Modal logics: overview

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Chapter 4. Completeness of $G(k, l, m, n)$ logics, and decidability of the basic modal logics

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Outline of chapter

- completeness of $G(k, l, m, n)$ logics
  - for each logic Λ, build a the canonical model M_Λ such that:
    - for every Λ-consistent φ there is w in M_Λ such that $M_\Lambda \Vdash \varphi$
  - difficulty: prove that M_Λ is a $G(k, l, m, n)$ model
- decidability of the basic modal logics built with D, T, B, 4, 5
  - idea: enumerate the set of Λ-models
    - can be done when models are finite
    - prove finite model property
  - tool: filtration method
  - difficulty: prove that the filtrated model is a Λ-model

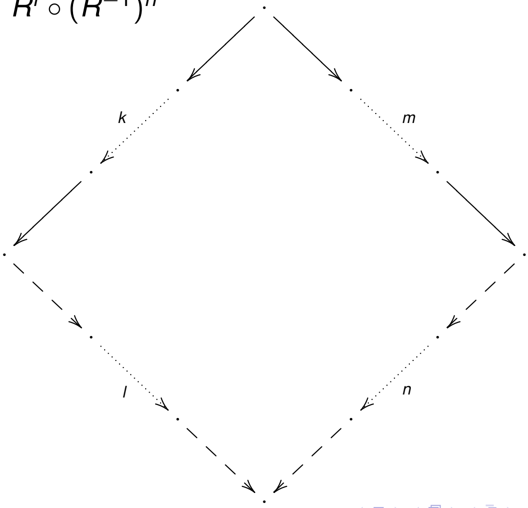
Introduction: soundness and completeness

$$\vdash_\Lambda \varphi \overset{?}{\Leftrightarrow} \models_\Lambda \varphi$$

- $\vdash_\Lambda \varphi \overset{?}{\Rightarrow} \models_\Lambda \varphi$ (soundness)
  - standard proof:
    1. every instance of every axiom schema of Λ is Λ-valid
    2. every inference rule of Λ preserves Λ-validity
- $\vdash_\Lambda \varphi \overset{?}{\Leftarrow} \models_\Lambda \varphi$ (completeness)
  - equivalently: $\not\vdash_\Lambda \varphi \;\Rightarrow\; \not\models_\Lambda \varphi$
    - actually proved: $\not\vdash_\Lambda \neg\varphi \;\Rightarrow\; \not\models_\Lambda \neg\varphi$ (‘φ consistent $\Rightarrow$ φ satisfiable’)
  - proof is ‘logic-by-logic’, and requires some creativity
  - fairly general method: via canonical models [Henkin]
    proof uses infinite model $\Rightarrow$ not constructive
  - here: general proof for $G(k, l, m, n)$ logics

The axioms $G(k, l, m, n)$

- axioms $\mathrm{G}(k, l, m, n) = \Diamond^k \Box^l \varphi \rightarrow \Box^m \Diamond^n \varphi$
- $\mathrm{g}(k, l, m, n) =$
  $(R^{-1})^k \circ R^m \subseteq R^l \circ (R^{-1})^n$



$G(k, l, m, n)$ logics: completeness

- logics $KX_1, \ldots, X_n$, where the (X_i) are axioms of the form G(k, l, m, n)
- frame property corresponding to (X_i): (x_i)

Theorem [Lemmon/Scott], [Sahlqvist].
Let the (X_i) be axioms of the form G(k, l, m, n).
Then $\vdash_{KX_1,\ldots,X_n} \varphi$ iff $\models_{KX_1,\ldots,X_n} \varphi$.

Idea of proof (L. Henkin):

- build a *canonical model* $M_{KX_1,\ldots,X_n}$ for $KX_1, \ldots, X_n$, such that for every consistent φ there is w in $M_{KX_1,\ldots,X_n}$ such that $M_{KX_1,\ldots,X_n}, w \Vdash \varphi$
- worlds of $M_{KX_1,\ldots,X_n}$ = maximal consistent sets of formulas. . .

