

On the Relationship Between Qualitative Choice Logic and Possibilistic Logic

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Abstract

Qualitative choice logic (*QCL*) is a propositional logic for representing prioritized alternatives. The logic adds to classical propositional logic a new connective called ordered disjunction: $A \vec{\times} B$ intuitively means: if possible A , but if A is not possible then at least B . *QCL* inference is obtained by defining a preference relation between interpretations of a propositional language. This paper shows that when this preference relation is total, then it is possible to encode *QCL* in the framework of possibilistic logic, and conversely.

1 Introduction

In many applications, there is a need to compactly and qualitatively represent agent's preferences and choices.

Possibilistic logic [2] offers a reasonably expressive framework for representing agent's choices. It is a logic of weighted formulas, equipped with a semantics where the weights can be understood as lower bounds of necessity degrees in the sense of possibility theory.

Recently, a new nonmonotonic propositional logic for representing qualitative choices, called qualitative choice logic (*QCL*), has been proposed [5]. The non-standard part of this logic is a new logical connective $\vec{\times}$ which is fully embedded in the logical language. Intuitively, if A and B are formulas then $A \vec{\times} B$

says: if possible A , but if A is impossible then (at least) B .

This paper investigates relationships between qualitative choice logic and possibilistic logic. More precisely, we show that a variant of *QCL*, called ranking functions-based *QCL*, can be encoded in a possibilistic logic framework. This encoding is obtained by viewing a *QCL* theory as a fusion of simple *QCL* theories that can be easily represented with guaranteed possibility distributions.

The rest of this paper is organised as follows. Sections 2 and 3 provide a brief reminder on possibilistic logic and introduce *QCL*. Section 4 establishes the relationships between ranking functions-based *QCL* and possibilistic logic. Section 5 concludes the paper.

2 A brief reminder on possibilistic logic

2.1 Possibility distributions

As in standard propositional logic, an interpretation I is an assignment of the classical truth values *true* and *false* to the atoms. We identify I with the subset of true atoms. \top represents a (classical) tautology, \perp a (classical) contradiction.

Possibilistic logic provides a tool for performing uncertainty reasoning, where uncertain information is semantically represented by means of possibility distributions. Possibility distributions are means to rank order different interpretations of a language. More precisely, a possibility distribution, denoted

by π , is a function from a set of mutually exclusive situations (solutions or interpretations) to the interval $[0,1]$. By convention $\pi(I) = 1$ means that I is among the most normal (or preferred) situations, $\pi(I) = 0$ means that I is impossible or excluded as a possible solution. More generally, $\pi(I) \geq \pi(I')$ means that I is at least as preferred as I' .

2.2 Guaranteed possibilistic knowledge bases

There are several compact (or syntactic) encodings of possibility distributions [1]: necessity based knowledge bases, min-based possibilistic graphs, product-based possibilistic graphs, etc. Recently, another type of compact representation, called guaranteed possibilistic knowledge bases or simply Δ -knowledge bases, has been investigated [6, 3]. It is based on the notion of guaranteed possibility measures, which are defined on formulas from a possibility distribution π in the following way: Let ϕ be a formula.

$$\Delta(\phi) = \min\{\pi(I) \mid I \models \phi\}.$$

A Δ -knowledge base is composed of a set of weighted formulas of the form $[\phi_i, \alpha_i]$, where ϕ_i denotes a propositional formula, and α_i is a real number between 0 and 1. The pair $[\phi_i, \alpha_i]$ means that any model of ϕ_i is satisfactory to a degree at least equal to α_i , namely:

$$\Delta(\phi_i) \geq \alpha_i$$

or

$$(1) \quad \forall I, \text{ if } I \models \phi_i \text{ then } \pi(I) \geq \alpha_i.$$

Definition 1 *Each Δ -knowledge base Δ induces a unique possibility distribution π where, for all I , $\pi(I) =$*

$$0 \text{ if } I \text{ falsifies all formulas of } \Delta, \\ \max\{\alpha_i : [\phi_i, \alpha_i] \in \Delta, I \models \phi_i\} \text{ otherwise.}$$

In [6] it has been shown that this possibility distribution corresponds to the most specific possibility distribution¹ satisfying (1) for each

¹ π is said to be more specific than π' , if $\forall I, \pi(I) \leq \pi'(I)$.

weighted formula in Δ . As we will show later, any basic *QCL* formula can be equivalently represented by a Δ -knowledge base. However, in order to show the encoding of a set of basic *QCL* formulas, we need to use possibilistic fusion operators which are recalled in next subsection.

