Modal logics: overview

Chapter 2. The basic multimodal logic K

Language

- primitive symbols:
  - set of propositional atoms *Atms*
  - set of parameters *Prms*
- BNF:

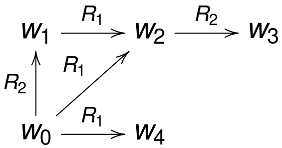
$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi$$

where p ranges over *Atms* and i over *Prms*

- abbreviations:
  - $\varphi \rightarrow \psi \stackrel{\text{def}}{=} \neg(\varphi \wedge \neg\psi)$
  - ...
  - $\Diamond_i \varphi \stackrel{\text{def}}{=} \neg\Box_i \neg\varphi$

Semantics of K: frames

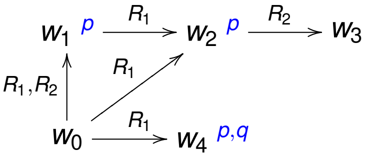
- K = 'Saul **K**ripke' [1959]
- ***K*-frame** $\mathcal{L}_l eW, R\rangle$ where:
  - W nonempty set — 'possible worlds'
    - ★ W possibly infinite (even uncountable)
  - $R : Prms \longrightarrow 2^{W \times W}$
    - ★ write R_i instead of $R(i)$ — 'accessibility relation for i'
    - ★ $R_i(w) = \{w' \ : \ \mathcal{L}_l ew, w'\rangle \in R(i)\}$ — 'worlds accessible from w'
- example:



Semantics of K: models

- K**-model** $M = \mathcal{L}_l eW, R, V\rangle$ where:
    - $\mathcal{L}_l eW, R\rangle$ is a frame
    - $V : Atms \longrightarrow 2^W$ 'valuation'
        - ⋆ $V(p) \subseteq W$

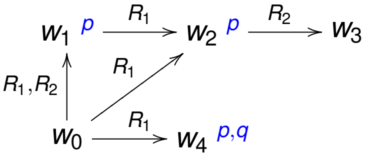
- example:



    - $V(p) = \{w_1, w_2, w_4\}$
    - $V(q) = \{w_4\}$

Semantics of K: truth conditions

- truth in a pointed model:
  - $M, w \Vdash p$ iff $w \in V(p)$
  - $M, w \Vdash \neg\varphi$ iff $M, w \nVdash \varphi$
  - $M, w \Vdash \varphi \wedge \psi$ iff $M, w \Vdash \varphi$ and $M, w \Vdash \psi$
  - **$M, w \Vdash \Box_i\varphi$ iff $M, w' \Vdash \varphi$ for all $w' \in R_i(w)$**
    - hence: $M, w \Vdash \Diamond_i\varphi$ iff $M, w' \Vdash \varphi$ for some $w' \in R_i(w)$
- example:



$M, w_0 \Vdash \neg p \wedge \neg q$	$M, w_0 \nVdash \Box_1\neg q$	$M, w_0 \Vdash \Box_3\bot$
$M, w_0 \Vdash \Box_1 p$	$M, w_0 \Vdash \Diamond_1 q$	$M, w_0 \Vdash \Diamond_1\Diamond_2\neg p$
$M, w_0 \Vdash \Box_2 p$	$M, w_0 \Vdash \Box_2\neg q$	$M, w_4 \nVdash \Diamond_1\bot$
$M, w_4 \Vdash \Box_1\bot$	$M, w_0 \Vdash \Box_1(q \rightarrow \Box_1\bot)$	

Semantics of K: exercises (1)

- find K-models for the following formulas:
  - $\Diamond_1 p \wedge \Diamond_1 \neg p$
  - $\Box_1 p \wedge \neg \Box_1 \Box_1 p$
  - $\Box_1 p \wedge \Box_2 \neg p$
  - $\Diamond_1 p \wedge \Diamond_1 \neg p \wedge \Box_1 (p \leftrightarrow q)$