Maximal consistent sets

let Λ be any logic

- a set of formulas Γ is (locally) Λ-*consistent* iff $\Gamma \not\vdash_{\Lambda}^{loc} \bot$
- Γ is Λ-*maximal consistent* ('maxcons') iff
  - Γ is Λ-consistent
  - $\Gamma \cup \{\varphi\}$ is Λ-inconsistent for every $\varphi \notin \Gamma$

Properties of Λ-maximal consistent sets Γ:

1. $\Gamma \vdash_{\Lambda} \varphi$ iff $\varphi \in \Gamma$ (suppose $\varphi \notin \Gamma$; then $\Gamma \cup \varphi$ is consistent. . .)
2. if $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$ (by 1. . . .)
3. for all φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ (suppose $\varphi \notin \Gamma$ and $\neg\varphi \notin \Gamma$; by 1. . . . ; as Γ is Λ maxcons . . .)
4. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$ (suppose $\varphi \notin \Gamma$ and $\neg\varphi \notin \Gamma$; by 3. . . .)

Lindenbaum Lemma

If Γ_0 is Λ-consistent then there is a Λ-maxcons Γ containing Γ_0

Proof:

1. enumerate the formulas of the language: $\langle \varphi_1, \varphi_2, \varphi_3, \ldots \rangle$;
2. starting with Γ_0 define inductively:

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \varphi_{i+1} & \text{if } \Gamma_i \cup \varphi_{i+1} \text{ is } \Lambda\text{-consistent} \\ \Gamma_i & \text{else} \end{cases}$$

The set $\bigcup_i \Gamma_i$ is Λ-consistent (because ...), and Λ-maxcons: suppose $\varphi \notin \bigcup_i \Gamma_i$; then $\varphi = \varphi_i$ for some i; ...

Property of any set Γ_0.

5. If $\Gamma \vdash_\Lambda \varphi$ for every Λ-maxcons Γ such that $\Gamma_0 \subseteq \Gamma$, then $\Gamma_0 \vdash_\Lambda \varphi$.

Proof: suppose $\Gamma_0 \not\vdash_\Lambda \varphi$; then $\Gamma_0 \cup \neg\varphi$ is Λ-consistent; by the Lindenbaum Lemma ...

Canonical model for Λ

canonical model = **unique** model $M_\Lambda = \langle W, R, V \rangle$ such that:

- W = the set of all Λ-maxcons sets of formulas
- $V(p) = \{w \in W \ : \ p \in w\}$
- $R(w) = \{v \ : \ v$ is Λ-maxcons and $w^{-\Box} \subseteq v\}$

Notation:

$$\begin{array}{rcl} w^{-\Box} & = & \{\varphi \ : \ \Box\varphi \in w\} \\ v^{+\Diamond} & = & \{\Diamond\varphi \ : \ \varphi \in v\} \end{array}$$

Property. wRv iff $w^{-\Box} \subseteq v$ iff $v^{+\Diamond} \subseteq w$

(suppose $w^{-\Box} \subseteq v$, and there is $\varphi \in v$ such that $\Diamond\varphi \notin w; \ldots$)

remains to be proved:

- M_Λ is a Λ-model
  - has to be checked case by case
- for every Λ-consistent φ there is w in M_Λ such that $M_\Lambda, w \Vdash \varphi$
  - idea: prove that $\varphi \in w$ iff $M_\Lambda, w \Vdash \varphi$

Fundamental Lemma

$$\varphi \in w \text{ iff } M_\Lambda, w \Vdash \varphi$$

Proof: by induction on structure of φ

- $\varphi \in Atms$: holds by definition of M_Λ
- $\varphi = \neg\psi$: ...
- $\varphi = \psi \wedge \chi$: ...
- $\varphi = \Box\psi$:

  $\Box\psi \in w \Rightarrow$ for every $v \in R(w)$, $\psi \in v$
  $\Rightarrow$ for every $v \in R(w)$, $M_\Lambda, v \Vdash \psi$ (by I.H.)
  $\Rightarrow$ $M_\Lambda, w \Vdash \Box\psi$

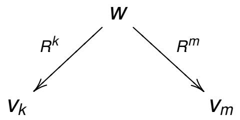
  $M_\Lambda, w \Vdash \Box\psi$
  $\Rightarrow$ for every $v \in R(w)$, $M_\Lambda, v \Vdash \psi$
  $\Rightarrow$ for every $v \in R(w)$, $\psi \in v$ (by I.H.)
  $\Rightarrow$ ψ in every Λ-maxcons extension of $w^{-\Box}$
  $\Rightarrow$ $w^{-\Box} \vdash_\Lambda \psi$ (Property **5.**)
  $\Rightarrow$ $w \vdash_\Lambda \Box\psi$ (apply RM(□))
  $\Rightarrow$ $\Box\psi \in w$ (Property **1.**)

Structural Lemma for $G(k, l, m, n)$ logics

If logic Λ is $KX_1 \dots X_j$ and every (X_i) is $G(k, l, m, n)$
then M_Λ is a $KX_1 \dots X_j$-model.

