Syntactic combination of uncertain information: A possibilistic approach[1]

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Abstract: This paper proposes syntactic combination rules for merging uncertain propositional knowledge bases provided by different sources of information, in the framework of possibilistic logic. These rules are the counterparts of combination rules which can be applied to the possibility distributions (defined on the set of possible worlds), which represent the semantics of each propositional knowledge base. Combination modes taking into account the levels of conflict, the relative reliability of the sources, or having reinforcement effects are considered.

1 - Introduction

In many situations, relevant information is provided by different sources. Information is also often pervaded with uncertainty. This uncertainty may directly reflect the reliability of the source, or may be attached to the information provided by the source itself. Taking advantage of the different sources of information usually requires to perform some combination operation on the pieces of information, and leads to a data fusion problem. The way this problem is tackled depends on the way the information is represented, which in turn is contingent on the nature of the information. On the one hand, pieces of information pertaining to numerical parameters are usually represented by distribution functions (in the sense of some uncertainty theory). These distributions are directly combined by means of operations which are in agreement with the uncertainty theory used, and which yield a new distribution on the set of the possible values of the considered parameter. On the other hand, information may be also naturally expressed in logical terms, especially in case of (symbolic) information pertaining to properties, which may be, however, pervaded with uncertainty. In this case, some uncertainty weights are attached to the logical formulas. Although similar issues are raised in the two frameworks, like the handling of conflicting information, the two lines of research in numerical data fusion (e.g., Abidi and Gonzalez, 1992; Flamm and Luisi, 1992) and in symbolic information combination (e.g., Baral et al., 1992; Cholvy, 1992; Dubois et al., 1992; Benferhat et al., 1993) have been investigated independently, and are unequally developed (the second trend being much more recent and the proposals still preliminary).

Possibility theory (Zadeh, 1978; Dubois and Prade, 1988a) offers an uncertainty modelling framework for dealing with possibility distributions representing fuzzy

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information. This information may pertain to the value of numerical parameters, or to variables ranging in discrete and finite domains. Moreover, possibilistic logic formulas (Dubois et al., 1994) can be handled in a purely syntactic way (as classical logic formulas associated with a weight) in complete agreement with a semantic interpretation in terms of a possibility distribution over the set of possible worlds, which represents the set of possibilistic logic formulas under consideration. The combination of possibility distributions representing ill-known data has been studied by two of the authors in the last past years (Dubois and Prade, 1988b, 1992), and different combination modes have been proposed depending on the nature of the conflict between the sources and their reliability.

In this paper, we apply these combination modes to the possibility distributions associated with the sets of uncertain propositions provided by each source of information and we look for the syntactic counterparts of these combination on the sets of possibilistic formulas. Section 2 gives the necessary background on possibilistic logic. Section 3 presents the semantical combination modes on possibility distributions, while Section 4 gives their syntactic counterparts applied to the possibilistic propositional logic bases corresponding to each sources.

2 - Possibilistic Logic

2.1 - Possibility Distribution and Possibilistic Entailment

In this paper, we only consider a finite propositional language denoted by $\mathscr{L}$. We denote by $\vDash$ the classical consequence relation, Greek letters $\alpha, \beta, \ldots$ represent formulas. Let Ω be the finite set of interpretations of the propositional logic language $\mathscr{L}$.

A possibility distribution is a mapping π from Ω to the interval [0,1]. π is said to be normal if $\exists \omega \in \Omega$, such that $\pi(\omega)=1$. By convention, π represents some background knowledge about the possible states of the real world; $\pi(\omega)=0$ means that the state ω is impossible, and $\pi(\omega)=1$ means that nothing prevents ω from being the real world. When $\pi(\omega)>\pi(\omega')$, ω is a preferred candidate to ω' for being the real state of the world.

A possibility distribution π induces two mappings grading respectively the possibility and the certainty of a formula ϕ:

– the possibility degree $\Pi(\phi) = \max\{\pi(\omega) \mid \omega \vDash \phi\}$ which evaluates to what extent ϕ is consistent with the available knowledge expressed by π (Zadeh, 1978). It satisfies the characteristic property:
$$\forall \phi\ \forall \psi\ \ \Pi(\phi \vee \psi) = \max(\Pi(\phi), \Pi(\psi));$$
– the necessity (or certainty) degree $N(\phi) = \inf\{1 - \pi(\omega) \mid \omega \vDash \neg\phi\}$ which evaluates to what extent ϕ is entailed by the available knowledge. We have:
$$\forall \phi\ \forall \psi\ \ N(\phi \wedge \psi) = \min(N(\phi), N(\psi)).$$

The duality between necessity and possibility is expressed by $N(\phi) = 1 - \Pi(\neg\phi)$. Given a possibility distribution π, a notion of preferential entailment $\vDash_\pi$ can be defined.

