Bi-capacities for decision making on bipolar scales

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Abstract

We present the concept of bi-capacity as a generalization of capacities (or fuzzy measures). The Choquet integral w.r.t. bi-capacities is defined, and is a generalization of Cumulative Prospect Theory models. This new model can bring much more flexibility in representing preferences. Lastly, we introduce the Möbius transform of bi-capacities.

Keywords: Choquet integral, capacity, bipolar scale, decision making

1 Introduction

In the field of decision theory, capacities and the Choquet integral have been extensively used as a powerful tool to model preferences (see the pioneering works of Schmeidler in decision under uncertainty [9], which have lead to Choquet Expected Utility, and the introduction of the concept of interaction between criteria, very useful in decision under multiple criteria [7, 2]).

In many situation in decision making, it is useful to consider underlying scales (for utility functions, or evaluations, scores) as bipolar scales, i.e. with a central neutral value (usually 0), considering values above the neutral level as “good” or “gains”, and values below it as “bad” or “losses”. The motivation for doing so is that human decision makers do effectively distinguish gains and losses and behave differently. This has lead to models based on the symmetric Choquet integral, or Šipoš integral (see a construction of a multicriteria decision model based on it in [4]), or more generally on CPT (Cumulative Prospect Theory) models in decision under risk or uncertainty [10].

Despite the ability of these models to cope with many decision behaviours, it is not uncommon to meet practical situations where these models fail to represent preferences, even though these preferences seem rather natural. The rest of this section is devoted to the presentation of such an example.

We consider the problem of evaluating students in a high school according to their levels in mathematics (M), physics (P) and literature (L). The director thinks that scientific subjects are more important than literature, however he would not be satisfied by a student having an important flaw in literature, even though excellent in sciences. As mathematics and physics have some redundance in skills, the following rules seems to be reasonable and reflect the director’s strategy in decision:

> **(R1)**: For a student good at mathematics (M), L is more important than P.
> **(R2)**: For a student bad in mathematics (M), P is more important than L.

According to these rules, it is easy to compare students A, B, C, D whose marks between 0 and 20 are given in the following table.

	Math.	Physics	Literature
student A	15	18	6
student B	15	16	8
student C	6	13	2
student D	6	11	4

Clearly, $A \prec B$ by Rule **R1**, and $C \succ D$ by Rule **R2**.

Let us try to represent these preferences by the Choquet integral, or a CPT model (definitions are given in Section 2). An easy calculation shows that the capacity v in the Choquet integral should satisfy both:

$$\begin{aligned} v(\{\mathrm{M},\mathrm{P}\}) + v(\{\mathrm{P}\}) &> 1 \\ v(\{\mathrm{M},\mathrm{P}\}) + v(\{\mathrm{P}\}) &< 1, \end{aligned}$$

which is impossible. Since the evaluation scale $[0, 20]$ is clearly bipolar with a neutral level 10, we should use the CPT model, which is the most general. Denoting by v_1, v_2 the capacities for positive and negative parts, an easy calculation leads to the following conditions:

$$\begin{aligned} v_1(\{\mathrm{P}\}) &> v_2(\{\mathrm{L}\}) \\ v_1(\{\mathrm{P}\}) &< v_2(\{\mathrm{L}\}), \end{aligned}$$

which shows that CPT too is unable to model such preferences. The underlying reason for this is that none of the above models is able to define a specific preference when alternatives have both positive and negative scores.

2 Background

We introduce necessary concepts for the sequel. We consider a finite set $N = \{1, \ldots, n\}$ which can be thought as the set of criteria, states of nature, voters, etc.