Proof:
Consider axiom $(X_i) = \Diamond^k \Box^l \varphi \to \Box^m \Diamond^n \varphi$ for some k, l, m, n.
to be shown: M_Λ satisfies constraint $(x_i) = (R^{-1})^k \circ R^m \subseteq R^l \circ (R^{-1})^n$

- suppose $wR^k v_k$ and $wR^m v_m$



- it suffices to show that $v_k^{-\Box^l} \cup v_m^{-\Box^n}$ is Λ-consistent $(v_k^{-\Box^l} \stackrel{\text{def}}{=} \{\varphi : \Box^l \varphi \in v_k\})$
  - $\Rightarrow$ there exists some Λ-maxcons u s.th. $v_k^{-\Box^l} \cup v_m^{-\Box^n} \subseteq u$ (by Prop.1 of maxcons sets)
  - $\Rightarrow$ $v_k R u$ because $v_k^{-\Box^l} \subseteq u$
  - $\Rightarrow$ $v_m R u$ because $v_m^{-\Box^n} \subseteq u$

Structural Lemma for $G(k, l, m, n)$ logics, ctd.

- so the only thing that remains to be shown is that $v_k^{-\Box^l} \cup v_m^{-\Box^n}$ is Λ-consistent
  - suppose $v_k^{-\Box^l} \cup v_m^{-\Box^n}$ is Λ-inconsistent

$\Rightarrow$ $v_k^{-\Box^l} \cup v_m^{-\Box^n} \vdash_\Lambda \bot$

$\Rightarrow$ $\exists$ finite sets (those used in the deductions) $\{\varphi_{k_1}, \ldots, \varphi_{k_i}\} \subseteq v_k^{-\Box^l}$ and $\{\varphi_{m_1}, \ldots, \varphi_{m_j}\} \subseteq v_m^{-\Box^n}$ (the formulas used in the deductions) s.th.

$$\{\varphi_{k_1}, \ldots, \varphi_{k_i}\} \cup \{\varphi_{m_1}, \ldots, \varphi_{m_j}\} \vdash_\Lambda \bot$$

$\Rightarrow$ $\{\varphi_{k_1}, \ldots, \varphi_{k_i}\} \vdash_\Lambda \neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j})$ (deduction theorem: remember $\vdash_\Lambda$ is local deduction)

$\Rightarrow$ $\{\Box^l\varphi_{k_1}, \ldots, \Box^l\varphi_{k_i}\} \vdash_\Lambda \Box^l\neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j})$ (apply RM($\Box$), k_i times)

$\Rightarrow$ $v_k \vdash_\Lambda \Box^l\neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j})$

$\Rightarrow$ $\Box^l\neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j}) \in v_k$ (by Fundamental Lemma)

$\Rightarrow$ $\Diamond^k\Box^l\neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j}) \in w$ (because $v_k^{+\Diamond^k} \subseteq w$)

$\Rightarrow$ $\Box^m\Diamond^n\neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j}) \in w$ (by Prop.2 of maxcons sets, because $\Diamond^k\Box^l\varphi \to \Box^m\Diamond^n\varphi \in w$)

$\Rightarrow$ $\Diamond^n\neg(\varphi_{m_1} \wedge \ldots \wedge \varphi_{m_j}) \in v_m$ (because $w^{-\Box^m} \subseteq v_m$)

$\Rightarrow$ but $\{\Box^n\varphi_{m_1}, \ldots, \Box^n\varphi_{m_j}\} \subseteq v_m$!

$\Rightarrow$ v_m is inconsistent: contradiction

Completeness theorem for $G(k, l, m, n)$ logics

If logic Λ is $KX_1 \dots X_j$ and every (X_i) is $G(k, l, m, n)$ and φ is Λ-consistent then φ is Λ-satisfiable

Proof:
suppose φ is Λ-consistent

- $\Rightarrow$ there is a Λ-maxcons set Γ such that $\varphi \in \Gamma$ (by Lindenbaum Lemma)
- $\Rightarrow$ Γ is a possible world in the canonical model M_Λ for Λ (by construction of M_Λ)
- $\Rightarrow$ $M_\Lambda, \Gamma \Vdash \varphi$ (by Fundamental Lemma)
- $\Rightarrow$ φ is Λ-satisfiable (because M_Λ is a $G(k, l, m, n)$-model of Λ by the Structural Lemma)

Introduction: decidability

"is there a *decision procedure* for satisfiability in Λ-models?"

