Modal logics: overview

Chapter 3.
The basic monomodal logics

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Language

- set of propositional atoms *Atms*
- set of parameters $\mathit{Prms} = \{1\}$
- BNF:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_1 \varphi$$

where p ranges over *Atms*

- convention: $\Box\varphi = \Box_1\varphi$

Monomodal axioms

- the basic monomodal axiom schemas [Chellas 80]:
  - $\Box\varphi \to \varphi$ T($\Box$)
  - $\Box\varphi \to \Diamond\varphi$ D($\Box$)
  - $\Box\varphi \to \Box\Box\varphi$ 4($\Box$)
  - $\Diamond\varphi \to \Box\Diamond\varphi$ 5($\Box$)
  - $\varphi \to \Box\Diamond\varphi$ B($\Box$)

Axiom T($\Box$): $\Box\varphi \to \varphi$

alternatively: $\varphi \to \Diamond\varphi$

applications:

- "what is known is true"
- "if *henceforth* φ then φ is true now" (temporal reasoning)
- "if φ is true *everywhere* then φ is true here" (spatial reasoning)
- "if φ is true *everywhere else* then φ is true here" (???)
- "if I *believe* that φ then φ is true" (???)
- "if φ is *obligatory* then φ is true" (???)

Axiom D($\Box$): $\Box\varphi \to \Diamond\varphi$

CPL-equivalent to $\neg(\Box\varphi \land \Box\neg\varphi)$

applications:

- "what is obligatory is permitted"
- "knowledge is consistent"
- "belief is consistent"
- "if henceforth φ then eventually φ"
- "if everywhere φ then somewhere φ"
- "if everywhere else φ then somewhere else φ"
- "if φ is true after every execution of program π, then φ is true after some execution of program π" (???)

Axiom 4($\Box$): $\Box\varphi \to \Box\Box\varphi$

alternatively: $\Diamond\Diamond\varphi \to \Diamond\varphi$

applications:

- "if I know that φ, then I know that I know that φ" ('positive introspection')
- "if I believe that φ, then I believe that I believe that φ" ('positive introspection')
- "if φ is true after π, then φ is true after executing π twice" (???)
- "if everywhere φ then everywhere it is the case that everywhere φ"

Axiom 5($\Box$): $\Diamond\varphi \to \Box\Diamond\varphi$

alternatively: $\Diamond\Box\varphi \to \Box\varphi$, or $\neg\Box\varphi \to \Box\neg\Box\varphi$

applications:

- "if I don't believe that φ, then I believe that I don't believe that φ" ('negative introspection')
- "I know what I don't know" (?, v.i.)
- "what is permitted is obligatorily so" (???)
- "if eventually φ, then henceforth eventually φ" (???)
- "if somewhere φ then everywhere it is the case that somewhere φ"

Axiom B($\Box$): $\varphi \to \Box\Diamond\varphi$

alternatively: $\Diamond\Box\varphi \to \varphi$

applications:

- rather in its multimodal version $\varphi \to \Box_1\Diamond_2\varphi$:
  "if φ then henceforth ($\Box_1$) there is a point in the past ($\Diamond_2$) such that φ"

(axiomatizes converse relationship)

Proofs and theorems

- logics $KX_1, \ldots, X_n$, where (X_1), ..., (X_n) are basic modal axiom schemas
  - inference rules for $KX_1, \ldots, X_n$: rules (MP) and RM($\Box$)
  - axiom schemas for $KX_1 \ldots X_n$:
    - axioms C($\Box$), N($\Box$) of K, plus (X_1), ..., (X_n)
- definitions as usual:
  - proof of φ in logic $KX_1, \ldots, X_n$
  - $KX_1, \ldots, X_n$-theorem
  - $KX_1, \ldots, X_n$-consistency
- *Notation:*
  $\vdash_{KX_1 \ldots X_n} \varphi$ iff φ is a theorem of logic $KX_1 \ldots X_n$
  - two logics have historic names [Lewis&Langford 32]:
    - $S4 = KT4$
    - $S5 = KT45 = KT5$
  - sometimes used in the literature: $T = KT$, $D = KD$, $B = KB$