A *capacity* $v : \mathcal{P}(N) \longrightarrow [0, 1]$ is a set function satisfying $v(\emptyset) = 0$, $v(N) = 1$, and $A \subset B$ implies $v(A) \leq v(B)$. The *conjugate* capacity is defined by $\overline{v}(A) = 1 - v(A^c)$. *Unanimity games* u_B, $\emptyset \neq B \subset N$, are particular capacities defined by:

$$u_B(A) = \begin{cases} 1, & \text{if } A \supset B \\ 0, & \text{otherwise.} \end{cases}$$

The *Möbius transform* of a capacity is defined by $m^v(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} v(B)$, for all $A \subset N$. Any set function v vanishing on the empty set can be represented in the basis of unanimity games:

$$v(A) = \sum_{B \subset N} m^v(B) u_B(A), \forall A \subset N. \qquad (1)$$

The Möbius transform represents the coordinates of v in the basis of unanimity games.

A non negative function defined on N is assimilated with a point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n_+$. The *Choquet integral* of a function $x \in \mathbb{R}^n_+$ w.r.t a capacity v is defined by:

$$\mathcal{C}_v(x) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)}) v(A_{(i)})$$

where $(\cdot)$ is a permutation such that $0 =: x_{(0)} \leq x_{(1)} \leq \cdots \leq x_{(n)}$, and $A_{(i)} := \{(i), \ldots, (n)\}$.

An important remark is that v needs not to be a capacity in order that the Choquet integral is properly defined: any set function vanishing on the empty set could do. In particular, we have the following

$$\mathcal{C}_{-v}(x) = -\mathcal{C}_v(x). \qquad (2)$$

Let us consider a real-valued function $x \in \mathbb{R}^n$. The *symmetric Choquet integral* or *Šipoš* integral is defined by

$$\check{\mathcal{C}}_v(x) = \mathcal{C}_v(x^+) - \mathcal{C}_v(x^-)$$

with $x^+ = x \vee 0$, and $x^- = (-x)^+$. The *CPT* model is a generalization with two capacities v_1, v_2:

$$\mathrm{CPT}(x) = \mathcal{C}_{v_1}(x^+) - \mathcal{C}_{v_2}(x^-).$$

The *asymmetric* Choquet integral (which is the usual definition of the Choquet integral for real-valued integrands, hence we keep the same symbol) is defined by

$$\mathcal{C}_v(x) = \mathcal{C}_v(x^+) - \mathcal{C}_{\overline{v}}(x^-).$$

A fundamental property of the Choquet integral is the following: for any $A \subset N$, we have $\mathcal{C}_v(1_A, 0_{A^c}) = v(A)$, where $(1_A, 0_{A^c})$ stands for the function taking value 1 on A and 0 outside. In the framework of multicriteria decision making, the function $(1_A, 0_{A^c})$ could be thought as being the scores over all criteria of an alternative being "satisfactory" on criteria in A, and "neither good nor bad" for all other criteria (see [4] for details), and $v(A)$ represents the overall score of this alternative, which we call a *binary* alternative. The set of all binary alternatives being the set of vertices of the $[0, 1]^n$ hypercube, the Choquet integral can be viewed as a particular extension of v (viewed as a function over the vertices) on the whole hypercube (again see [4] for details).

3 Bi-capacities

Since the Choquet integral is entirely defined if one fixes the overall scores of binary alternatives, it becomes clear that the Šipoš integral is determined too, while the CPT model is defined solely by fixing the overall scores on binary alternatives *and* on *negative* binary alternatives, i.e. $(-1_A, 0_{A^c})$.

A natural generalization is to enrich this model by fixing the overall score of all *ternary* alternatives $(1_A, -1_B, 0_{(A\cup B)^c})$, for all disjoint pairs (A, B) in N^2. We denote $v(A, B)$ the overall score of $(1_A, -1_B, 0_{(A\cup B)^c})$. The "new" Choquet integral should then extend the scores $v(A, B)$ on $[-1, 1]^n$.