2.3 Possibilistic fusion

In [3] several possibilistic fusion operators have been proposed to merge a set of Δ -knowledge bases. In this section we restrict the class of merging operators to those which are useful for establishing relationships between *QCL* and possibilistic logic. More precisely, let \oplus be a function from a vector of real numbers in $[0,1]$ to a real number between $[0,1]$. \oplus will be called a $[0,1]$ -based merging operator and satisfy:

Unanimity: If $\forall i \in \{1, \dots, n\} \alpha_i \geq \alpha'_i$ then $\oplus(\alpha_1, \dots, \alpha_n) \geq \oplus(\alpha'_1, \dots, \alpha'_n)$.

Model preservation: $\oplus(\alpha_1, \dots, \alpha_n) = 0$ iff for some $i \in \{1, \dots, n\} \alpha_i = 0$.

The second requirement is stronger than the one used in [3] which simply requires that $\oplus(0, \dots, 0) = 0$. It is strengthened here in order to have an easier connection with *QCL*. Let $\Delta_1, \dots, \Delta_n$ be the Δ -knowledge bases to be merged. We denote by $[\phi_{ij}, \alpha_{ij}]$ the j^{th} weighted formula in Δ_i .

Definition 2 *The result of merging $\Delta_1, \dots, \Delta_n$, denoted by Δ_{\oplus} , is defined as:*

$$\Delta_{\oplus} = \{[\phi_{1i} \wedge \dots \wedge \phi_{nk}, \oplus(\alpha_{1i}, \dots, \alpha_{nk})] \mid [\phi_{1i}, \alpha_{1i}] \in \Delta_1, \dots, [\phi_{nk}, \alpha_{nk}] \in \Delta_n\}.$$

It can be checked that the possibility distribution associated with Δ_{\oplus} using the above definition can be characterized as follows:

Lemma 1 *Let $\Delta_1, \dots, \Delta_n$ be a set of Δ -knowledge bases. Let π_1, \dots, π_n be their respective associated possibility distributions. Let \oplus be a $[0,1]$ -based merging operator, and Δ_{\oplus} be the Δ -knowledge base given by Definition 2. Then:*

$$\forall I, \pi_{\Delta_{\oplus}}(I) = \oplus(\pi_1(I), \dots, \pi_n(I)).$$

3 Syntax and semantics of *QCL*

3.1 Syntax

We start with standard propositional logic and add a new associative non-standard kind of disjunction: given formulas A_1, \dots, A_n , we will use

$$A_1 \vec{\times} \dots \vec{\times} A_n$$

to express: some A_j must be true, preferably A_1 , but if this is not possible then A_2 , if this is not possible A_3 , etc. The idea is that

- from $A \vec{\times} B$ you get A ,
- from $A \vec{\times} B, \neg A$ you get B , and
- $A \vec{\times} B, \neg A, \neg B$ is inconsistent.

Clearly, the order matters. We, therefore, call $\vec{\times}$ ordered disjunction.

Ordered disjunction is fully embedded in the logical language to make sure that context dependent options can be expressed, that is we may have formulas like $A \rightarrow (B \vec{\times} C)$ and $\neg A \rightarrow (C \vec{\times} B)$. The precise definition of the syntax is as follows:

Definition 3 *Let \mathcal{V} be a set of atoms. The set of well-formed formulas of *QCL* is inductively defined as follows*

1. every element of \mathcal{V} is a well-formed formula,
2. if F_1 and F_2 are well-formed formulas, then $(\neg F_1)$, $(F_1 \vee F_2)$, $(F_1 \wedge F_2)$ and $(F_1 \vec{\times} F_2)$ are well-formed formulas.

The connectives \vee and \wedge behave like classical conjunction and disjunction when they are applied on propositional formulas. Since $\vec{\times}$ is an associative binary operator[5], the formula $A_1 \vec{\times} \dots \vec{\times} A_n$ should be read as shorthand for $(A_1 \vec{\times} (\dots \vec{\times} A_n \dots))$. Despite the connective $\vec{\times}$ being defined as a binary operator, we use it as a n-ary

3.2 Semantics

The semantics of *QCL* is based on the degree of satisfaction of a formula in a particular (classical) model. Intuitively, the degree can be viewed as a measure of disappointment: the higher the degree the more disappointing the model (or the farther away from complete satisfaction).