Axiomatics: examples of theorems

- $\vdash_{KT} \Box\Box\varphi \to \varphi$
  proof:
  1. $\Box\Box\varphi \to \Box\varphi$ — axiom T($\Box$)
  2. $\Box\varphi \to \varphi$ — axiom T($\Box$)
  3. $\Box\Box\varphi \to \varphi$ — from 1. and 2. by *CPL*

- $\vdash_{KT} \varphi \to \Diamond\varphi$
  proof:
  1. $\Box\neg\varphi \to \neg\varphi$ — axiom T($\Box$)
  2. $\varphi \to \neg\Box\neg\varphi$ — from 1. by *CPL*

Axiomatics: examples of theorems

- $\vdash_{KT} \Box\varphi \to \Diamond\varphi$
  proof:
  1. $\Box\varphi \to \varphi$ — axiom T($\Box$)
  2. $\Box\neg\varphi \to \neg\varphi$ — axiom T($\Box$)
  3. $\varphi \to \Diamond\varphi$ — from 2. by *CPL*
  4. $\Box\varphi \to \Diamond\varphi$ — from 1. and 3. by *CPL*

  Hence *KDT* = *KT*.

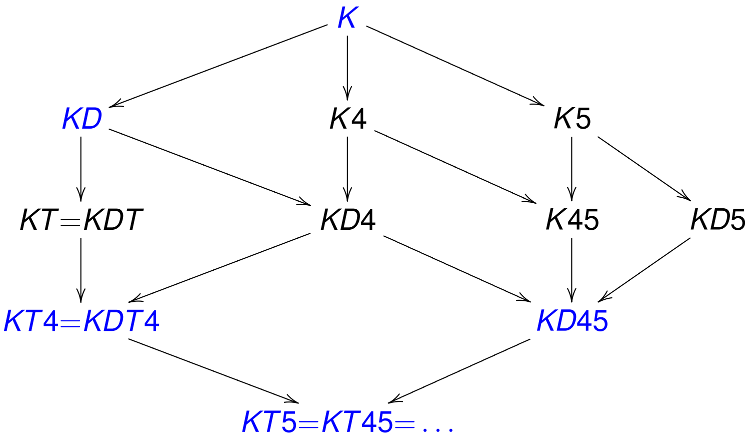
Axiomatics: examples of theorems, ctd.

- $\vdash_{KT5} \Box\varphi \to \Box\Box\varphi$
  proof:
  1. $\Diamond\Diamond\varphi \to \Box\Diamond\Diamond\varphi$ — axiom 5($\Box$)
  2. $\Diamond\varphi \to \Box\Diamond\varphi$ — axiom 5($\Box$)
  3. $\Diamond\Diamond\varphi \to \Diamond\Box\Diamond\varphi$ — from 2. by RM($\Diamond$)
  4. $\Box\Diamond\Diamond\varphi \to \Box\Diamond\Box\Diamond\varphi$ — from 3. by RM($\Box$)
  5. $\Diamond\Box\Diamond\varphi \to \Box\Diamond\varphi$ — axiom 5($\Box$)
  6. $\Box\Diamond\Box\Diamond\varphi \to \Box\Box\Diamond\varphi$ — from 5. by RM($\Box$)
  7. $\Box\Box\Diamond\varphi \to \Box\Diamond\varphi$ — axiom T($\Box$)
  8. $\Box\Diamond\varphi \to \Diamond\varphi$ — axiom T($\Box$)
  9. $\Diamond\Diamond\varphi \to \Diamond\varphi$ — from 1., 4., 6., 7., 8. by *CPL*
  10. $\Box\varphi \to \Box\Box\varphi$ — from 9. by *CPL*

  Hence $KT45 = KT5$ (you won't get such proofs in the exams...)