The condition of dominance in decision making leads us to impose that $v(\cdot, \cdot)$ should be increasing w.r.t set inclusion in the first coordinate, and decreasing in the second. Also it is natural to impose $v(\emptyset, \emptyset) = 0$, $v(N, \emptyset) = 1$ and $v(\emptyset, N) = -1$. Such a set function defined on $\mathcal{Q}(N) := \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) | A \cap B = \emptyset\}$ is called a *bi-capacity*. Note that $v(\cdot, \emptyset) \geq 0$ and $v(\emptyset, \cdot) \leq 0$.

A particular case of interest is when left and right part can be separated. We say that a bi-capacity is of the *CPT type* if there exists two capacities ν_1, ν_2 such that

$$v(A, B) = \nu_1(A) - \nu_2(B), \forall (A, B) \in \mathcal{Q}(N).$$

When $\nu_1 = \nu_2$, we say that the capacity is *symmetric*, and asymmetric when $\nu_2 = \overline{\nu}_1$. By analogy with the classical case, we say that the bi-capacity is *additive* if it satisfies for all $(A, B) \in \mathcal{Q}(N)$:

$$v(A, B) = \sum_{i\in A} v(i, \emptyset) + \sum_{i\in B} v(\emptyset, i).$$

Decomposable bi-capacities, bi-measures of possibility, etc. can be defined as well (see [3]). Unanimity games can be generalized as follows (we call them *bi-unanimity games*)

$$u_{(B,B')}(A, A') = \begin{cases} 1, & \text{if } A \supset B \text{ and } A' \subset B' \\ 0, & \text{otherwise.} \end{cases}$$

We give some insight into the structure of $\mathcal{Q}(N)$, which is of primary importance. It is easy to see that $(\mathcal{Q}(N), \sqsubseteq)$ is the lattice 3^n, with $(A, B) \sqsubseteq (C, D)$ if $A \subset C$ and $B \supset D$. Supremum and infimum are respectively

$$(A, B) \vee (C, D) = (A \cup C, B \cap D)$$
$$(A, B) \wedge (C, D) = (A \cap C, B \cup D).$$

We give as an illustration $(\mathcal{Q}(N), \sqsubseteq)$ for $n = 3$ in Fig. 1. As a general feature, $\mathcal{Q}(N)$ is formed by 2^n Boolean sub-lattices 2^n: each sub-lattice corresponds to a given partition of N into two parts, one for positive scores, the other for negative ones, which contain all subsets of non-zero scores, including the empty set. Hence, all these sublattices have as a common point $(\emptyset, \emptyset)$. It can be shown [1] that every element of $\mathcal{Q}(N)$ can be obtained as a supremum over the set of elements $\{(\emptyset, i^c), (i, i^c), \forall i \in N\}$ ($\vee$-irreducible elements).

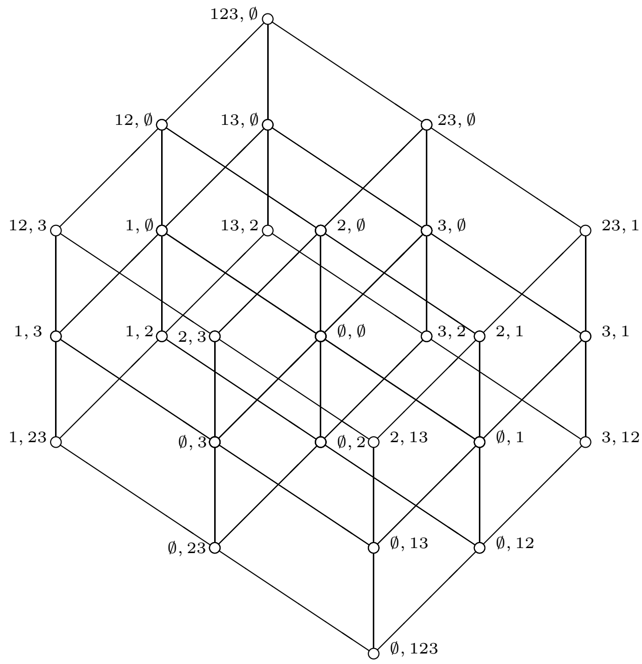

Figure 1: The lattice $\mathcal{Q}(N)$ for $n = 3$

4 Möbius transform of bi-capacities

An important concept for capacities is the Möbius transform, since it represents the coordinates of the capacity in the basis of unanimity games (see (1)). We shall find the equivalent notion for bi-capacities. The general theory of Möbius transform starts from the Möbius function, which is an inversion operator over posets [8]. More specifically, we consider f, g two real-valued functions on a locally finite poset $(X, \leq)$ such that