Definition 1: An interpretation ω is a π-preferential model of a formula ϕ w.r.t. π, which is denoted by $\omega \vDash_\pi \phi$, iff:

i) $\omega \vDash \phi$, ii) $\pi(\omega)>0$ and iii) $\nexists \omega'$, $\omega' \vDash \phi$ and $\pi(\omega')>\pi(\omega)$.

A π-preferential model of ϕ is a normal state of affairs where ϕ is true.

Definition 2: A formula ψ is said to be a *conditional conclusion* of a fact ϕ, with background knowledge π, which is denoted by $\phi \vDash_\pi \psi$, iff

i) $\prod(\phi)>0$, and

ii) each π-preferential model of ϕ satisfies ψ, i.e., $\forall\omega$, $\omega \vDash_\pi \phi \Rightarrow \omega \vDash \psi$.

2.2 - From Possibilistic Knowledge Bases to Possibility Distributions

In particular, a possibilistic knowledge base made of one formula $\{(\phi\ a)\}$ is represented by the possibility distribution:

$$\forall\omega\in\Omega,\ \pi_{\{(\phi\ a)\}}(\omega) \begin{cases} = 1 & \text{if } \omega\vDash\phi \\ = 1-a & \text{otherwise.}\end{cases}$$

Thus, π_Σ can be viewed as the result of the conjunctive combination of the $\pi_{\{(\phi_i\ a_i)\}}$'s using the min operator, that is, a fuzzy intersection. The possibility distribution π_Σ is not necessarily normal, and $\mathrm{Inc}(\Sigma) = 1 - \max_{\omega\in\Omega} \pi_\Sigma(\omega)$ is called the degree of inconsistency of the knowledge base Σ.

Lastly, note that several syntactically different possibilistic knowledge bases may have the same possibility distribution as a semantics counterpart. In such a case, it can be shown that these knowledge bases are equivalent in the following sense: their a-cuts, which are classical knowledge bases, are logically equivalent in the usual sense, where the a-cut of a possibilistic knowledge base Σ is the set of classical formulas whose level of certainty is greater than or equal to a.

2.3 - Possibilistic Inference

The possibilistic logic inference can be performed at the syntactical level by means of a weighted version of the resolution principle:

$$\frac{\begin{array}{c}(\phi\vee\psi\ a)\\ (\neg\phi\vee\delta\ b)\end{array}}{(\psi\vee\delta\ \min(a,b))}$$

It has been shown that $\mathrm{Inc}(\Sigma)$ corresponds to the greatest lower bound that can be obtained for the empty clause by the repeated use of the above resolution rule and using a refutation strategy. Proving $(\psi\ a)$ from a possibilistic knowledge base Σ comes down to deriving the contradiction $(\perp\ a)$ from $\Sigma\cup\{(\neg\psi\ 1)\}$ with a weight $a > \mathrm{Inc}(\Sigma)$. It will be denoted by $\Sigma\vdash(\psi\ a)$. This inference method is as efficient as classical logic refutation by resolution, and can be implemented in the form of an A*-like algorithm (Dubois et al., 1987). This inference method is sound and complete with respect to the possibilistic semantics of the knowledge base. Namely (Dubois et al., 1994):

$\phi \vDash_{\pi_\Sigma} \psi$ if and only if $\Sigma' \vdash (\psi\ a)$ with $\Sigma' = \Sigma \cup \{(\phi\ 1)\}$, and $a > \mathrm{Inc}(\Sigma \cup \{(\phi\ 1)\})$.

Clearly possibilistic reasoning copes with partial inconsistency. It yields non-trivial conclusions by using a consistent sub-part of Σ, consistent with ϕ, which corresponds to formulas belonging to the layers having sufficiently high levels of certainty. Moreover, possibilistic logic offers a syntactic inference whose complexity is similar to classical logic.

3 - Merging Possibility Distributions

In (Dubois and Prade, 1992), several propositions have been made to address the problem of combining n uncertain pieces of information represented by n possibility distributions $\pi_{i=1,n}$ (encoding the knowledge of n experts or sources of information about some parameters of interest), into a new possibility distribution. All the proposed combination modes are defined at the semantical level, and their syntactic counterpart will be investigated in Section 4. We assume that the experts who provide the π_i's use the same universe of discourse to describe their information, and also use the same scale to evaluate the levels of uncertainty.

Let us now review the different combination modes on possibility distributions.