The lattice of basic modal logics

(all combinations of D, T, 4, 5)



Axiomatics: important *KD*45 theorems

In *KD*45, sequences of modal operators can be reduced to length 1:

- $\vdash_{KD45} \Box\Box\varphi \leftrightarrow \Box\varphi$
  proof:
  1. $(\Box\Box\varphi \land \Box\bot) \to \Box\varphi$ — theorem of K
  2. $(\Box\Box\varphi \land \neg\Box\bot) \to \Diamond\Box\varphi$ — theorem of K
  3. …
- $\vdash_{KD45} \Box\Diamond\varphi \leftrightarrow \Diamond\varphi$
  proof:
  1. $\Box\Diamond\varphi \to \Diamond\Diamond\varphi$ — (D)
  2. $\Diamond\Diamond\varphi \to \Diamond\varphi$ — (4)
  3. …
- $\vdash_{KD45} \Diamond\Box\varphi \leftrightarrow \Box\varphi$
- $\vdash_{KD45} \Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$

N.B.: a fortiori also holds for *KT*5 = *S*5 (why?)

Axiomatics: important $K45$ theorems

- $\vdash_{K45} \Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow (\Diamond\varphi \wedge \Diamond\psi)$
  proof:
  1. $\Diamond(\varphi \wedge \Diamond\psi) \rightarrow (\Diamond\varphi \wedge \Diamond\Diamond\psi)$ — theorem of *K*
  2. $\Diamond\Diamond\psi \rightarrow \Diamond\psi$ — (4)
  3. $\Diamond\psi \rightarrow \Box\Diamond\psi$ — (5)
  4. $(\Diamond\varphi \wedge \Diamond\psi) \rightarrow (\Diamond\varphi \wedge \Box\Diamond\psi)$ — from 3. by *CPL*
  5. $(\Diamond\varphi \wedge \Box\Diamond\psi) \rightarrow \Diamond(\varphi \wedge \Diamond\psi)$ — theorem of *K*
  6. $\Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow (\Diamond\varphi \wedge \Diamond\psi)$ — from 1.,2.,4.,5. by *CPL*
- $\vdash_{K45} \Diamond(\varphi \wedge \Box\psi) \leftrightarrow (\Diamond\varphi \wedge \Box\psi)$
  proof: homework
- $\vdash_{K45} \Box(\varphi \vee \Box\psi) \leftrightarrow (\Box\varphi \vee \Box\psi)$
  proof: … (use that $\vdash_{K45} \Diamond(\varphi \wedge \Diamond\psi) \leftrightarrow (\Diamond\varphi \wedge \Diamond\psi)$)
- $\vdash_{K45} \Box(\varphi \vee \Diamond\psi) \leftrightarrow (\Box\varphi \vee \Diamond\psi)$
  proof: …

Remark. Differently from *KD*45, $\not\vdash_{K45} \Box\Diamond\varphi \leftrightarrow \Diamond\varphi$

but: $\vdash_{K45} \Box\Diamond\varphi \leftrightarrow (\Box\bot \vee \Diamond\varphi)$, because $\vdash_{K45} \Box\Diamond\varphi \leftrightarrow \Box(\bot \vee \varphi)$

Normal form for $K45$ (and $KD45$ and $KT5 = S5$)

- classical clause $C = L_1 \vee \ldots \vee L_k$,
  - L_i = literals: $L_i = p$ or $L_i = \neg p$, for some $p \in Atms$
- classical CNF $K = C_1 \wedge \ldots \wedge C_m$, where C_j is a classical clause
- $K45$-clause = $C_0 \vee \Box C_1 \vee \ldots \vee \Box C_n \vee \Diamond K$ where $C_0, C_1, \ldots$ are classical clauses and K is a classical CNF

Normal Form Theorem for $K45$: For every φ there are $K45$-clauses $C_1, \ldots, C_n$ such that $\vdash_{K45} \varphi \leftrightarrow (C_1 \wedge \ldots \wedge C_n)$.

Proof: rewrite φ applying logical equivalences:

$\neg\neg\varphi$	$\leftrightarrow$	φ	*CPL*	$\Box(\varphi \vee \Box\psi)$	$\leftrightarrow$	$\Box\varphi \vee \Box\psi$	*K45*
$\neg(\varphi \wedge \psi)$	$\leftrightarrow$	$\neg\varphi \vee \neg\psi$	*CPL*	$\Box(\varphi \vee \Diamond\psi)$	$\leftrightarrow$	$\Box\varphi \vee \Diamond\psi$	*K45*
$\neg(\varphi \vee \psi)$	$\leftrightarrow$	$\neg\varphi \wedge \neg\psi$	*CPL*	$\Diamond(\varphi \wedge \Box\psi)$	$\leftrightarrow$	$\Diamond\varphi \wedge \Box\psi$	*K45*
$\varphi \vee (\psi \wedge \chi)$	$\leftrightarrow$	$(\varphi \vee \psi) \wedge (\varphi \vee \chi)$	*CPL*	$\Diamond(\varphi \wedge \Diamond\psi)$	$\leftrightarrow$	$\Diamond\varphi \wedge \Diamond\psi$	*K45*
$\Box(\varphi \wedge \psi)$	$\leftrightarrow$	$\Box\varphi \wedge \Box\psi$	*K*				
$\Diamond(\varphi \vee \psi)$	$\leftrightarrow$	$\Diamond\varphi \vee \Diamond\psi$	*K*				

to be proved: the max. number of nested modal operators decreases

Normal form for $K45$, examples

- example:
  $$\Diamond(p \wedge \Box\neg p) \quad \leftrightarrow \quad \Diamond p \wedge \Box\neg p$$
- example:
  $$\begin{aligned} \Box(\Box p \to p) \quad &\leftrightarrow \quad \Box(\neg\Box p \vee p) \\ &\leftrightarrow \quad \Box(\Diamond\neg p \vee p) \\ &\leftrightarrow \quad \Diamond\neg p \vee \Box p \end{aligned}$$
- example:
  $$\begin{aligned} \Box(p \wedge \neg\Box p) \quad &\leftrightarrow \quad \Box(p \wedge \Diamond\neg p) \\ &\leftrightarrow \quad \Box p \wedge \Box\Diamond\neg p \\ &\leftrightarrow \quad \Box p \wedge \Box(\bot \vee \Diamond\neg p) \\ &\leftrightarrow \quad \Box p \wedge (\Box\bot \vee \Diamond\neg p) \end{aligned}$$
  - is equivalent to $\Box\bot$
- example:
  $$\begin{aligned} \Box\Diamond(p \to \Diamond(q \wedge r)) \quad &\leftrightarrow \quad \Box\Diamond(\neg p \vee \Diamond(q \wedge r)) \\ &\leftrightarrow \quad \Box(\Diamond\neg p \vee \Diamond(q \wedge r)) \\ &\leftrightarrow \quad \Box\Diamond\neg p \vee \Diamond(q \wedge r) \\ &\leftrightarrow \quad \Box\bot \vee \Diamond\neg p \vee \Diamond(q \wedge r) \end{aligned}$$

Semantics: second-order models for *KT*

- φ is **valid in a set ('class') of K-models** $\mathcal{K}$ iff
  $$M \Vdash \varphi \text{ for every model } M \in \mathcal{K}$$
- a K-model M is a **second-order model of** KT iff
  $$M \Vdash \Box\varphi \to \varphi \text{ for \textit{every formula} } \varphi$$

Soundness and Completeness Theorem.
$\vdash_{KT} \varphi$ iff φ is valid in the class of second-order KT-models.

Proof: omitted

...but the KT-theorems are also valid in the class of **reflexive** K-frames ...

- if a K-model M is reflexive then $M \Vdash_{KT} \Box\varphi \to \varphi$ for all φ
  - the converse is wrong: $M \Vdash_{KT} \Box\varphi \to \varphi$ for all φ does not imply that M is reflexive
    - ⋆ proof: ...
  - so, in which sense does axiom T($\Box$) *characterize* reflexivity?

Correspondence Theorem for *KT*.
A ***K*-frame** $\langle W, R\rangle$ is reflexive iff $\langle W, R\rangle \Vdash_{KT} \Box\varphi \to \varphi$ for all φ.

Proof.