- decision procedure: stops for every input φ, and
  - output = "no" if $\models_\Lambda \neg\varphi$
  - output = "yes" if $\not\models_\Lambda \neg\varphi$
- finite axiomatizations provide **negative satisfiability test**
  - procedure: enumerate all theorems

    $$\langle \varphi_1, \varphi_2, \varphi_3, \ldots \rangle$$

    - if $\varphi_i = \neg\varphi$ then $\vdash_\Lambda \neg\varphi$: output = "no" and stop
    - else $i := i+1$
  - loops if $\not\vdash_\Lambda \neg\varphi$
- missing: **positive satisfiability test**
  - procedure: recursively enumerate all models (???) and check whether M_i is a Λ-model of φ
    - would work if suffices to only check *finite models* ...

Decidability via finite model property (fmp)

finite model property (fmp):

if there is a Λ-model for φ then there is a *finite* Λ-model for φ

- if Λ has fmp then decision procedure for Λ:
  1. enumerate in parallel
     - ★ the set of all Λ-theorems: $\langle \varphi_1, \varphi_2, \varphi_3, \ldots \rangle$
     - ★ the set of all finite Λ-models: $\langle M_1, M_2, M_3, \ldots \rangle$
  2. if $\varphi_i = \neg\varphi$ then $\vdash_\Lambda \neg\varphi$ then output = "*unsatisfiable!*" and stop
  3. if M_i is a Λ-model (check axiom schema instances and rules)
     and there is a w in M s.th. $M, w \Vdash \varphi$
     then output = "*satisfiable!*" and stop
  4. else $i := i{+}1$
- procedure stops at some i (because Λ has fmp)
- here: proof of fmp for K, KT, $K4$ by the *filtration method*
  - ▸ does not work for all $G(k, l, m, n)$ logics
  - ▸ many simple modal logics are undecidable
  - ▸ for many simple modal logics decidability still unknown

fmp via the filtration method

- suppose $M, w \not\Vdash \varphi$
- problem: M may have infinite W
- idea: identify worlds where φ and all subformulas of φ have the same truth value
  - $w \approx w'$ iff for every $\psi \in sf(\varphi)$: ($M, w \Vdash \psi$ iff $M, w \Vdash \psi$)
    - $\approx$ is an equivalence relation
    - equivalence class of $w \stackrel{\text{def}}{=} \{w' : w' \approx w\} = |w|_\approx$
    - the number of equivalence classes in $\approx$ is finite (proof: if $|w|_\approx \neq |w'|_\approx$ then there is $\psi \in sf(\varphi)$ s.th. . .)
  - reduce W by $\approx$:
    - $W^\approx \stackrel{\text{def}}{=} \{|w|_\approx : w \in W\}$
  - valuation:
    - $V^\approx(p) \stackrel{\text{def}}{=} \begin{cases} \emptyset & \text{if } p \notin sf(\varphi) \\ \{|w|_\approx : w \in V(p)\} & \text{if } p \in sf(\varphi) \end{cases}$
  - relation $R^\approx \stackrel{\text{def}}{=} \ldots$

fmp via the filtration method, ctd.

- several possibilities for $R^{\approx}$
  - minimal conditions for a filtration:
    1. if wRv then $|w|_{\approx} R^{\approx} |v|_{\approx}$
    2. if $|w|_{\approx} R^{\approx} |v|_{\approx}$ then for all $\Box\psi \in sf(\varphi)$, if $M, w \Vdash \Box\psi$ then $M, v \Vdash \psi$

       (weaker than bisimulation)
  - look for a definition satisfying semantic constraints of logic Λ ...
- $M^{\approx} = \langle W^{\approx}, R^{\approx}, V^{\approx} \rangle$ = "*filtration* of M by $sf(\varphi)$"
  - a K-model
  - $W^{\approx}$ finite

Filtration Theorem.
Let M be a filtration of M by $sf(\varphi)$. Then for every $\psi \in sf(\varphi)$ and every w in M: $M, w \Vdash \psi$ iff $M^{\approx}, |w|_{\approx} \Vdash \psi$.