$$g(x) = \sum_{y \leq x} f(y). \tag{3}$$

The solution of this equation in term of g is given through the Möbius function $\mu(x, y)$ by

$$f(x) = \sum_{y \leq x} \mu(y, x) g(y) \tag{4}$$

where μ is defined inductively and depends solely on $(X, \leq)$. When $(X, \leq)$ is a Boolean lattice, isomorphic to $(\mathcal{P}(N), \subset)$, it is well known that the Möbius function becomes, for any $A, B \in \mathcal{P}(N)$

$$\mu(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subset B \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

g being a capacity, f is its Möbius transform. Taking now $(\mathcal{Q}(N), \sqsubseteq)$ as a finite poset, it can be shown that the corresponding Möbius function is

$$\mu((A, A'), (B, B')) = \begin{cases} (-1)^{|B \setminus A| + |A' \setminus B'|}, & \text{if } (A, A') \sqsubseteq (B, B') \\ & \text{and } A' \cap B = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the Möbius transform of v is expressed by

$$m(A, A') = \sum_{\substack{B \subset A \\ A' \subset B' \subset A^c}} (-1)^{|A \setminus B| + |B' \setminus A'|} v(B, B'). \tag{6}$$

By construction, we have

$$v(A, A') = \sum_{(B, B') \sqsubseteq (A, A')} m(B, B').$$

This can be rewritten under the following form, showing that, as in the classical case, the Möbius transform represents the coordinates of the bi-capacity in the basis of bi-unanimity games.

$$v(A, A') = \sum_{(B, B') \in \mathcal{Q}(N)} m(B, B') u_{(B, B')}(A, A').$$

The following result is fundamental.

Proposition 1 *Let v be a bi-capacity of the CPT type, with $v(A, B) = \nu_1(A) - \nu_2(B)$. Then its Möbius transform is given by:*

$$\begin{aligned} m(A, A^c) &= m^{\nu_1}(A), \forall A \subset N, A \neq \emptyset \\ m(\emptyset, B) &= m^{\overline{\nu_2}}(B^c), \forall B \subsetneq N \\ m(\emptyset, N) &= -1 \\ m(A, B) &= 0 \text{ otherwise.} \end{aligned}$$

We get as immediate corollaries the expression of the Möbius transform of symmetric and asymmetric bi-capacities, and also of additive bi-capacities.

Proposition 2 *Let v be an additive bi-capacity on $\mathcal{Q}(N)$. Then its Möbius transform is non null only for the $\vee$-irreducible elements and the bottom of $\mathcal{Q}(N)$. Specifically,*

$$\begin{aligned} m(\emptyset, i^c) &= -v(\emptyset, i), \forall i \in N \\ m(i, i^c) &= v(i, \emptyset), \forall i \in N \\ m(\emptyset, N) &= -1. \end{aligned}$$

5 The Choquet integral w.r.t. bi-capacities

Let us consider a bi-capacity v on $\mathcal{Q}(N)$, and define a generalized Choquet integral w.r.t v of some $x \in \mathbb{R}^n$, which we denote $F_v(x)$. We define $N^+ \subset N$ as being the set of elements i where $x_i \geq 0$. $N^- := N \setminus N^+$. Our definition should satisfy three requirements:

