Modal logics: overview

- Part I: Introduction to modal and multimodal logics
  1. Motivation and introduction
  2. The basic multimodal logic K
  3. The basic monomodal logics
  4. Completeness of $G(k, l, m, n)$ logics, and decidability of the basic modal logics
  5. Basic multimodal logics
  6. Other modal logics
- Part II: Applications
  7. Knowledge and announcements
  8. Belief
  9. **Common knowledge and common belief**
  10. Action and propositional dynamic logic
  11. Goals and intentions
  12. Ability, agency and branching time
- Part II: Proof methods
  13. Translation method
  14. Tableau method

Chapter 9. Common knowledge and common belief

Nov.4, 2008

Common knowledge and common belief: overview

- common knowledge
  - induction axiom
- common belief

Common knowledge: introduction

- $Agts = \{i_1, i_2, \ldots, i_{\mathrm{card}(Agts)}\}$ ('*i*ndividuals')
- $\mathrm{CK}_{i,j}\,\varphi$ = "it is common knowledge of i and j that φ"
  - required in situations of coordination
    - ★ conventions in societies ('drive on the right')
    - ★ coordinated attack problem ('Byzantine Generals')
    - ★ remember: *Agts* finite (else 2^{Agts} uncountable)
- expected to be valid:
  - $(\mathrm{CK}_{i,j}\,\varphi \wedge \mathrm{CK}_{i,j}\,\psi) \leftrightarrow \mathrm{CK}_{i,j}\,(\varphi \wedge \psi)$
  - $\mathrm{CK}_{i,j}\,\top$
  - $\mathrm{CK}_{i,j}\,\varphi \rightarrow \mathrm{K}_i\,\varphi \wedge \mathrm{K}_j\,\varphi$
  - $\mathrm{CK}_{i,j}\,\varphi \rightarrow \mathrm{K}_i\,\mathrm{K}_j\,\varphi \wedge \mathrm{K}_j\,\mathrm{K}_i\,\varphi$
  - $\mathrm{CK}_{i,j}\,\varphi \rightarrow \varphi$
  - $\mathrm{CK}_{i,j,k}\,\varphi \rightarrow \mathrm{CK}_{i,j}\,\varphi$
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- informally:
  - $\mathsf{CK}_{i,j}\,\varphi = \mathsf{K}_i\,\varphi \wedge \mathsf{K}_j\,\varphi \wedge \mathsf{K}_i\,\mathsf{K}_j\,\varphi \wedge \mathsf{K}_j\,\mathsf{K}_i\,\varphi \wedge \mathsf{K}_i\,\mathsf{K}_j\,\mathsf{K}_i\,\varphi \wedge \ldots$

Common knowledge: semantics

- $M, w \Vdash \mathsf{CK}_{i,j}\,\varphi$ iff $M, w \Vdash \mathsf{K}_i\,\varphi \wedge \mathsf{K}_j\,\varphi \wedge \mathsf{K}_i\,\mathsf{K}_j\,\varphi \wedge \ldots$
- accessibility relation for $\mathsf{CK}_{i,j}$ is 'big':
  - $R_{\mathsf{CK}_{i,j}}(w) = R_{\mathsf{K}_i}(w) \cup R_{\mathsf{K}_j}(w) \cup (R_{\mathsf{K}_i} \circ R_{\mathsf{K}_j})(w) \cup (R_{\mathsf{K}_j} \circ R_{\mathsf{K}_i})(w) \cup \ldots$
- formally:
  - $R_{\mathsf{CK}_{i,j}}(w) \stackrel{\text{def}}{=} (R_{\mathsf{K}_i} \cup R_{\mathsf{K}_j})^*(w)$
- generalizes to groups:
  - $R_{\mathsf{CK}_J}(w) \stackrel{\text{def}}{=} (\bigcup_{i \in J} R_{\mathsf{K}_i})^*(w)$

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- abbreviation:
  - $\mathrm{EK}_{i_1,\ldots,i_n}\,\varphi \stackrel{\text{def}}{=} \mathrm{K}_{i_1}\,\varphi \wedge \ldots \wedge \mathrm{K}_{i_n}\,\varphi$ 'everybody knows'
- properties:
  - $\models (\mathrm{EK}_{J_1}\,\varphi \wedge \mathrm{EK}_{J_2}\,\varphi) \leftrightarrow \mathrm{EK}_{J_1 \cup J_2}\,\varphi$
  - $\not\models \mathrm{EK}_J\,\varphi \rightarrow \mathrm{EK}_J\,\mathrm{EK}_J\,\varphi$

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  - cannot be defined as an abbreviation
  - new modal operator