- from the left to the right: see above
- from the right to the left:
  let $\langle W, R\rangle$ be a non-reflexive K-frame, i.e. $w \notin R(w)$ for some w;
  let V be a valuation on $\langle W, R\rangle$ such that $V(p) = W \setminus \{w\}$;
  then $\langle W, R, V\rangle, w \Vdash \Box p$, but $\langle W, R, V\rangle, w \nVdash p$;
  hence $\langle W, R\rangle \nVdash \Box p \to p$

Semantics: correspondences for the basic modal logics

name	axiom	first-order frame condition
T($\Box$)	$\Box\varphi \rightarrow \varphi$	reflexivity of R
D($\Box$)	$\Box\varphi \rightarrow \Diamond\varphi$	seriality of R: $\forall w \exists w' R(w, w')$
4($\Box$)	$\Box\varphi \rightarrow \Box\Box\varphi$	transitivity of R
5($\Box$)	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$	Euclidianity of R:
		$\forall w \forall w_1 \forall w_2((R(w, w_1) \wedge R(w, w_2)) \rightarrow R(w_1, w_2))$
B($\Box$)	$\varphi \rightarrow \Box\Diamond\varphi$	symmetry of R

Correspondence Theorem.
Let (X_i) be a basic modal axiom schema, and let (x_i) be the associated first-order frame condition.
A K**-frame** $\langle W, R \rangle$ satisfies condition (x_i) on R iff every instance of (X_i) is valid in $\langle W, R \rangle$.

Remarks:

- more general algorithms: 2nd-order quantifier elimination [van Benthem 76, ..., Goranko, Vakarelov et col. 04], SQEMA algorithm

Excursion: second-order quantifier elimination

second-order condition for reflexivity:

$$\begin{aligned}\forall p \forall w((\Box p \to p)(w)) &= \forall p \forall w((\Box p)(w) \to p(w)) \\ &= \forall p \forall w(\forall v(\neg R(w,v) \lor p(v)) \to p(w))\end{aligned}$$

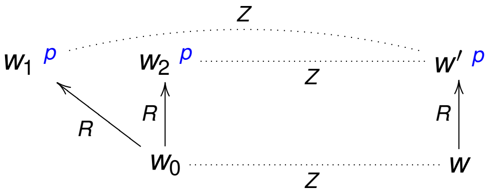
SCAN algorithm [Gabbay&Ohlbach 94]:

negate:	$\exists p \exists x(\forall y(\neg R(x,y) \lor p(y)) \land \neg p(x))$
skolemize:	$\forall y(\neg R(\mathbf{c},y) \lor p(y)) \land \neg p(\mathbf{c}))$
put in clausal form:	1. $\neg R(c,y) \lor p(y))$
	2. $\neg p(c)$
apply resolution:	3. $\neg R(c,c)$
purify (elim. p-clauses):	$\{\neg R(c,c)\}$
unskolemize:	$\exists x \neg R(x,x)$
negate:	$\forall x\{R(x,x)\}$

- …but does not work for every input axiom schema:
  - resolution may loop
  - unskolemization may fail

Semantics of K: bisimulations

- Let $M = \{W, R, V\}$ and $M' = \{W', R', V'\}$ be K-models. A relation $Z \subseteq W \times W'$ is a **bisimulation** between M and M', noted $Z : M \leftrightarrows M'$, iff for every $w \in W, w' \in W'$ such that wZw':
  - $w \in V(p)$ iff $w' \in V'(p)$ (atomic)
  - for every $u \in R(w)$ there is $u' \in R'(w')$ such that uZu' (forth)
  - for every $u' \in R'(w')$ there is $u \in R(w)$ such that uZu' (back)



Theorem.
Let $Z : M \leftrightarrows M'$. For all $w \in W$ and $w' \in W'$:
if wZw' then $M, w \Vdash \varphi$ iff $M', w' \Vdash \varphi$ for all formulas φ.

Proof: by induction on structure of φ (homework)

Semantics of K: irreflexivity is not modally definable

Theorem.
There is no axiom φ_{irr} such that $\langle W, R\rangle \Vdash \varphi_{irr}$ iff R is irreflexive.

Proof: Suppose such a φ_{irr} exists. Consider the two frames:

- $\langle W, R\rangle$ such that $W = \{w_0, w_1\}$, $R = \{\langle w_0, w_1\rangle, \langle w_1, w_0\rangle\}$, and
- $\langle W', R'\rangle$ such that $W' = \{w'\}$, $R' = \{\langle w', w'\rangle\}$.