3.1 - Idempotent Conjunctive and Disjunctive Modes

The basic combination modes in the possibilistic setting are the conjunction (i.e., the minimum) and the disjunction (i.e., the maximum) of possibility distributions. Namely define:

$$\forall\omega,\ \pi_{cm}(\omega) = \min_{i=1,n} \pi_i(\omega), \qquad \text{(CM)}$$

$$\forall\omega,\ \pi_{dm}(\omega) = \max_{i=1,n} \pi_i(\omega). \qquad \text{(DM)}$$

If the information provided by a source k is less (resp. more) specific than the information given by all the others then $\pi_{dm}=\pi_k$ (resp. $\pi_{cm}=\pi_k$). The conjunctive aggregation makes sense if all the sources are regarded as equally and fully reliable since all values that are considered as impossible by one source but possible by the others are rejected, while the disjunctive aggregation corresponds to a weaker reliability hypothesis, namely, in the group of sources there is at least one reliable source for sure, but we do not know which one. If $\forall\omega$, $\pi_{cm}(\omega)$ is significantly smaller than 1 the conjunctive mode of combination is debatable since in that case at least one of the sources or experts is likely to be wrong, and a disjunctive combination might be more advisable. It is why disjunctive combinations of possibility distributions should be regarded as *cautious* operations, while conjunctive combinations are more *adventurous*. Besides, if two sources provide the same information $\pi_1=\pi_2$, the result of the conjunctive, or of the disjunctive combination is still the same distribution; indeed min (resp. max) is the only idempotent conjunction (resp. disjunction) connective.

3.2 - Other t-Norm and t-Conorm-Based Combination Modes

The previous combination modes based on maximum and minimum operators have no reinforcement effect. Namely, if expert 1 assigns possibility $\pi_1(\omega)<1$ to interpretation ω, and expert 2 assigns possibility $\pi_2(\omega)<1$ to this interpretation then overall, in the conjunctive mode, $\pi(\omega)=\pi_1(\omega)$ if $\pi_1(\omega)<\pi_2(\omega)$, regardless of the value of $\pi_2(\omega)$. However since both experts consider ω as rather impossible, and if these opinions are independent, it may sound reasonable to consider ω as less possible than what each of the experts claims. More generally, if a pool of independent experts is divided into two unequal groups that disagree, we may want to favor the opinion of the biggest group. This type of combination cannot be modelled by the minimum operation, nor by any idempotent operation. What is needed is a reinforcement effect. A reinforcement effect can be obtained using a triangular norm operation other than min in case of conjunctive, thus adventurous, combination, and a triangular conorm operation other than max for disjunctive, thus cautious, combination.

Definition 3: A triangular norm (for short t-norm) tn is a two place real-valued function whose domain is the unit square $[0,1]\times[0,1]$ and which satisfies the following conditions:

1. $\mathrm{tn}(0,0) = 0$, $\mathrm{tn}(a,1) = \mathrm{tn}(1,a) = a$ (boundary conditions);
2. $\mathrm{tn}(a,b) \le \mathrm{tn}(c,d)$ whenever $a\le c$ and $b\le d$ (monotonicity);
3. $\mathrm{tn}(a,b) = \mathrm{tn}(b,a)$ (symmetry);
4. $\mathrm{tn}(a, \mathrm{tn}(b,c)) = \mathrm{tn}(\mathrm{tn}(a,b), c)$ (associativity).

A triangular conorm (for short t-conorm) ct is a two place real-valued function whose domain is the unit square $[0,1] \times [0,1]$ and which satisfies the conditions 2-4 given in the previous definition plus the following boundary conditions:

5. $\mathrm{ct}(1,1)=1$, $\mathrm{ct}(a,0)=\mathrm{ct}(0,a)=a$.

Any t-conorm ct can be generated from a t-norm through the duality transformation:

$$\mathrm{ct}(a,b) = 1 - \mathrm{tn}(1-a, 1-b)$$

and conversely. The basic t-norms are the minimum operator, the product operator and the t-norm $\max(0, a+b-1)$ sometimes called "Lukasiewicz t-norm" (since it is directly related to Lukasiewicz many-valued implication). The duality relation respectively yields the following t-conorms: the maximum operator, the "probabilistic sum" $a+b-ab$, and the "bounded sum" $\min(1, a+b)$.

In particular, several authors (e.g., Dubois and Prade, 1992; Boldrin and Sossai, 1995) have proposed to use Lukasiewicz operator for combining possibility distributions defined by (for two possibility distributions π_1 and π_2):

$$\forall\omega,\ \pi_{LM}(\omega) = \max(0, \pi_1(\omega) + \pi_2(\omega) - 1). \qquad \text{(LM)}$$

Like the product-based combination, the (LM) combination mode is not idempotent; moreover the reinforcement effect with Lukasiewicz t-norm is more drastic since the combination of low degrees (i.e., $\pi_1(\omega) + \pi_2(\omega) < 1$) may result into a null degree of possibility which expresses complete impossibility.