Interpretations can satisfy formulas to a certain degree. For classical formulas without ordered disjunction no suboptimal way of satisfaction exists. Hence, such formulas can only be satisfied to degree 1 or not satisfied at all.

For formulas containing ordered disjunction let us first consider a simple special case. Let F be a formula of the form

$$A_1 \vec{\times} \dots \vec{\times} A_n$$

where each A_i is a classical propositional formula without $\vec{\times}$ and $n > 1$. Such formulas are called basic choice logic formulas. In this case the satisfaction degree of F in an interpretation I is simply the smallest k such that I satisfies A_k . If none of the ordered disjuncts is satisfied, then also F is not satisfied. More formally, using an index to express the degree of satisfaction, we define the satisfaction relation for formulas of this type as (\models is classical propositional satisfaction):

$$I \models_k (A_1 \vec{\times} \dots \vec{\times} A_n) \text{ iff} \\ I \models (A_1 \vee \dots \vee A_n) \text{ and } k = \min\{j \mid I \models A_j\}.$$

For arbitrary formulas determining the degree of disappointment is somewhat more involved. We first need to introduce the notion of optionality of a formula, which somewhat represents the degree of disappointment of a formula in the worst case:

Definition 4 *The optionality of a formula is defined as follows:*

$$\begin{aligned} \text{opt}(A) &= 1 \text{ if } A \text{ is an atom,} \\ \text{opt}(\neg F) &= 1, \\ \text{opt}(F_1 \vee F_2) &= \max(\text{opt}(F_1), \text{opt}(F_2)), \\ \text{opt}(F_1 \wedge F_2) &= \max(\text{opt}(F_1), \text{opt}(F_2)), \text{ and} \\ \text{opt}(F_1 \vec{\times} F_2) &= \text{opt}(F_1) + \text{opt}(F_2). \end{aligned}$$

Intuitively, if the optionality of F is n , then there may be a best way, a second best way,

etc. and an n -th best way of satisfying F . For classical formulas there is only one way to satisfy them, hence they all have optionality 1.

The optionality of a negated formula may seem puzzling at first, but there is only one way of making $\neg(A \vec{\times} B)$ true: you must make A and B false, there is no second best solution for this. In a sense negation transforms nested ordered disjunctions into standard disjunctions. Hence $\neg F$ behaves like a classical formula which explains why $opt(\neg F) = 1$.

For conjunction and disjunction we obtain the maximum optionality of the subformulas. For instance, if $opt(F_1) = j$ and $opt(F_2) = k$ with $j < k$, then an interpretation which does not satisfy F_1 but satisfies F_2 to degree k will satisfy $F_1 \vee F_2$ to degree k . Similarly, an interpretation which does satisfy F_1 to some degree and satisfies F_2 to degree k will satisfy $F_1 \wedge F_2$ to degree k .

If $F = A \vec{\times} B$, where A and B are classical, 2 degrees are needed. The best way of satisfying F is making A true, but if this doesn't work there is still the second best option, namely making B true. This generalizes to the sum of optionalities if A and B are arbitrary formulas. Note that for formulas of the form $A_1 \vec{\times} \dots \vec{\times} A_n$ where all A_i are classical the optionality is n .

We now define the satisfaction relation. The relation is indexed according to the degree of satisfaction of a formula in a model.

Definition 5

1. $I \models_k A$ iff $k = 1$ and $A \in I$ (for propositional atoms A)
2. $I \models_k P \wedge Q$ iff $I \models_m P$ and $I \models_n Q$ and $k = \max(m, n)$
3. $I \models_k P \vee Q$ iff $I \models_m P$ or $I \models_n Q$ and $k = \min\{r \mid I \models_r P \text{ or } I \models_r Q\}$
4. $I \models_k \neg P$ iff $k = 1$ and for no m : $I \models_m P$
5. $I \models_k P \vec{\times} Q$ iff $I \models_k P$ or $[I \models_1 \neg P, I \models_m Q, \text{ and } k = m + opt(P)]$.

Let us illustrate the satisfiability relation using a simple example. Let $F = A \vee (B \vec{\times} C)$ where A, B and C are atoms. Now all interpretations I containing A as well as those containing B satisfy F with degree 1, that is, we have $I \models_1 F$. Now assume I is an interpretation which contains C but makes A and B false. In this case F is satisfied with degree 2, that is $I \models_2 F$. Interpretations satisfying none of the three atoms do not satisfy F to any degree; they satisfy $\neg F$ with degree 1.

