Modal logics: overview

Chapter 5.
Basic multimodal logics

Oct.7, 2008

Basic multimodal logics: overview

- simple multimodal axiom schemas
  - inclusion, conversion, confluence, permutation
- generalize axiom schemas $G(k, l, m, n)$ to multimodal logics
  - multidimensional modal logics

Examples of interaction axioms

extensions of the basic multimodal logic K by:

- Incl($\Box_i, \Box_j$): $\Box_i\varphi \to \Box_j\varphi$
  - hierarchy of knowledge or belief
  - 'always φ implies next φ'
  - 'belief φ implies (realistic) preference φ'
- Conv($\Box_i, \Box_j$): $\varphi \to \Box_i\Diamond_j\varphi$
  - $\varphi \to \Box_{\to}\Diamond_{\leftarrow}\varphi$ — temporal reasoning with past and future
- Confl($\Box_i, \Box_j$): $\Diamond_i\Box_j\varphi \to \Box_j\Diamond_i\varphi$ — alternatively: Confl(j, i)
  - $\Diamond_{horiz}\Box_{vertic}\varphi \to \Box_{vertic}\Diamond_{horiz}\varphi$ — spatial reasoning
- Perm($\Box_i, \Box_j$): $\Box_i\Box_j\varphi \to \Box_j\Box_i\varphi$
  - $\mathrm{K}\,\Box_a\varphi \to \Box_a\mathrm{K}\,\varphi$ — 'perfect recall', 'no forgetting'
  - $\Box_a\mathrm{K}\,\varphi \to \mathrm{K}\,\Box_a\varphi$ — 'no learning'
  - $\Box_{horiz}\Box_{vertic}\varphi \to \Box_{vertic}\Box_{horiz}\varphi$ — spatial reasoning

Examples of interaction axioms: correspondence

- incl($\Box_i, \Box_j$): $R_j \subseteq R_i$
  - $\Diamond_j \varphi \to \Diamond_i \varphi$
- conv($\Box_i, \Box_j$): $(R_i)^{-1} \subseteq R_j$ $\qquad id_W \subseteq R_i \circ R_j$
- confl($\Box_i, \Box_j$): $(R_i)^{-1} \circ R_j \subseteq R_j \circ (R_i)^{-1}$
- perm($\Box_i, \Box_j$): $R_j \circ R_i \subseteq R_i \circ R_j$
  - $\Diamond_j \Diamond_i \varphi \to \Diamond_i \Diamond_j \varphi$

Correspondence Theorem.
Let X_k be among the axioms Incl($\Box_i, \Box_j$), Perm($\Box_i, \Box_j$), Confl($\Box_i, \Box_j$), and let x_k be the associated condition among incl($\Box_i, \Box_j$), perm($\Box_i, \Box_j$), confl($\Box_i, \Box_j$).
A multimodal K-frame $\langle W, R \rangle$ satisfies condition (x_k) on R iff every instance of (X_k) is valid in $\langle W, R \rangle$.

will follow from more general result (... that won't be proved)

The axioms of generalized confluence

$$G(\alpha, \beta, \gamma, \delta) = \Diamond_\alpha \Box_\beta \varphi \to \Box_\gamma \Diamond_\delta \varphi$$

- generalization of axiom schemas $G(k, l, m, n)$ to multimodal logics
- inductive definition of expressions α:

$$\alpha \quad ::= \quad a \mid \text{skip} \mid (\alpha;\alpha) \mid (\alpha \cup \alpha)$$

  where *a* ranges over set of parameters *Prms*
- possible intuition:
  - *a* = abstract atomic program, atomic action
  - skip = empty program
  - $\alpha;\alpha$ = sequential composition
  - $\alpha \cup \alpha$ = nondeterministic composition

  cf. dynamic logic (Chapter 10)

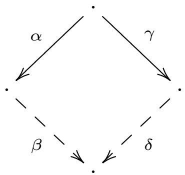
The axioms of generalized confluence

$$G(\alpha, \beta, \gamma, \delta) = \Diamond_\alpha \Box_\beta \varphi \to \Box_\gamma \Diamond_\delta \varphi$$

- define as abbreviations:
  - $\Box_{\text{skip}} \varphi \stackrel{\text{def}}{=} \varphi$
  - $\Box_{\alpha;\beta} \varphi \stackrel{\text{def}}{=} \Box_\alpha \Box_\beta \varphi$
  - $\Box_{\alpha \cup \beta} \varphi \stackrel{\text{def}}{=} \Box_\alpha \varphi \wedge \Box_\beta \varphi$
- examples:
  - $\text{Incl}(\Box_i, \Box_j) = \ldots$
  - $\text{Perm}(\Box_i, \Box_j) = \ldots$
  - $\text{Confl}(\Box_i, \Box_j) = \ldots$

