation in evaluating $[\varphi]\psi$ for the remainder of this paper. The semantic condition reduces to:

$$\forall y \in W_U \cap V(\varphi), \exists z \in W_u \cap V(\varphi)[z \leq_u y \& \forall w <_u z, w \notin V(\varphi) \& z \in V(\psi)].$$

3.2. **Proof System.** The following set of axioms, with the same set of rules as for minimal relational logic presented in section 2.2, is complete for *MCL* (cf. [2]):

(1) Classical tautologies
(2) $[\varphi]\varphi$
(3) $[\varphi]\psi \wedge [\varphi]\theta \rightarrow [\varphi](\psi \wedge \theta)$
(4) $[\varphi](\psi \wedge \theta) \rightarrow [\varphi]\psi$
(5) $[\varphi]\psi \wedge [\varphi]\theta \rightarrow [\varphi \wedge \psi]\theta$
(6) $[\varphi]\psi \wedge [\theta]\psi \rightarrow [\varphi \vee \theta]\psi$

3.3. **Minimal Conditional Logic is Broccoli Logic.** Let $\mathfrak{M} = (U, R, V)$ be a finite minimal conditional logic model. We will transform this model into a broccoli model, by constructing a broccoli at each world of $\mathfrak{M}$, taking the downward closed sets of worlds according to the underlying order. More precisely, let $C_x(y) = \{z \in U : z \leq_x y\}$, then $BROC(x) = \{C_x(y) : y \in W_x\}$ is the induced broccoli at world x. In particular, since $\mathfrak{M}$ is finite, the generalized limit condition of definition 1.1 holds. An induced broccoli model $BROC(\mathfrak{M})$ is then given by:

$$BROC(\mathfrak{M}) = \{BROC(x) : x \in U\}$$

The semantics of $[\varphi]\psi$ in the induced broccoli model is given by the following:

$$BROC(\mathfrak{M}), x \models [\varphi]\psi \text{ iff } \forall Z\mu(BROC(x) \bullet |\varphi|)(Z \cap |\varphi| \subseteq |\psi|).$$

The main result of this section now follows from lemma 3.2.

Lemma 3.2. *$\mathfrak{M}, x \models [\varphi]\psi$ iff $BROC(\mathfrak{M}), x \models [\varphi]\psi$.*

Proof. In the one direction, assume that $\mathfrak{M}, x \models [\varphi]\psi$. Let $C_w\mu(BROC(x) \bullet |\varphi|)$, and let $v \in C_w \cap |\varphi|$. By the truth definition for $[\varphi]\psi$, $\exists z \leq_x v$ such that $\mathfrak{M}, z \models \varphi \wedge \psi$ and $\forall y <_x z, \mathfrak{M}, y \not\models \varphi$. But z must be equal to v. Otherwise, $C_z \subset C_v \subseteq C_w$ (the latter inclusion uses the transitivity of $\leq_x$), which implies that C_z would be a proper subset of C_w intersecting $|\varphi|$, contradicting our assumption. Thus, $v \in |\psi|$, which implies that $C_w \cap |\varphi| \subseteq |\psi|$. Therefore, as C_w was chosen arbitrarily, $BROC(\mathfrak{M}), x \models [\varphi]\psi$.

In the other direction, assume that $BROC(\mathfrak{M}), x \models [\varphi]\psi$ and suppose that $\mathfrak{M}, y \models \varphi$ for some $y \in U$. Then $C_y \cap |\varphi| \neq \emptyset$. Hence, $\exists C_w \subseteq C_y$ such that $C_w\mu(BROC(x) \bullet |\varphi|)$ (by the limit condition!) and $C_w \cap |\varphi| \subseteq |\psi|$. But since $C_w \subseteq C_y, w \leq_x y$. Assume that w is not a minimal world satisfying $\varphi \wedge \psi$ with respect to $\leq_x$, then $\exists w' <_x w$ such that $\mathfrak{M}, w' \models \varphi \wedge \psi$. This implies that $C'_w \subset C_w$ and $C'_w \cap |\varphi| \cap |\psi| \neq \emptyset$, contradicting the minimality of C_w. Therefore, w is a minimal world satisfying $\varphi \wedge \psi$ and since $w \leq_x y$, we get that $\mathfrak{M}, x \models [\varphi]\psi$. □

We are now ready for our main theorem.

Theorem 3.3. *Broccoli logic = MCL.*

Proof. To show that *MCL* is *BL*, we need to show 1) that all axioms of section 3.1 are valid in *BL*, whose semantics was given in section 1 and 2) that if a principle is not derivable in *MCL*, then there is a broccoli countermodel.

Showing that the *MCL* axioms are valid in the *BL*-models of section 1 is straightforward. We show that axiom (5) is valid and leave the others to the reader. Let $\mathfrak{M}$ be an arbitrary broccoli model and let $u \in U$ be arbitrary. If $\neg\langle\varphi]\top \notin u$, i.e., if there is no revision by φ, then the thesis is vacuously true. Hence, assume that there is a revision by φ. Assume furthermore that $\mathfrak{M}, u \vDash [\varphi]\psi \wedge [\varphi]\theta$. Since $\mathfrak{M}, u \vDash [\varphi]\psi$, $|\varphi| \cap |\psi| \neq \emptyset$. Let $Z\mu(\mathcal{B} \bullet |\varphi \wedge \psi|)$ be a minimal set of $\mathcal{B}$ intersecting $|\varphi \wedge \psi|$. Then for every $z \in Z$, $x \in |\varphi| \cap |\psi|$ implies that $z \in |\varphi| \subseteq |\theta|$. Hence, $\mathfrak{M}, u \vDash [\varphi \wedge \psi]\theta$.

To show that if a principle is not provable in MCL, then there is a broccoli countermodel to φ, we use the completeness result of Burgess. If $MCL \nvdash \varphi$ for some φ, then there is a finite model $\mathfrak{M} = (U, R, V)$ and a world $u \in U$ such that $\mathfrak{M}, u \nvDash \varphi$. [3] By lemma 3.2, $BROC(\mathfrak{M}), u \nvDash \varphi$. Therefore, $BROC(\mathfrak{M})$ is a broccoli countermodel to φ. This completes the proof of theorem 3.3. □

Corollary 3.4. *BL is decidable.*

4. Conclusion

Our goal was to generalize onion semantics to capture relational belief revision; the result was BL. It turns out that BL is equivalent to a well-known conditional logic, the Burgess-Veltman minimal conditional logic. This is a fortunate outcome, as it saves a lot of work in coming up with a completeness result expanding on the minimal revisional logic of section 2. The major difficulty along the latter line is to devise an appropriate generalized arrow condition yielding generalized selection functions, and this is still an open question. Another open question is the rôle played by the $[\varphi\rangle$ modality in BL: what is the complete minimal logic of $[\varphi]\psi$ and $[\varphi\rangle\psi$ over the Burgess-Veltman models? An advantage of MCL over BL is that it avoids the problem of choosing an appropriate generalized limit condition by dropping the sphere representation altogether. A lesson should be drawn here, namely that, as so often over the past years, we see that logics of belief revision are largely conditional logics.

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