Definition 6 Let T be a set of formulas. An interpretation I is a model of T if it satisfies each formula in T to some degree.

4 Ranking functions-based QCL

The satisfaction degrees of formulas help us to determine preferred models. There are different ways of doing this. In [5] we defined entailment for QCL in terms of a lexicographic ordering on models, based on the number of formulas satisfied to a certain degree. This leads to an approach where solutions are preferred when they contain the highest number of most preferred options. For instance, if there are three choices with three options each, say

$$A_1 \vec{\times} A_2 \vec{\times} A_3, B_1 \vec{\times} B_2 \vec{\times} B_3, C_1 \vec{\times} C_2 \vec{\times} C_3$$

then a model M_1 satisfying A_1, B_1 and C_3 is preferred over a model M_2 satisfying A_2, B_2 and C_1 because the number of formulas satisfied in M_1 with degree 1 is 2, the number in M_2 is 1.

This section investigates an alternative way for defining preferred models using ranking functions. Ranking functions assign an integer rank with a model M of a set of formulas

$$T = \{F_1, \dots, F_n\}$$

based on the vector of satisfaction degrees

$$(deg^M(F_1), \dots, deg^M(F_n))$$

of the formulas in T , where:

$deg^M(f) = k$ iff $M \models_k f$ and $deg^M(f) = 0$ iff $M \models_1 \neg f$.

Let \ominus be a function from a vector of integers to an integer. \ominus will be called a ranking function if it satisfies the following requirements (similar to the one required for \oplus):

Unanimity: If $\forall i = 1, \dots, n, j_i \geq k_i$ then:

$$\ominus(j_1, \dots, j_n) \geq \ominus(k_1, \dots, k_n).$$

Model preservation:

$$\ominus(j_1, \dots, j_n) = 0 \text{ iff } j_i = 0 \text{ for some } i = 1, \dots, n.$$

Intuitively, the second requirement means that if a given interpretation falsifies some formula of T , then it should not be considered as a model of T . Ranking functions induce preferences on models in a straightforward way:

Definition 7 Let $T = \{F_1, \dots, F_n\}$ be a set of QCL formulas, \ominus a ranking function. A model M_1 of T is a \ominus -preferred model of T iff there is no model M_2 of T such that $\ominus(\text{deg}^{M_2}(F_1), \dots, \text{deg}^{M_2}(F_n)) < \ominus(\text{deg}^{M_1}(F_1), \dots, \text{deg}^{M_1}(F_n))$.

The inference relation induced by this preference relation will be denoted by \vdash_{\ominus} , and is restricted here to classical conclusions, that is, the consequence relation we are interested in will be a relation between sets of QCL formulas and classical formulas.

Definition 8 Let T be a set of formulas, and let A be a classical formula. Let \ominus be a ranking function. $T \vdash_{\ominus} A$ iff A is satisfied in all \ominus -preferred models of T .

As an example consider the following ranking function:

$$\begin{aligned} \ominus(j_1, \dots, j_n) = \\ 0 \text{ if } j_k = 0 \text{ for some } k \in \{1, \dots, n\} \\ \sum_{1 \leq k \leq n} j_k \text{ otherwise.} \end{aligned}$$

This ranking function just adds up the degrees of (dis)satisfaction of all formulas in T for each model and prefers those models whose sum is minimal, that is, whose overall dissatisfaction is minimal. In the example from the beginning of this section, both M_1 and M_2 have overall degree 5 ($1 + 1 + 3$ and $2 + 2 + 1$, respectively), thus none of the models is preferred to the other.

4.1 Ranking functions-based QCL and possibilistic logic

We restrict ourselves to sets of basic choice formulas. This is not a limitation since first it has been shown in [5] that any QCL formula can be equivalently transformed into a basic choice logic formula. Moreover, any classical propositional formula p can be represented as a basic choice formula $p \vec{\times} \perp$ without changing the ranking of interpretations. The following lemma is immediate, noticing that models of $\{p_i \vec{\times} \perp \mid p_i \text{ is a classical formula of } T\}$ are exactly the same as the ones of $\{p_i \mid p_i \text{ is a classical formula of } T\}$.

Lemma 2 Let T be a set of formulas. Let T' be obtained from T by replacing each classical formula p in T by a new basic choice formula $p \vec{\times} \perp$. Then T and T' induce the same ranking on the set of interpretations.

The following lemma establishes a first connection between QCL and possibilistic logic when we only have one basic choice formula.