Semantics of K: exercises (2)

- auxiliary definition: a formula φ is **valid** on a frame $\mathcal{L}_l eW, R\rangle$, noted $\mathcal{L}_l eW, R\rangle \Vdash \varphi$, iff for every valuation V on $\mathcal{L}_l eW, R\rangle$ and for every $w \in W$ it holds that $\mathcal{L}_l eW, R, V\rangle, w \Vdash \varphi$
- find K-frames $\mathcal{L}_l eW, R\rangle$ such that
  - $\mathcal{L}_l eW, R\rangle \Vdash \Box_1 p \to p$
  - $\mathcal{L}_l eW, R\rangle \Vdash \Diamond_1 \Diamond_1 p \to \Diamond_1 p$
  - $\mathcal{L}_l eW, R\rangle \Vdash p \to \Box_1 \Diamond_1 p$
  - $\mathcal{L}_l eW, R\rangle \Vdash \Diamond_1 p \to \Box_1 \Diamond_1 p$
  - $\mathcal{L}_l eW, R\rangle \Vdash \Box_1 p \to \Box_2 p$
  - $\mathcal{L}_l eW, R\rangle \Vdash \Diamond_1 p \to \Diamond_1 \Diamond_1 p$
  - $\mathcal{L}_l eW, R\rangle \Vdash p \to \Box_1 \Diamond_2 p$
  - $\mathcal{L}_l eW, R\rangle \Vdash \Box_1(\Box_1 p \to p)$
- for each of the above formulas φ, find the biggest class of frames on which φ is valid

Semantics of K: exercises (3)

- find formulas φ such that for every K-frame $\mathcal{L}_l eW, R\rangle$:
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $\mathcal{L}_l eW, R\rangle$ is reflexive
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $\mathcal{L}_l eW, R\rangle$ is transitive
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $R_1(w) = \emptyset$ for every $w \in W$
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $R_1(w) \neq \emptyset$ for every $w \in W$
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $\text{card}(R_1(w)) \geq 1$ for every $w \in W$
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $\text{card}(R_1(w)) \leq 1$ for every $w \in W$
  - $\mathcal{L}_l eW, R\rangle \Vdash \varphi$ iff $\text{card}(R_1(w)) \leq 2$ for every $w \in W$

Semantics of K: satisfiability and validity

- φ is ***K*-satisfiable** iff $M, w \Vdash \varphi$ for *some* K-model $M = \mathcal{L}_l eW, R, V\rangle$ and *some* possible world $w \in W$
  - $\neg p \wedge \Box_1 p$ is K-satisfiable (witness: M, w_0 above)
  - $\Box_1 p \wedge \Box_1 \neg p$ is K-satisfiable (witness: M, w_3 and M, w_4 above)
  - $\Diamond_1 q \wedge \Diamond_1 \neg q$ is K-satisfiable (witness: M, w_0 above)

- φ is ***K*-valid** ($\models_K \varphi$) iff $M, w \Vdash \varphi$ for *every* K-model $M = \mathcal{L}_l eW, R, V\rangle$ and *every* possible world $w \in W$
  - $\models_K \Box_1 p \leftrightarrow \neg \Diamond_1 \neg p$
  - $\models_K \Box_1 \top$
  - $\models_K \Box_1 (p \wedge q) \leftrightarrow (\Box_1 p \wedge \Box_1 q)$
  - $\models_K \Diamond_1 (p \vee q) \leftrightarrow (\Diamond_1 p \vee \Diamond_1 p)$

Semantics of K: *two* kinds of logical consequence

- auxiliary definition:
  - $M, w \Vdash \Gamma$ iff $M, w \Vdash \varphi_i$ for every $\varphi_i \in \Gamma$
- **local** logical consequence:

  $\Gamma \models^{loc}_K \varphi$ iff for every K-model M and every $w \in W$: $M, w \Vdash \Gamma$ implies $M, w \Vdash \varphi$