Proof: induction on structure of ψ

- $M, v \Vdash \Box\psi_1 \;\Rightarrow\; M^{\approx}, |v|_{\approx} \Vdash \Box\psi_1$: use condition (2.)
- $M^{\approx}, |v|_{\approx} \Vdash \Box\psi_1 \;\Rightarrow\; M, v \Vdash \Box\psi_1$: use condition (1.)

Filtration Theorem

Let M be a filtration of M by $sf(\varphi)$.
Then for every $\psi \in sf(\varphi)$ and every w in M:
$M, w \Vdash \psi$ iff $M^{\approx}, |w|_{\approx} \Vdash \psi$.

Proof: induction on structure of ψ

- case $\psi = p$: ...
- case $\psi = \neg\psi_1$: ...
- case $\psi = \psi_1 \wedge \psi_2$: ...
- case $\psi = \Box\psi_1$:

$$\begin{array}{llll}
M, v \Vdash \Box\psi_1 & \Rightarrow & M, u \Vdash \psi_1 \text{ for all } u \in R^{\approx}(|v|_{\approx}) & \text{(by 2.)} \\
 & \Leftrightarrow & M^{\approx}, |u|_{\approx} \Vdash \psi_1 \text{ for all } u \in R^{\approx}(|v|_{\approx}) & \text{(by I.H.)} \\
 & \Leftrightarrow & M^{\approx}, |v|_{\approx} \Vdash \Box\psi_1 &
\end{array}$$

$$\begin{array}{llll}
M^{\approx}, |v|_{\approx} \Vdash \Box\psi_1 & \Leftrightarrow & M^{\approx}, |u|_{\approx} \Vdash \psi_1 \text{ for all } u \in R^{\approx}(|v|_{\approx}) & \\
 & \Leftrightarrow & M, u \Vdash \psi_1 \text{ for all } u \in R^{\approx}(|v|_{\approx}) & \text{(by I.H.)} \\
 & \Rightarrow & M, u \Vdash \psi_1 \text{ for all } u \in R^{\approx}(|v|_{\approx}) & \text{(by 1.)} \\
 & \Leftrightarrow & M, v \Vdash \Box\psi_1 &
\end{array}$$

Decidability using filtration

- remains to be proved: $M^{\approx}$ is still a Λ-model
  - conditions on the definition of $R^{\approx}$ do not guarantee this!
- logic-by-logic:
  - $\Lambda = K$: every $M^{\approx}$ is a K-model
  - $\Lambda = KT$: if M is reflexive then every $M^{\approx}$ is reflexive (by condition 1)
  - $\Lambda = K4$: if M is transitive then not every $M^{\approx}$ is transitive!
    $\Rightarrow$ modify 2nd condition on $R^{\approx}$:
    2.′ if $|w|_{\approx} R |v|_{\approx}$ then for all $\Box\psi \in sf(\varphi)$, if $M, w \Vdash \Box\psi$ then $M, v \Vdash \psi \wedge \Box\psi$
  - ...
- most general result:
  - $\Lambda = KX_1 \ldots X_k$: when all X_i are in $\{D, T, B, 4, 5\}$ then definition of $R^{\approx}$ can be 'arranged' such that $M^{\approx}$ is a $KX_1 \ldots X_k$-model
- ...but filtration fails for some $G(k, l, m, n)$ logics:
  - $\Lambda = KD4.De$ has fmp, but filtration fails (axiom De($\Box$): $\Diamond\varphi \to \Diamond\Diamond\varphi$)

Decidability theorem for the basic modal logics

Let logic $KX_1 \dots X_n$ be such that all X_k are in $\{D, T, B, 4, 5\}$.
Then $KX_1 \dots X_n$ has the fmp, and $KX_1 \dots X_n$-satisfiability is decidable.

Sketch of proof:

- If φ is $KX_1 \dots X_n$-unsatisfiable then there is a $KX_1 \dots X_n$-proof of $\neg\varphi$.
- If φ is $KX_1 \dots X_n$-satisfiable then by the Filtration Theorem there must be a finite K-model of φ;
  - arrange the filtration such that the result of the filtration is a $KX_1 \dots X_n$-model.

    (to be done logic-by-logic)
  - Enumerate in parallel:
    - the set of all $KX_1 \dots X_n$-theorems
    - the set of all finite K-models
  - After a finite number of steps ...

$\Rightarrow$ impractical $\Rightarrow$ find efficient algorithms (matching theoretical complexity)