Lemma 3 Let $F = A_1 \vec{\times} \dots \vec{\times} A_n$ be a basic choice formula (A_i 's are propositional formulas). Let $\alpha_i = \varepsilon^i$ for $i=1, \dots, n$, where $0 < \varepsilon < 1$.² Let $\Delta = \{[A_1, \alpha_1], \dots, [A_n, \alpha_n]\}$ be the Δ -knowledge base associated with F . Then $\forall I, I \models_k F$ if and only if $\pi(I) = \alpha_k$ and $I \models_1 \neg F$ iff $\pi(I) = 0$.

Hence, each basic choice logic formula can be represented by a Δ -knowledge base.

As a corollary of Lemmas 1 and 3, it is possible to provide a possibilistic encoding of a general QCL^{\ominus} theory:

Lemma 4 Let $T = \{F_1, \dots, F_n\}$ be a QCL theory, and let \ominus be a ranking function. Let us again denote by $\alpha_i = \varepsilon^i$ for $i \in \aleph$. Let Δ_i be the Δ -knowledge base associated with F_i using Lemma 3. Then the Δ -knowledge base associated with T is the one obtained from Def. 2 by merging Δ_i 's with $\oplus(\alpha_i, \dots, \alpha_k) = \varepsilon^{\ominus(i, \dots, k)}$.

Lemma 4 is interesting since it means that, for any ranking function \ominus , any QCL theory can

²In fact, any function such that $\alpha_i > \alpha_j$ if and only if $i < j$ is appropriate.

be transformed into one basic choice formula. Indeed, let $T = \{F_1, \dots, F_n\}$ be a set of basic choice formulas. Again, we denote by A_{ij} the propositional formula which is in position j (namely after $j - 1$ occurrences of $\vec{\times}$) in the basic choice formula F_i . We denote by

$$\mathcal{C} = A_{1i} \wedge \dots \wedge A_{nk}$$

any conjunction of exactly one A_{ij} from each F_i . \mathcal{C} is called a complex cube. We denote by $degree(\mathcal{C})$ the rank associated to \mathcal{C} defined as equal to $\ominus(i, \dots, k)$. Then, it can be checked that for each ranking function \ominus , it is possible to replace T by one basic choice formula defined as $B_1 \vec{\times} \dots \vec{\times} B_m$ where:

$$m = \ominus(opt(F_1), \dots, opt(F_n)),$$

and

$B_i = \perp$ if there is no \mathcal{C} such that $degree(\mathcal{C}) = i$,
 $B_i = \bigvee_{degree(\mathcal{C})=i} \mathcal{C}$, otherwise.

Example 1 Let us assume that T is composed of two basic choice formulas $F_1 = A_{11} \vec{\times} A_{12}$ and $F_2 = A_{21} \vec{\times} A_{22}$. Then we have 4 complex cubes:

$$A_{11} \wedge A_{21}, A_{11} \wedge A_{22}, A_{12} \wedge A_{21}, A_{12} \wedge A_{22}$$

Let us assume that the ranking function \ominus yields the sum of the satisfaction degrees of the single formulas (as described in previous section). Then, we have:

$$\begin{aligned} degree(A_{11} \wedge A_{21}) &= 2 \\ degree(A_{11} \wedge A_{22}) &= 3 \\ degree(A_{12} \wedge A_{21}) &= 3 \\ degree(A_{12} \wedge A_{22}) &= 4 \end{aligned}$$

Therefore, it can be checked that if we rank-order interpretations with respect to sum of degree then this ranking can be recovered from the basic choice formula $C_1 \vec{\times} C_2 \vec{\times} C_3 \vec{\times} C_4$ with:

$$\begin{aligned} C_1 &= \perp, \\ C_2 &= A_{11} \wedge A_{21}, \\ C_3 &= (A_{12} \wedge A_{21}) \vee (A_{12} \wedge A_{22}), \text{ and} \\ C_4 &= A_{12} \wedge A_{22}. \end{aligned}$$