  - Γ = local hypotheses
  - mainly used in literature
- **global** logical consequence:

  $\Gamma \models^{glb}_K \varphi$ iff for every K-model M: $M, w \Vdash \Gamma$ for every $w \in W$ implies $M, w \Vdash \varphi$ for every world $w \in W$

  - Γ = global hypotheses

Semantics of K: differences between the two kinds of logical consequence

- global and local consequence coincide when $\Gamma = \emptyset$:
  - $\emptyset \models^{glb}_K \varphi$ iff $\emptyset \models^{loc}_K \varphi$ iff $\models_K \varphi$ (proof?)
- else global consequence *strictly stronger* :
  - if $\Gamma \models^{loc}_K \varphi$ then $\Gamma \models^{glb}_K \varphi$ (proof?)
  - $\{p\} \models^{glb}_K \Box_i p$, but $\{p\} \not\models^{loc}_K \Box_i p$ (proof?)
- status of the inference rule of *necessitation* $\frac{\varphi}{\Box_i \varphi}$:
  - $\{\varphi\} \not\models^{loc}_K \Box_1 \varphi$ (proof?)
  - $\{\varphi\} \models^{glb}_K \Box_1 \varphi$ (proof?)
- status of the *deduction theorem* (semantical version):
  - N.B.: actually rather a meta-theorem (because theorems = provable formulas)
  - $\{\varphi\} \models^{loc}_K \psi$ iff $\models_K \varphi \to \psi$ (proof?)
  - $\{p\} \models^{glb}_K \Box_1 p$, but $\not\models^{glb}_K p \to \Box_1 p$ (proof?)

Axiomatics of *K*

- axiom schemas for *K*:
  - every theorem schema of *classical propositional logic* (*CPL*)
  - $(\Box_i\varphi \land \Box_i\psi) \to \Box_i(\varphi \land \psi)$ conjunction C($\Box_i$)
  - $\Box_i\top$ necessity of truth N($\Box_i$)

- inference rules for *K*:
  - $\frac{\varphi, \varphi\to\psi}{\psi}$ modus ponens (MP)
  - $\frac{\varphi\to\psi}{\Box_i\varphi\to\Box_i\psi}$ rule of monotony RM($\Box_i$)

- N.B.: in axiom schemas and rules, φ, ψ and *i* are meta-variables
- *K*-proof, *K*-theorem: as usual

- we say: "*CPL*+C($\Box_i$)+N($\Box_i$)+RM($\Box_i$) *axiomatizes* the logic *K*"

Axiomatics of K: examples of theorems

- $\vdash_K \Box_i\varphi \rightarrow \Box_i\varphi$
  - proof:
    1. $\Box_i\varphi \rightarrow \Box_i\varphi$ (CPL)
- $\vdash_K \Box_i(\varphi \wedge \psi) \rightarrow \Box_i\varphi$
  - proof:
    1. $(\varphi \wedge \psi) \rightarrow \varphi$ (CPL)
    2. $\Box_i(\varphi \wedge \psi) \rightarrow \Box_i\varphi$ from 1. by RM($\Box_i$)
- $\vdash_K \Box_i(\varphi \wedge \psi) \rightarrow \Box_i\psi$
  - proof: ...

Axiomatics of K: examples of theorems, ctd.

- $\vdash_K \Box_i(\varphi \land \psi) \to (\Box_i\varphi \land \Box_i\psi)$
  - proof:
    1. $\Box_i(\varphi \land \psi) \to \Box_i\varphi$ — v.s.
    2. $\Box_i(\varphi \land \psi) \to \Box_i\psi$ — v.s.
    3. $1 \to (2 \to (\Box_i(\varphi \land \psi) \to (\Box_i\varphi \land \Box_i\psi)))$ — (CPL)
    4. $2 \to (\Box_i(\varphi \land \psi) \to (\Box_i\varphi \land \Box_i\psi))$ — from 1. and 3. by (MP)
    5. $\Box_i(\varphi \land \psi) \to (\Box_i\varphi \land \Box_i\psi)$ — from 2. and 4. by (MP)
- $\vdash_K \Box_i(\varphi \land \psi) \leftrightarrow (\Box_i\varphi \land \Box_i\psi)$
  - proof: ...

