

Modal logics: overview

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Chapter 5.

Basic multimodal logics

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Basic multimodal logics: overview

- simple multimodal axiom schemas
 - ▶ inclusion, conversion, confluence, permutation
- generalize axiom schemas $G(k, l, m, n)$ to multimodal logics
 - ▶ multidimensional modal logics

Examples of interaction axioms

extensions of the basic multimodal logic K by:

- $\text{Incl}(\Box_i, \Box_j)$: $\Box_i\varphi \rightarrow \Box_j\varphi$
 - ▶ hierarchy of knowledge or belief
 - ▶ 'always φ implies next φ '
 - ▶ 'belief φ implies (realistic) preference φ '
- $\text{Conv}(\Box_i, \Box_j)$: $\varphi \rightarrow \Box_i\Diamond_j\varphi$
 - ▶ $\varphi \rightarrow \Box_{\rightarrow}\Diamond_{\leftarrow}\varphi$ temporal reasoning with past and future
- $\text{Confl}(\Box_i, \Box_j)$: $\Diamond_i\Box_j\varphi \rightarrow \Box_j\Diamond_i\varphi$ alternatively: $\text{Confl}(j, i)$
 - ▶ $\Diamond_{\text{horiz}}\Box_{\text{vertic}}\varphi \rightarrow \Box_{\text{vertic}}\Diamond_{\text{horiz}}\varphi$ spatial reasoning
- $\text{Perm}(\Box_i, \Box_j)$: $\Box_i\Box_j\varphi \rightarrow \Box_j\Box_i\varphi$
 - ▶ $\mathbf{K}\Box_a\varphi \rightarrow \Box_a\mathbf{K}\varphi$ 'perfect recall', 'no forgetting'
 - ▶ $\Box_a\mathbf{K}\varphi \rightarrow \mathbf{K}\Box_a\varphi$ 'no learning'
 - ▶ $\Box_{\text{horiz}}\Box_{\text{vertic}}\varphi \rightarrow \Box_{\text{vertic}}\Box_{\text{horiz}}\varphi$ spatial reasoning

Examples of interaction axioms: correspondence

- $\text{incl}(\Box_i, \Box_j): R_j \subseteq R_i$
 - ▶ $\Diamond_j \varphi \rightarrow \Diamond_i \varphi$
- $\text{conv}(\Box_i, \Box_j): (R_i)^{-1} \subseteq R_j$ $\text{id}_W \subseteq R_i \circ R_j$
- $\text{confl}(\Box_i, \Box_j): (R_i)^{-1} \circ R_j \subseteq R_j \circ (R_i)^{-1}$
- $\text{perm}(\Box_i, \Box_j): R_j \circ R_i \subseteq R_i \circ R_j$
 - ▶ $\Diamond_j \Diamond_i \varphi \rightarrow \Diamond_i \Diamond_j \varphi$

Correspondence Theorem.

Let X_k be among the axioms $\text{Incl}(\Box_i, \Box_j)$, $\text{Perm}(\Box_i, \Box_j)$, $\text{Confl}(\Box_i, \Box_j)$, and let x_k be the associated condition among $\text{incl}(\Box_i, \Box_j)$, $\text{perm}(\Box_i, \Box_j)$, $\text{confl}(\Box_i, \Box_j)$.

A multimodal K -frame $\langle W, R \rangle$ satisfies condition (x_k) on R iff every instance of (X_k) is valid in $\langle W, R \rangle$.

will follow from more general result (... that won't be proved)

The axioms of generalized confluence

$$G(\alpha, \beta, \gamma, \delta) = \Diamond_{\alpha} \Box_{\beta} \varphi \rightarrow \Box_{\gamma} \Diamond_{\delta} \varphi$$

- generalization of axiom schemas $G(k, l, m, n)$ to multimodal logics
- inductive definition of expressions α :

$$\alpha ::= a \mid \text{skip} \mid (\alpha; \alpha) \mid (\alpha \cup \alpha)$$

where a ranges over set of parameters $Prms$

- possible intuition:
 - ▶ a = abstract atomic program, atomic action
 - ▶ skip = empty program
 - ▶ $\alpha; \alpha$ = sequential composition
 - ▶ $\alpha \cup \alpha$ = nondeterministic composition

cf. dynamic logic (Chapter 10)

The axioms of generalized confluence

$$G(\alpha, \beta, \gamma, \delta) = \diamond_{\alpha} \square_{\beta} \varphi \rightarrow \square_{\gamma} \diamond_{\delta} \varphi$$

- define as abbreviations:

- ▶ $\square_{\text{skip}} \varphi \stackrel{\text{def}}{=} \varphi$
- ▶ $\square_{\alpha; \beta} \varphi \stackrel{\text{def}}{=} \square_{\alpha} \square_{\beta} \varphi$
- ▶ $\square_{\alpha \cup \beta} \varphi \stackrel{\text{def}}{=} \square_{\alpha} \varphi \wedge \square_{\beta} \varphi$