The converse transformation from a guaranteed possibilistic knowledge base to a *QCL*

theory is immediate using again Lemma 2, provided that a finite totally ordered scale is used for quantifying the uncertainty degree of a formula. Namely, let $\mathcal{L} = \{\alpha_1, \dots, \alpha_n\}$ be a finite ordered scale using with $\alpha_1 > \dots > \alpha_n$. Let $\Delta = \{[A_1, \alpha_1], \dots, [A_m, \alpha_m]\}$ be a Δ -knowledge base. We assume without loss of generality that there is no two formulas having a same weight. Indeed, two formulas having a same weight can be equivalently replaced by their disjunction with a same weight. Moreover, if for some degree α_i ($i < m$) there is no $[A_i, \alpha_i]$ in Δ , then we add to Δ without changing induced possibility distributions the formula $[\perp, \alpha_i]$. Then letting $F = A_1 \vec{\times} \dots \vec{\times} A_m$, we can show that $\forall I, I \models_k F$ if and only if $\pi(I) = \alpha_k$ and $I \models_1 \neg F$ iff $\pi(I) = 0$.

5 Conclusions

This paper investigates relationships between possibilistic logic and a class of *QCL* based on ranking functions \ominus . In particular, we show that i) any basic choice logic formula can be viewed as a guaranteed possibilistic knowledge base, and ii) for each ranking function \ominus , the entailment \vdash_{\ominus} can be recovered in a possibility theory framework. A corollary of this result is that, for any ranking function, any *QCL* theory can be equivalently transformed into a basic choice logic formula. The converse transformation from a guaranteed possibilistic knowledge base to a *QCL* theory is provided. And more generally, it is possible to provide a transformation from a necessity-based knowledge base to a *QCL* theory, exploiting the correspondences between necessity-based knowledge bases and guaranteed-based knowledge bases [3].

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Appendix: Proofs of propositions

Proof of Lemma 1 (possibility distribution associated with Δ_{\oplus})

We only show the case where two Δ -knowledge bases are merged. The general case follows in a similar way.

Recall that :

$$\pi_{\Delta_{\oplus}} = \{[\phi_i \wedge \psi_j, \oplus(\alpha_i, \beta_j)] : [\phi_i, \alpha_i] \in \Delta_1 \text{ and } [\psi_j, \beta_j] \in \Delta_2\}$$

Note first that if for some interpretation I , $\pi_1(I) = 0$ or $\pi_2(I) = 0$ then this means that I falsifies all propositional formulas of Δ_1 , or I falsifies all propositional formulas of Δ_2 . Hence, I also falsifies all propositional formulas of Δ_{\oplus} . Then, $\pi_{\Delta_{\oplus}} = 0 = \oplus(\pi_1(I), \dots, \pi_n(I))$, since $\oplus(0, 0) = 0$.

Now assume that I satisfies at least one propositional formula from Δ_1 and at least one propositional formula from Δ_2 . Then $\pi_{\Delta_{\oplus}}$ is computed as follows:

$$\begin{aligned} \pi_{\Delta_{\oplus}}(I) &= \max\{\oplus(\alpha_i, \beta_j) : I \models \phi_i \wedge \psi_j \text{ and } [\phi_i \wedge \psi_j, \oplus(\alpha_i, \beta_j)] \in \Delta_{\oplus}\} \\ &= \max\{\oplus(\alpha_i, \beta_j) : I \models \phi_i \wedge \psi_j \text{ and } [\phi_i, \alpha_i] \in \Delta_1, [\psi_j, \beta_j] \in \Delta_2\} \\ &= \max\{\oplus(\alpha_i, \beta_j) : I \models \phi_i, [\phi_i, \alpha_i] \in \Delta_1 \text{ and } I \models \psi_j, [\psi_j, \beta_j] \in \Delta_2\}. \end{aligned}$$

Since \oplus satisfies the unanimity condition then when α_i and β_j are maximal then $\oplus(\alpha_i, \beta_j)$ is also maximal.

$$\text{Hence, } \pi_{\Delta_{\oplus}}(I) = \oplus(\max\{\alpha_i : I \models \phi_i, [\phi_i, \alpha_i] \in \Delta_1\}', \max\{\beta_j : I \models \psi_j, [\psi_j, \beta_j] \in \Delta_2\}) = \oplus(\pi_1(I), \pi_2(I)).$$

Proof of Lemma 3 (basic choice formulas as Δ -knowledge bases)

The proof is immediate. Indeed, if for all i , $I \not\models A_i$, then from Definition 1 we have $\pi(I) = 0$, and from definition 5 we have $I \models_{\oplus} \neg F$. Now, assume that I satisfies some A_i . Then $\pi(I) = \alpha_k$ means that $I \models A_k$, and for $i = 1, \dots, k-1$, we have $I \not\models A_i$, and this is exactly equivalent to $I \models_k F$.