Axiomatics of K: some useful theorems

- **Rule of Necessitation** RN($\Box_i$): $\frac{\varphi}{\Box_i\varphi}$
  (“for all φ, if $\vdash_K \varphi$ then $\vdash_K \Box_i\varphi$”)
  - proof:
    1. φ — by hyp.
    2. $\varphi \to (\top \to \varphi)$ — (*CPL*)
    3. $\top \to \varphi$ — from 1. and 2. by (MP)
    4. $\Box_i\top \to \Box_i\varphi$ — from 3. by RM($\Box_i$)
    5. $\Box_i\top$ — N($\Box_i$)
    6. $\Box_i\varphi$ — from 4. and 5. by (MP)
  - N.B.: shorter proof using *derived CPL inference rules*:
    1. φ — by hyp.
    2. $\top \to \varphi$ — from 1. **by (*CPL*)**
    3. $\Box_i\top \to \Box_i\varphi$ — from 2. by RM($\Box_i$)
    4. $\Box_i\top$ — N($\Box_i$)
    5. $\Box_i\varphi$ — from 3. and 4. **by (*CPL*)**

- **Rule of $\Diamond$-Monotony** RM($\Diamond_i$): $\frac{\varphi\to\psi}{\Diamond_i\varphi\to\Diamond_i\psi}$
  - proof: …(homework)

Axiomatics of K: some useful theorems

- **Rule of Equivalence** RE($\Box_i$): $\frac{\varphi \leftrightarrow \psi}{\Box_i \varphi \leftrightarrow \Box_i \psi}$
  ("for all φ, if $\vdash_K \varphi \leftrightarrow \psi$ then $\vdash_K \Box_i \varphi \leftrightarrow \Box_i \psi$")
  - proof:
    1. $\varphi \leftrightarrow \psi$ — by hyp.
    2. $\varphi \rightarrow \psi$ — from 1. by (*CPL*)
    3. $\Box_i \varphi \rightarrow \Box_i \psi$ — from 2. by RM($\Box_i$)
    4. $\psi \rightarrow \varphi$ — from 1. by (*CPL*)
    5. $\Box_i \psi \rightarrow \Box_i \varphi$ — from 4. by RM($\Box_i$)
    6. $\Box_i \varphi \leftrightarrow \Box_i \psi$ — from 3. and 5. by (*CPL*)

- **Rule of Replacement of Proved Equivalents** (REq): $\frac{\psi \leftrightarrow \psi'}{\varphi[\psi] \leftrightarrow \varphi[\psi']}$
  ("if $\vdash_K \psi \leftrightarrow \psi'$ then for all φ and φ' such that $\psi \in sf(\varphi)$ and φ' is obtained from φ by replacing that occurrence of ψ by ψ' it holds that $\vdash_K \varphi \leftrightarrow \varphi'$")
  - proof by induction on the *structure* of φ:
    1. φ atomic: then $\psi = \varphi$, and $\varphi' = \psi'$
    2. $\varphi = \neg\varphi_1$: if $\psi = \varphi$ then $\varphi' = \psi'$; else $\psi \in sf(\varphi_1)$; ...
    3. $\varphi = \varphi_1 \wedge \varphi_2$: ...
    4. $\varphi = \Box_i \varphi_1$: ...
  - N.B.: replacement $\neq$ (uniform) substitution:
    - replacement in (REq): applies to *one* occurrence of a *subformula*;
    - substitution: applies to *all* occurrences of an *atom*

Axiomatics of K: some useful theorems, ctd.