$G(\alpha, \beta, \gamma, \delta)$ logics: semantics

- frame property corresponding to axiom $G(\alpha, \beta, \gamma, \delta)$:
  g$(\alpha, \beta, \gamma, \delta)$ = $R_\alpha^{-1} \circ R_\gamma \subseteq R_\beta \circ R_\delta^{-1}$
  - $R_{\text{skip}} = \delta_W = \{\langle w, w \rangle : w \in W\}$
  - $R_{\alpha;\beta} = R_\alpha \circ R_\beta$
  - $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$



$G(\alpha, \beta, \gamma, \delta)$ logics: completeness

- logics $KX_1, \ldots, X_n$, where $(X_1), \ldots, (X_n)$ are axioms of the form $G(\alpha, \beta, \gamma, \delta)$
  - $\vdash_{KX_1,\ldots,X_n} \varphi$ = "φ provable from axioms of K and $(X_1), \ldots, (X_n)$"
- frame property corresponding to (X_i): (x_i)
  - $\models_{KX_1,\ldots,X_n} \varphi$ = "φ valid in the class of $(x_1), \ldots, (x_n)$-frames"

Theorem [Lemmon/Scott], [Sahlqvist], [Catach].
Let the (X_i) be axioms of the form $G(\alpha, \beta, \gamma, \delta)$.
Then $\vdash_{KX_1,\ldots,X_n} \varphi$ iff $\models_{KX_1,\ldots,X_n} \varphi$.

proof similar to that for $G(k, l, m, n)$ logics; not proved here

other mathematical properties:

- decidability: unknown for many $G(\alpha, \beta, \gamma, \delta)$ logics
- complexity of (decidable) $G(\alpha, \beta, \gamma, \delta)$ logics: often very high
  - multidimensional modallogics...

Multidimensional modal logics

- alias product logics
- any modal logic having permutation and confluence axioms for every couple of modal operators:
  - Perm($\Box_i, \Box_j$): $\Box_i \Box_j \varphi \to \Box_j \Box_i \varphi$
  - Confl($\Box_i, \Box_j$): $\Diamond_i \Box_j \varphi \to \Box_j \Diamond_i \varphi$
- alternative semantics: $M = \langle W, R, V \rangle$ where
  - $W = W_1 \times \ldots \times W_n$
    - notation: $\{\vec{w}, \vec{v}, \ldots\}$
    - $\vec{w}_i = \langle w_1, \ldots, w_n \rangle_i = w_i$ (ith element of $\vec{w}$)
  - $R_i(\vec{w}) = \{\vec{v} \ : \ \vec{v}_i = \vec{w}_i\}$
- logics: $K^2 = K \otimes K$, $K^3 = K \otimes K \otimes K$, $\ldots KT5^2, \ldots, K \otimes KT5, \ldots$
- N.B.: the two semantics do not always match!
  - every product model of $KT5(1) \otimes KT5(2) \otimes KT5(3)$ is a $KT5(1) + KT5(2) + KT5(3)$ model satisfying perm($\Box_i, \Box_j$) and confl($\Box_i, \Box_j$) for every i, j;
  - …but not the other way round!

Multidimensional modal logics, ctd.

- applications:
  - spatial modal logics
    - $K^2, K^3, \ldots, KT4^2, KT4^3, \ldots, KT5^2, \ldots$
  - logics of time and knowledge (or belief)
    - $K \otimes KT5, KT4 \otimes SKT5, K \otimes KD45, KT4 \otimes SKD45, \ldots$
  - logics of agency with independence of agents
    - $KT5^2 = KT5 \otimes KT5, KT5^3, \ldots$
- properties: sometimes decidable; and if so very complex
  - K^2: decidable; complexity unknown
  - $KT5^2$: NEXPTIME
  - $KT5^3$: undecidable & not finitely axiomatizable
  - ...
  - technique to prove undecidability: encode tiling problems
- [Gabbay, Kurucz, Wolter and Zakharyaschev 03]

Chapter 6.
Other modal logics

Oct.21, 2008

Other modal logics: overview

- intuitionistic logic
- logic of provability
  - no infinite descending chains (no FO-property)
- temporal logic
  - transitive closure (no FO-property)
  - also for common knowledge, common belief
- epistemic logics without omniscience
  - 'non-normal modal logics' $\Rightarrow$ no Kripke models!
  - 'neighborhood models'
- first-order modal logics

Intuitionistic logic (excursion)

- language: primitives $\bot$, $\to$, $\vee$, $\wedge$
  - N.B.: only negation is definable
    - ★ $\neg\varphi \stackrel{\text{def}}{=} \varphi \to \bot$
- motivation: design a logic without $(\neg\varphi \to \bot) \to \varphi$

  'reductio ad absurdum'
  - also excludes:
    - ★ $\varphi \vee \neg\varphi$ 'tertio non datur'
    - ★ $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$ contraposition
  - but keep $\varphi \to \neg\neg\varphi$
  - ...and keep all the other principles not related to negation
    - ★ well, not exactly: Pearce's law $((\varphi \to \psi) \to \varphi) \to \varphi$ also has to be abandoned

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$\Rightarrow$ axiomatics: $\vdash_{LJ} \varphi$

- semantics?