- **case** $N^+ = N$**:** $F_v(x)$ must coincide with the Choquet integral for all $x \in \mathbb{R}^n_+$, i.e. $F_v(x) = \mathcal{C}_{\nu_N}(x)$ for some capacity ν_N.
- **case** $N^+ = \emptyset$**:** $F_v(x)$ must coincide with the Šipoš integral for all $x \in \mathbb{R}^n_-$, i.e. $F_v(x) = \check{\mathcal{C}}_{\nu'_\emptyset}(x) = -\mathcal{C}_{\nu'_\emptyset}(-x)$ for some capacity $\nu'_\emptyset$.
- **general case:** F_v must be an extension of the scores of ternary alternatives. Specifically, for all $(A, B) \in \mathcal{Q}(N)$

$$F_v(1_A, -1_B, 0_{(A \cup B)^c}) = v(A, B). \tag{7}$$

Let us consider a ternary alternative with $N^+ = N$. Third and first conditions impose for any $A \subset N$:

$$F_v(1_A, 0_{A^c}) = v(A, \emptyset) = \mathcal{C}_{\nu_N}(1_A, 0_{A^c}) = \nu_N(A).$$

Hence ν_N is the restriction of v to the Boolean sublattice $\{(A,\emptyset), A \subset N^+\}$.

Let us consider now the case $N^+ = \emptyset$ (negative ternary alternative). Similarly we get

$$F_v(-1_B, 0_{B^c}) = v(\emptyset, B) = -\mathcal{C}_{\nu'_\emptyset}(1_B, 0_{B^c}) = -\nu'_\emptyset(B).$$

Using (2) and defining $\nu_\emptyset := -\nu'_\emptyset$, we obtain that the set function $\nu_\emptyset$ is the restriction of v to the Boolean sublattice $\{(\emptyset, B), B \subset N^-\}$. Moreover,

$$F_v(-1_B, 0_{B^c}) = \mathcal{C}_{\nu_\emptyset}(1_B, 0_{B^c}).$$

Considering now any ternary alternative $(1_A, -1_B, 0_{(A\cup B)^c})$, a natural generalization is to write

$$\begin{aligned} F_v(1_A, -1_B, 0_{(A\cup B)^c}) &= v(A,B) \\ &= \mathcal{C}_{\nu_{N^+}}(1_{A\cup B}, 0_{(A\cup B)^c}) = \nu_{N^+}(A\cup B). \end{aligned}$$

We remark that (A,B) belongs to the Boolean sublattice $\{(C,D), C \subset N^+, D \subset N^-\}$. Then splitting any $C \subset N$ on N^+ and N^-, we can rewrite $\nu_{N^+}(A\cup B)$ as $\nu_{N^+}(A\cup B) = v((A\cup B)\cap N^+, (A\cup B)\cap N^-)$. Hence we should adopt for any $C \subset N$

$$\nu_{N^+}(C) := v(C\cap N^+, C \cap N^-), \qquad (8)$$

Then we are lead to

$$F_v(x) := \mathcal{C}_{\nu_{N^+}}(x_{N^+}, -x_{N^-}) \qquad (9)$$

where $\nu_{N^+}(C)$ is defined by (8). Note that ν_{N^+} is no more a capacity in general, as it could be negative and non monotonic.

We present a characterization of the generalized Choquet integral [5]. In the sequel, F_v denotes any functional on $\mathbb{R}^n$ defined w.r.t. a bi-capacity v. We introduce the following axioms.

(LM): For any bi-capacities v, v' on $\mathcal{Q}(N)$, for all $x \in \mathbb{R}^n$ and $\gamma, \delta \in \mathbb{R}$,

$$F_{\gamma v + \delta v'}(x) = \gamma F_v(x) + \delta F_{v'}(x)$$

(In): For any bi-capacity v on $\mathcal{Q}(N)$, $\forall x, x' \in \mathbb{R}^n$,

$$x_i \le x'_i \,,\ \forall i \in N \ \Rightarrow F_v(x) \le F_v(x')$$

The next axiom **(PW)** is merely our third requirement above (see Eq. (7)).