Common knowledge: axiomatization

- axiomatization of $KT5(\mathsf{K}_i)$ with common knowledge:
  - axiomatics $KT5(i)$
  - fixpoint axiom:
    - $\mathsf{CK}_J\,\varphi \leftrightarrow (\varphi \wedge \mathsf{EK}_J\,\mathsf{CK}_J\,\varphi)$
    - N.B.: right-to-left direction already a theorem:
      $\vdash_{KT5(\mathsf{K}_i)} \mathsf{EK}_J\,\mathsf{CK}_J\,\varphi \rightarrow \mathsf{K}_i\,\mathsf{CK}_J\,\varphi$, and
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  - least fixpoint axiom (alias induction axiom):
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  - only weakly complete, but not strongly:
    - $\{\mathsf{EK}_J{}^n\varphi \ : \ n \geq 0\} \models \mathsf{CK}_J\,\varphi$, but
      $\{\mathsf{EK}_J{}^n\varphi \ : \ n \geq 0\} \not\vdash \mathsf{CK}_J\,\varphi$
  - '$KT5(\mathsf{K}_i)$ with common knowledge not compact'
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- complexity of satisfiability: EXPTIME complete

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  $\not\models \mathsf{CK}_{i,j}\,(|n_i - n_j| = 1) \rightarrow \mathsf{CK}_{i,j}\,(n_i \leq 100)$
- prove that the logic of common knowledge has all principles of $KT5$
  - axiomatically: ...
  - semantically: prove that the reflexive and transitive union of equivalence relations is an equivalence relation
    - ★ $(\bigcup_{i\in J} R_{\mathsf{K}_i})^*$ is reflexive
    - ★ if some R_{K_i} is reflexive then $(\bigcup_{i\in J} R_{\mathsf{K}_i})^+$ is reflexive
    - ★ if every R_{K_i} is symmetric then $(\bigcup_{i\in J} R_{\mathsf{K}_i})^+$ is symmetric

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- $\mathrm{EB}_J \varphi \stackrel{\text{def}}{=} \bigwedge_{i \in J} \mathrm{B}_i \varphi$ 'everybody believes'
- axiomatization of $KD45(\mathrm{B}_i)$ with common belief:
  - axiomatics $KD45(\mathrm{B}_i)$
  - fixpoint axiom:
    - $\mathrm{CB}_J \varphi \leftrightarrow (\mathrm{EB}_J \varphi \wedge \mathrm{EB}_J \mathrm{CB}_J \varphi)$
  - least fixpoint inference rule (alias induction rule):
    - $\frac{\varphi \rightarrow \mathrm{EB}_J \varphi}{\mathrm{EB}_J \varphi \rightarrow \mathrm{CB}_J \varphi}$

    equivalent to least fixpoint axiom
    - $(\mathrm{EB}_J \varphi \wedge \mathrm{CB}_J (\varphi \rightarrow \mathrm{EB}_J \varphi)) \rightarrow \mathrm{CB}_J \varphi$
- sound, complete and decidable
- EXPTIME complete

Exercises

- prove that if R_{B_i} is serial then $(\bigcup_{i\in J} R_{\mathrm{B}_i})^+$ is serial
- prove that $(\bigcup_{i\in J} R_{\mathrm{B}_i})^+$ is transitive
- prove that $(\bigcup_{i\in J} R_{\mathrm{B}_i})^+$ is not necessarily Euclidean

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- prove that $(\bigcup_{i\in J} R_{\mathsf{B}_i})^+$ is transitive
- prove that $(\bigcup_{i\in J} R_{\mathsf{B}_i})^+$ is not necessarily Euclidean
  - $\not\models \neg \mathsf{CB}_{i,j}\,\varphi \rightarrow \mathsf{CB}_{i,j}\,\neg \mathsf{CB}_{i,j}\,\varphi$
    (no negative introspection)
  - logic of common belief weaker than $KD45$!

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- also studied: distributed knowledge
  - $M, w \Vdash \mathrm{DK}_{i,j}\,\varphi$ iff $M, v \Vdash \varphi$ for every $v \in R_{\mathrm{K}_i} \cap R_{\mathrm{K}_i}(w)$
  - 'the common knowledge i and j potentially have' (after having communicated all they know)
    - ★ intersection of accessibility relations: not modally definable
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    - ★ but a complete axiomatization exists
- check that for the extension of *PAL* by common knowledge, the reduction axiom $[\varphi!]\mathsf{K}_i\,\psi \leftrightarrow \neg\varphi \vee \mathsf{K}_i\,[\varphi!]\psi$ is not valid.