Intuitionistic logic (excursion, ctd.)

- translation into $KT4$ [Gödel]:
  - $t(p) = \Box p$
  - $t(\bot) = \bot$
  - $t(\neg\varphi) = \Box\neg t(\varphi)$
  - $t(\varphi_1 \land \varphi_2) = t(\varphi_1) \land t(\varphi_2)$
  - $t(\varphi_1 \lor \varphi_2) = t(\varphi_1) \lor t(\varphi_2)$
  - $t(\varphi_1 \to \varphi_2) = \Box(t(\varphi_1) \to t(\varphi_2))$

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  - $t(\varphi_1 \vee \varphi_2) = t(\varphi_1) \vee t(\varphi_2)$
  - $t(\varphi_1 \to \varphi_2) = \Box(t(\varphi_1) \to t(\varphi_2))$
- **Theorem.** $\vdash_{LJ} \varphi$ iff $\vdash_{KT4} t(\varphi)$

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- example: $t(p \vee \neg p) = \ldots$
  - find a countermodel

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  - in fact, PSPACE complete

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- $\vdash_{LJ}$-satisfiability problem is decidable, in PSPACE
  - in fact, PSPACE complete
- therefore the Kripke semantics of LJ is: ...

Intuitionistic logic (excursion, ctd.)

- semantics: Kripke models [Kripke]
  - *LJ*-model: $M = \langle W, R, V \rangle$ such that
    - R reflexive and transitive
    - if wRv and $w \in V(p)$ then $v \in V(p)$ 'hereditary'
  - truth conditions:
  - $M, w \Vdash p$ iff $w \in V(p)$
  - $M, w \Vdash \varphi_1 \wedge \varphi_2$ iff $M, w \Vdash \varphi_1$ and $M, w \Vdash \varphi_2$
  - $M, w \Vdash \varphi_1 \vee \varphi_2$ iff $M, w \Vdash \varphi_1$ or $M, w \Vdash \varphi_2$
  - $M, w \Vdash \neg\varphi$ iff $M, v \nVdash \varphi$ **for all** $v \in R(w)$
  - $M, w \Vdash \varphi_1 \to \varphi_2$ iff $M, v \Vdash \varphi_1$ implies $M, v \Vdash \varphi_2$ **for all** $v \in R(w)$

Logic of provability: no infinite descending chains

- class of models such that R is transitive and has no infinite chain
  - monomodal
  - complete axiomatization: K plus

$$\Diamond\varphi \to \Diamond(\varphi \land \Box\neg\varphi)$$

- allows to reason about provability in arithmetic
  - $\Box\varphi$ = "φ provable sentence in arithmetic"
  - $\Diamond\varphi$ = "φ consistent sentence in arithmetic"
- Gödel-Löb logic *GL*

LTL: transitive closure

- natural temporal operators:
  - "always φ"
  - "sometimes φ"
  - "always in the future φ" = "henceforth φ" = $\mathsf{G}\varphi$
  - "sometimes in the future φ" = "eventually φ" = $F\varphi$
  - "always in the past φ"
  - "sometimes in the past φ"
  - "φ until ψ" (binary modal operator)
  - "φ since ψ"
  - "φ (strictly) before ψ" = $\neg\psi$ until φ
  - "φ (strictly) after ψ" = ...
- if time is discrete then there naturally is a 'next' operator:
  - "next φ" = $\mathsf{X}\,\varphi$

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- accessibility relation R_{X} determines R_{G}:
  - $R_{\mathsf{G}} = R_{\mathsf{X}}^*$ (R_{G} = reflexive and transitive closure of R_{X})
- Linear-time Temporal Logic *LTL*

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- axiomatization?

LTL: transitive closure, ctd.