(weak SPL$^+$): For any bi-capacity v on $\mathcal{Q}(N)$, for all $A, C \subset N$, $\alpha > 0$, and $\beta \ge 0$,

$$F_v\left((\alpha+\beta)_A, \beta_{-A}\right) = \alpha F_v\left(1_A, 0_{-A}\right) + \beta v(N,\emptyset).$$

This axiom is a weak version of axiom **SPL** (invariance to shift and linear transform) proposed by Marichal [6] in order to characterize the (asymmetric) Choquet integral, which is compatible with interval scales (0 is not fixed). Since we deal here with the symmetric Choquet integral, compatible with ratio scales, the 0 should not be shifted. However, when only elements in $\mathbb{R}^n_+$ are considered, the real meaning of 0 is lost and it can be shifted. Hence, x can be changed into $\alpha x + \beta$, with $\alpha > 0$ and $\beta \ge 0$.

For $A \subset N$, consider the following application $\Pi_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(\Pi_A(x))_i = \begin{cases} x_i & \text{if } i \in A \\ -x_i & \text{otherwise} \end{cases}$$

By **(PW)**, $v(B, B')$ corresponds to the point $\left(1_B, -1_{B'}, 0_{(B\cup B')}\right)$. Define $\Pi_A \circ v(B,B')$ as the term of the bi-capacity associated to the point

$$\begin{aligned} &\Pi_A\left(1_B, -1_{B'}, 0_{(B\cup B')^c}\right) = \\ &\quad \left(1_{(B\cap A)\cup(B'\setminus A)}, -1_{(B\setminus A)\cup(B'\cap A)}, 0_{(B\cup B')^c}\right). \end{aligned}$$

Hence we set

$$\begin{aligned} &\Pi_A \circ v(B,B') := \\ &\quad v\left((B\cap A)\cup(B'\setminus A), (B\setminus A)\cup(B'\cap A)\right). \end{aligned}$$

By symmetry arguments, it is reasonable to have $F_{\Pi_A \circ v}(\Pi_A(x))$ being equal to $F_v(x)$.

(Sym): For any $v : \mathcal{Q}(N) \to \mathbb{R}$, we have for all $A \subset N$

$$F_v(x) = F_{\Pi_A \circ v}\left(\Pi_A(x)\right) \ .$$

The following can be shown.

Theorem 1 $\{F_v\}_v$ *satisfies* **(LM)**, **(In)**, **(PW)**, **(weak SPL$^+$)** *and* **(Sym)** *if and only if for any bi-capacity* v*, and for any* $N^+ \subset N$, $x \in \Sigma_{N^+}$,

$$F_v(x) = \mathcal{C}_{\nu_{N^+}}(x_{N^+}, -x_{N^-})$$

where $\nu_{N^+}(C) := v(C \cap N^+, C \cap N^-)$, *and* $\Sigma_{N^+} := \{x \in \mathbb{R}^n \ , x_{N^+} \geq 0 \ , x_{N^-} < 0\}$.

The following can be shown.

Proposition 3 *(i) If* v *is a bi-capacity of the CPT type, then* F_v *coincide with a CPT model.*

(ii) If v *is a symmetric bi-capacity, then* F_v *coincides with the Šipoš integral.*

Let us come back to our example, unsolved with the CPT model. Applying the definition of the generalized Choquet integral, the two preferences $A \prec B$ and $C \succ D$ implies

$$v(\{\mathrm{M},\mathrm{P}\},\emptyset) - v(\{\mathrm{M},\mathrm{P}\},\{\mathrm{L}\}) > v(\{\mathrm{P}\},\emptyset)$$
$$-v(\emptyset,\{\mathrm{L}\}) < v(\{\mathrm{P}\},\{\mathrm{M},\mathrm{L}\}) - v(\emptyset,\{\mathrm{M},\mathrm{L}\}).$$

There is no contradiction between these two inequalities, and so the preference can be properly represented.

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