- **Kripke's axiom** K($\Box_i$): $\vdash_K \Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$
  - proof:
    1. $(\Box_i\varphi \land \Box_i(\varphi \to \psi)) \to \Box_i(\varphi \land (\varphi \to \psi))$ — C($\Box_i$)
    2. $(\varphi \land (\varphi \to \psi)) \to \psi$ — (*CPL*)
    3. $\Box_i(\varphi \land (\varphi \to \psi)) \to \Box_i\psi$ — from 2. by RM($\Box_i$)
    4. $(\Box_i\varphi \land \Box_i(\varphi \to \psi)) \to \Box_i\psi$ — from 1. and 3. by (*CPL*)
    5. $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$ — from 4. by (*CPL*)
- $\vdash_K (\Box_i\varphi \land \Diamond_i\psi) \to \Diamond_i(\varphi \land \psi)$
  - proof: ... — hint: use (REq) and K($\Box_i$)

Axiomatics of K: soundness and completeness

Soundness Theorem.
If $\vdash_K \varphi$ then $\models_K \varphi$.

Proof.
We prove: if there is a K-proof $\mathcal{L}_l e \varphi_1, \ldots, \varphi_n\rangle$ of φ then $\models_K \varphi$.
We proceed by induction on n.

Base case: If $n = 1$ then φ is an instance of an axiom schema. We prove that every such instance is valid.
Let M be any K-model, and w any world in M.

- axiom N($\Box_i$) is K-valid:
  $M, w \Vdash \Box_i \top$ because $M, w' \Vdash \top$ for every w'.
- Every instance of axiom schema C($\Box_i$) : $(\Box_i\varphi \land \Box_i\psi) \to \Box_i(\varphi \land \psi)$ is K-valid:
  suppose $M, w \Vdash \Box_i\varphi \land \Box_i\psi$;
  then both φ and ψ are true in every world $w' \in R_i(w)$;
  therefore $\varphi \land \psi$ is true in every $w' \in R_i(w)$.

Axiomatics of K: soundness and completeness, ctd.

(*Proof of Soundness Theorem, ctd.*)

Induction hypothesis (I.H.): For all $m < n$, if $\mathcal{L}_l e\varphi_1, \ldots, \varphi_m\rangle$ is a K-proof of φ then $\models_K \varphi$.

Induction step: Let $\mathcal{L}_l e\varphi_1, \ldots, \varphi_n\rangle$ be a K-proof of φ. We do a case analysis, checking the possible ways φ_n is obtained:

- φ_n is an instance of an axiom schema.
  Then we already know that $\models_K \varphi$.
- φ_n is obtained from some φ_k, $k < n$, via RM($\Box_i$).
  Then $\varphi_k = \psi \to \chi$ and $\varphi_n = \Box_i(\psi \to \chi)$, and $\mathcal{L}_l e\varphi_1, \ldots, \varphi_k\rangle$ is a K-proof of φ_k.
  By I.H., $\models_K \psi \to \chi$, i.e. $M, w \Vdash \psi \to \chi$ for every K-model M and every world w in M. Therefore we must have $\models_K \Box_i(\psi \to \chi)$.
  "RM($\Box_i$) preserves validity"
- φ_n is obtained from some φ_k and $\varphi_l = \varphi_k \to \varphi_n$ via (MP).
  "(MP) preserves validity"
  …

Axiomatics of K: soundness and completeness, ctd.

Weak Completeness Theorem.
If $\models_K \varphi$ then $\vdash_K \varphi$.

Proof.
will follow from more general result (v. Chapter 5)

Decidability Theorem.
The problem of K-satisfiability of a formula φ can be decided in polynomial space (PSPACE).

Proof.
using the tableau procedure (v. Chapter 11)

Axiomatics of K: an equivalent axiomatization

Theorem.
The logic K is also axiomatized by CPL+K($\Box_i$)+RN($\Box_i$).