- $R_{\mathsf{G}} = R_{\mathsf{X}}^{*}$
- problem: not a first-order property
  - $R_a^+(w, v) \leftrightarrow$
    $\exists n(\exists\{w_1, \ldots, w_n\})(R_a(w_1, w_2) \wedge \ldots \wedge R_a(w_{n-1}, w_n) \wedge w_1 = w \wedge w_n = v)$
- axiomatization of *LTL* with G and X (no ‘until’):
  - axiomatics $K(\mathsf{X})$
  - D(X): $\mathsf{X}\,\varphi \rightarrow \neg\mathsf{X}\,\neg\varphi$
  - Alt(X): $\neg\mathsf{X}\,\neg\varphi \rightarrow \mathsf{X}\,\varphi$
  - axiomatics $KT4(\mathsf{G})$
  - fixpoint axiom:
    - ★ $\mathsf{G}\varphi \leftrightarrow (\varphi \wedge \mathsf{X}\,\mathsf{G}\varphi)$
  - least fixpoint axiom (alias induction axiom):
    - ★ $(\varphi \wedge \mathsf{G}(\varphi \rightarrow \mathsf{X}\,\varphi)) \rightarrow \mathsf{G}\varphi$
- sound, complete and decidable
  - proof ...
- PSPACE complete

Transitive closure: other applications

- generally (for any relation R_a):
  - $R_{a^*} \stackrel{\text{def}}{=} (R_a)^*$ (so G = X*)
  - $R_{a^+} \stackrel{\text{def}}{=} (R_a)^+$ (transitive closure of R_a)
- also relevant for other logics:
  - program iteration: $R_{[a^*]} = (R_{[a]})^*$
  - common knowledge: $R_C K = (\bigcup_{i \in Agts} R_{K_i})^+ = (\bigcup_{i \in Agts} R_{K_i})^*$
  - common belief: $R_C B = (\bigcup_{i \in Agts} R_{B_i})^+$
- similar axiomatizations: induction axiom, alias least fixpoint axiom (or rule)
- common problem in proof of fmp: filtration by $sf(\varphi)$ does not work
  - common technique: "Fischer-Ladner closure of $sf(\varphi)$"

LTL: standard semantics

- *LTL*-models can be restricted to:
  - models $M = \langle W, R, V \rangle$ such that
    - W = the set of natural numbers
    - R_X is the successor function: $R_X(n) = \{n+1\}$
    - $R_G = \leq$
    - V any valuation
  - $M, n \Vdash X\varphi$ iff $M, n+1 \Vdash \varphi$
  - $M, n \Vdash G\varphi$ iff $M, n+k \Vdash \varphi$ for all $k \geq 0$

Non-normal modal logics

- idea: invalidate axiom C(□): $(\Box\varphi_1 \wedge \Box\varphi_1) \rightarrow \Box(\varphi_1 \wedge \varphi_1)$
  - avoid omniscience
- abandon Kripke models

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  - avoid omniscience
- abandon Kripke models
- $M = \langle W, N, V \rangle$ where
  - W nonempty set of possible worlds
  - $N : W \longrightarrow 2^{2^W}$ 'neighborhood of w'
  - $V : Atms \longrightarrow 2^W$
- $M, w \Vdash \Box\varphi$ iff there is $V \in N(w)$ such that for all $v \in V$, $M, w \Vdash \varphi$

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- $M, w \Vdash \Box\varphi$ iff there is $V \in N(w)$ such that for all $v \in V$, $M, w \Vdash \varphi$
- basic monotonic modal logic M
  - validates RM($\Box$) and N($\Box$)
  - does not validate C($\Box$)
  - sound and complete, decidable
  - in general: less complex
    - reason: less expressive

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- exercise: can you think of a way of 'unplugging' N(□)? RM(□)?

First-order modal logics (excursion)

- add domain of objects *Obj*
- associate a domain to every world: $D : W \longrightarrow 2^{Obj}$
- free domains: no relation between domains at different worlds
  - difficult to axiomatize ...

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- free domains: no relation between domains at different worlds
  - difficult to axiomatize ...
- increasing domains: if wR_iv then $D(w) \subseteq D(v)$
  - validates *CBF*: $\exists x \Diamond \varphi \to \Diamond \exists x \varphi$ ('converse Barcan Formula')
  - ... already provable in any axiomatic extension of K:
    1. $\varphi \to \varphi$ *CPL*
    2. $\varphi \to \exists x \varphi$ $(\exists_r)$ of *FOL*
    3. $\Diamond \varphi \to \Diamond \exists x \varphi$ RM($\Diamond$)
    4. $\exists x \Diamond \varphi \to \Diamond \exists x \varphi$ $(\exists_l)$ of *FOL*
  - semantics for FO-intuitionistic logic
- decreasing domains: if wR_iv (for some i) then $D(v) \subseteq D(w)$
  - validates: $\Diamond \exists x \varphi \to \exists x \Diamond \varphi$ (BF = 'Barcan Formula')
  - optional, except for logics with symmetric accessibility relations
- uniform domains: $D(w) = D(v)$ for all w, v
- some are incomplete ...