- examples:

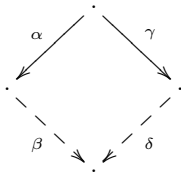
- ▶ $\text{Incl}(\square_i, \square_j) = \dots$
- ▶ $\text{Perm}(\square_i, \square_j) = \dots$
- ▶ $\text{Confl}(\square_i, \square_j) = \dots$

$G(\alpha, \beta, \gamma, \delta)$ logics: semantics

- frame property corresponding to axiom $G(\alpha, \beta, \gamma, \delta)$:

$$g(\alpha, \beta, \gamma, \delta) = R_\alpha^{-1} \circ R_\gamma \subseteq R_\beta \circ R_\delta^{-1}$$

- ▶ $R_{\text{skip}} = \delta_W = \{\langle w, w \rangle : w \in W\}$
- ▶ $R_{\alpha;\beta} = R_\alpha \circ R_\beta$
- ▶ $R_{\alpha \cup \beta} = R_\alpha \cup R_\beta$



$G(\alpha, \beta, \gamma, \delta)$ logics: completeness

- logics KX_1, \dots, X_n , where $(X_1), \dots, (X_n)$ are axioms of the form $G(\alpha, \beta, \gamma, \delta)$
 - ▶ $\vdash_{KX_1, \dots, X_n} \varphi =$ “ φ provable from axioms of K and $(X_1), \dots, (X_n)$ ”
- frame property corresponding to (X_j) : (x_j)
 - ▶ $\models_{KX_1, \dots, X_n} \varphi =$ “ φ valid in the class of $(x_1), \dots, (x_n)$ -frames”

Theorem [Lemmon/Scott], [Sahlqvist], [Catach].

Let the (X_j) be axioms of the form $G(\alpha, \beta, \gamma, \delta)$.

Then $\vdash_{KX_1, \dots, X_n} \varphi$ iff $\models_{KX_1, \dots, X_n} \varphi$.

proof similar to that for $G(k, l, m, n)$ logics; not proved here

other mathematical properties:

- decidability: unknown for many $G(\alpha, \beta, \gamma, \delta)$ logics
- complexity of (decidable) $G(\alpha, \beta, \gamma, \delta)$ logics: often very high
 - ▶ multidimensional modallogics...

Multidimensional modal logics

- alias product logics
- any modal logic having permutation and confluence axioms for every couple of modal operators:
 - ▶ Perm(\Box_i, \Box_j): $\Box_i \Box_j \varphi \rightarrow \Box_j \Box_i \varphi$
 - ▶ Confl(\Box_i, \Box_j): $\Diamond_i \Box_j \varphi \rightarrow \Box_j \Diamond_i \varphi$
- alternative semantics: $M = \langle W, R, V \rangle$ where
 - ▶ $W = W_1 \times \dots \times W_n$
 - ★ notation: $\{\vec{w}, \vec{v}, \dots\}$
 - ★ $\vec{w}_i = \langle w_1, \dots, w_n \rangle_{i = w_i}$ (ith element of \vec{w})
 - ▶ $R_i(\vec{w}) = \{ \vec{v} : \vec{v}_i = \vec{w}_i \}$
- logics: $K^2 = K \otimes K$, $K^3 = K \otimes K \otimes K$, ... $KT5^2$, ..., $K \otimes KT5$, ...
- N.B.: the two semantics do not always match!
 - ▶ every product model of $KT5(1) \otimes KT5(2) \otimes KT5(3)$ is a $KT5(1) + KT5(2) + KT5(3)$ model satisfying perm(\Box_i, \Box_j) and confl(\Box_i, \Box_j) for every i, j ;
 - ▶ ... but not the other way round!

Multidimensional modal logics, ctd.

- applications:
 - ▶ spatial modal logics
 - ★ $K^2, K^3, \dots, KT4^2, KT4^3, \dots, KT5^2, \dots$
 - ▶ logics of time and knowledge (or belief)
 - ★ $K \otimes KT5, KT4 \otimes SKT5, K \otimes KD45, KT4 \otimes SKD45, \dots$
 - ▶ logics of agency with independence of agents
 - ★ $KT5^2 = KT5 \otimes KT5, KT5^3, \dots$
- properties: sometimes decidable; and if so very complex
 - ▶ K^2 : decidable; complexity unknown
 - ▶ $KT5^2$: NEXPTIME
 - ▶ $KT5^3$: undecidable & not finitely axiomatizable
 - ▶ ...
 - ▶ technique to prove undecidability: encode tiling problems
- [Gabbay, Kurucz, Wolter and Zakharyashev 03]