Proof.
We have to show:

- φ can be proved from CPL+C($\Box_i$)+N($\Box_i$)+RM($\Box_i$) iff φ can be proved from CPL+K($\Box_i$)+RN($\Box_i$).

For that, it will suffice to prove:

- that CPL+C($\Box_i$)+N($\Box_i$)+RM($\Box_i$)
  - has theorem K($\Box_i$): $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$
  - has derived rules (MP) and RN($\Box_i$): $\frac{\varphi}{\Box_i\varphi}$
- that CPL+K($\Box_i$)+RN($\Box_i$)
  - has theorems C($\Box_i$) and N($\Box_i$)
  - has derived rules (MP) and RM($\Box_i$)

Axiomatics of K: deduction with *global* hypotheses

- generic definition (same for every modal logic Λ)
- a deduction of φ from **global** hypotheses Γ in logic Λ is a sequence $\mathcal{L}_l e \varphi_1, \ldots, \varphi_n \rangle$ such that $\varphi_n = \varphi$ and for every $k \leq n$:
  - $\varphi_k \in \Gamma$, or
  - φ_k is (an instance of) an axiom schema for Λ, or
  - there are φ_l, φ_m such that $l, m < k$ and $\varphi_m = \varphi_l \to \varphi_k$ (MP)
  - there is φ_l such that $l < k$ and $\varphi_k = \Box_i \varphi_l$ RN($\Box_i$)

  N.B.: uniform substitution (instantiation) can only be applied to axiom schemas, but not to hypotheses in Γ
- φ is deducible in Λ from **global** hypotheses Γ iff there exists a deduction of φ from global hypotheses Γ in Λ
  - notation: $\Gamma \vdash^{glb}_{\Lambda} \varphi$

Axiomatics of K: deduction with *local* hypotheses

- generic definition (same for every modal logic Λ)
- a deduction of φ from **local** hypotheses Γ in logic Λ is a sequence $\mathcal{L}_l e \varphi_1, \ldots, \varphi_n\rangle$ such that $\varphi_n = \varphi$ and for every $k \leq n$:
  - $\varphi_k \in \Gamma$, or
  - φ_k is (an instance of) an axiom schema for Λ, or
  - there are φ_l, φ_m such that $l, m < k$ and $\varphi_m = \varphi_l \to \varphi_k$ (MP)
  - there is φ_l such that $l < k$ and $\varphi_k = \Box_i \varphi_l$ **and** $\vdash_\Lambda \varphi_l$ RN($\Box_i$)

  N.B.: uniform substitution (instantiation) can only be applied to axiom schemas, but not to hypotheses in Γ
- φ is deducible in Λ from **local** hypotheses Γ iff there exists a deduction of φ from local hypotheses Γ in Λ
  - notation: $\Gamma \vdash_\Lambda^{loc} \varphi$

Properties of global and local deduction in K

Strong Completeness Theorem for local deduction in K.
$\Gamma \vdash_K^{loc} \varphi$ iff $\Gamma \models_K^{loc} \varphi$

Proof.
will follow from more general result (v. Ch.5)

Strong Completeness Theorem for global deduction in K.
$\Gamma \vdash_K^{glb} \varphi$ iff $\Gamma \models_K^{glb} \varphi$

not proved in this course

hence just as for semantical consequence:

- global and local deducibility coincide when $\Gamma = \emptyset$:
  - $\emptyset \vdash_K^{glb} \varphi$ iff $\emptyset \vdash_K^{loc} \varphi$ iff $\vdash_K \varphi$ (syntactical proof?)
- else global deducibility *strictly stronger* :
  - if $\Gamma \vdash_K^{glb} \varphi$ then $\Gamma \vdash_K^{loc} \varphi$ (syntactical proof?)
  - $\{p\} \vdash_K^{glb} \Box_i p$, but $\{p\} \not\vdash_K^{loc} \Box_i p$ (syntactical proof?)
- deduction theorem holds for $\vdash_K^{loc}$, but not for $\vdash_K^{glb}$