We should have $\langle W, R\rangle \Vdash \varphi_{irr}$ and $\langle W', R'\rangle \not\Vdash \varphi_{irr}$.
$\langle W', R'\rangle \not\Vdash \varphi_{irr}$ implies that there are V', w' s.th. $\langle W', R', V'\rangle, w' \not\Vdash \varphi_{irr}$.
Define V such that the relation $Z = W \times W'$ is a bisimulation:
$V(p) = \{w_0, w_1\}$ if $V'(p) = \{w'\}$, and $V(p) = \emptyset$ else.
Z is a bisimulation indeed, and $w_i Z w'$, therefore:

$$\langle W, R, V\rangle, w_i \Vdash \varphi \text{ iff } \langle W', R', V'\rangle, w' \Vdash \varphi \text{ for every } \varphi.$$

Hence $\langle W, R\rangle \not\Vdash \varphi_{irr}$.

Soundness and completeness

Let (X_i) be a basic modal axiom schema, and let (x_i) be its corresponding first-order frame condition.

Notation:
$\models_{KX_1 \ldots X_n} \varphi$ = "φ valid in the class of K-frames satisfying $(x_1), \ldots, (x_n)$"

Soundness Theorem.
If $\vdash_{KX_1 \ldots X_n} \varphi$ then $\models_{KX_1 \ldots X_n} \varphi$.

Proof: by induction on the length of the proof of φ
Remark. more generally, correspondence *always* implies soundness

Completeness Theorem.
If $\models_{KX_1 \ldots X_n} \varphi$ then $\vdash_{KX_1 \ldots X_n} \varphi$.

Proof: will follow from a more general result, v.i.
Remark. open problem: when does correspondence imply completeness?

The axioms $G(k, l, m, n)$

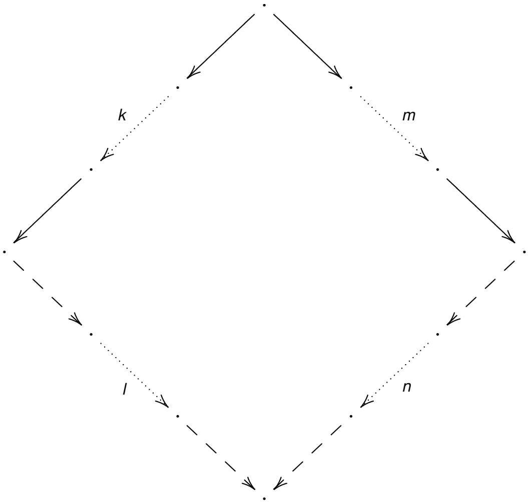
[Lemmon/Scott, Sahlqvist]

$$G(k, l, m, n) = \Diamond^k \Box^l \varphi \rightarrow \Box^m \Diamond^n \varphi$$

- examples:
  - axiom $T = \ldots$
  - $\ldots$
  - $G(1, 0, 1, 0) = \ldots$
  - $G(1, 0, 1, 1) = \ldots$
  - $G(1, 1, 1, 1) = \ldots$
  - $\ldots$
- corresponding frame property: generalized confluence
  $g(k, l, m, n) = (R^{-1})^k \circ R^m \subseteq R^l \circ (R^{-1})^n$

The logics of confluence $G(k, l, m, n)$: semantics

$\mathrm{g}(k, l, m, n) =$
$(R^{-1})^k \circ R^m \subseteq R^l \circ (R^{-1})^n$



The axioms of confluence $G(k, l, m, n)$: completeness

- logics $KX_1, \ldots, X_n$, where $(X_1), \ldots, (X_n)$ are axioms of the form $\mathrm{G}(k, l, m, n)$
  - $\vdash_{KX_1,\ldots,X_n} \varphi$ = "φ provable from axioms of K and $(X_1), \ldots, (X_n)$"
- frame property corresponding to (X_i): (x_i)
  - $\models_{KX_1,\ldots,X_n} \varphi$ = "φ valid in the class of $(x_1), \ldots, (x_n)$-frames"

Theorem [Lemmon/Scott], [Sahlqvist].
Let the (X_i) be axioms of the form $\mathrm{G}(k, l, m, n)$.
Then $\vdash_{KX_1,\ldots,X_n} \varphi$ iff $\models_{KX_1,\ldots,X_n} \varphi$.

Proof: see Chapter 5