We shall denote by π_{tn} and π_{ct} the possibility distributions resulting from the combination using a t-norm operator tn and a t-conorm operator ct respectively.

3.3 - Normalisation Rules

Disjunctive combination preserves normalisation (for any triangular conorm). When conjunctive combination mode $\mathcal{C}$ provides subnormal results ($\nexists\omega$, $\pi_{\mathcal{C}}(\omega)=1$), we may think of renormalizing $\pi_{\mathcal{C}}$. Indeed, let $h(\pi_{\mathcal{C}})=\max_{\omega}\{\pi_{\mathcal{C}}(\omega)\}$. It estimates to what extent there exists at least one interpretation which is possible according to each source. Thus, $h(\pi_{\mathcal{C}})$ is a degree of consistency of the pieces of information provided by the different sources. When $h(\pi_{\mathcal{C}})<1$ there is a partial inconsistency between the sources. There are several ways to renormalize $\pi_{\mathcal{C}}$ into $\pi_{N-\mathcal{C}}$. The minimal requirements for $\pi_{N-\mathcal{C}}$ are:

1) $\exists\omega$, $\pi_{N-\mathcal{C}}(\omega) = 1$,
2) if $\pi_{\mathcal{C}}$ is normal then $\pi_{N-\mathcal{C}} = \pi_{\mathcal{C}}$,
3) $\forall\omega,\omega'$, $\pi_{\mathcal{C}}(\omega)<\pi_{\mathcal{C}}(\omega')$ iff $\pi_{N-\mathcal{C}}(\omega)<\pi_{N-\mathcal{C}}(\omega')$.

Note that the third condition entails that only interpretations having possibility degrees equal to $h(\pi_{\mathcal{C}})$ can receive value 1 in the normalisation process. In the following we consider three noticeable renormalizing procedures defined by the three following equations where it is assumed that $h(\pi_{\mathcal{C}})>0$.[2]

- (N-1) $\quad \pi_{N1-\mathcal{C}}(\omega) = \dfrac{\pi_{\mathcal{C}}(\omega)}{h(\pi_{\mathcal{C}})}$.
- (N-2) $\quad \pi_{N2-\mathcal{C}}(\omega) = 1$ if $\pi_{\mathcal{C}}(\omega) = h(\pi_{\mathcal{C}})$
  $= \pi_{\mathcal{C}}(\omega)$ otherwise.
- (N-3) $\quad \pi_{N3-\mathcal{C}}(\omega) = \pi_{\mathcal{C}}(\omega) + (1 - h(\pi_{\mathcal{C}}))$.

All the normalization rules get rid of inconsistency since $h(\pi_{Ni-\mathcal{C}})=1$ for i=1,3, and thus the inconsistency degree of any possibilistic knowledge base associated to $\pi_{Ni-\mathcal{C}}$ for i=1,3 is zero. The normalisation based on the equation (N-1) is the most usual one and is anologuous to the one used in probability theory. As (N-1), normalisation (N-2) is related to a conditioning in possibility theory. Indeed, the conditioning of a possibility distribution π by a formula ϕ held for true is either defined by (Dubois and Prade, 1988a):

$$
\begin{aligned}
-\ \pi(\omega|\phi) &= 1 \text{ if } \pi(\omega) = \max\{\pi(\omega) \mid \omega\models\phi\} = \Pi(\phi)\\
&= \pi(\omega) \text{ if } \omega\models\phi \text{ and } \pi(\omega)<\Pi(\phi)\\
&= 0 \text{ if } \omega \not\models \phi;
\end{aligned}
$$

in a purely ordinal setting (where the grades are only encoding a linear ordering), or using [0,1] as a ratio scale, by

$$
\begin{aligned}
-\ \pi(\omega|\phi) &= \frac{\pi(\omega)}{\Pi(\phi)} \text{ if } \omega\models\phi.\\
&= 0 \text{ if } \omega \not\models \phi.
\end{aligned}
$$

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[2] When $h(\pi_{\mathcal{C}})=0$ there is a total disagreement between the sources, and conjunctive combination makes no sense in this case.

Clearly, these two types of conditioning are particular cases of (N-2) and (N-1) respectively (viewing the set of models of ϕ as a binary possibility distribution). The third normalisation rule (N-3), first introduced by Yager (1987) in the Dempster-Shafer framework, transforms the amount of conflict $1-h(\pi_{\mathfrak{C}})$ into a level of total ignorance. Indeed, any interpretation ω in Ω gets a possibility degree at least equal to $1-h(\pi_{\mathfrak{C}})$.

In general, an associative combination rule does not remain associative under normalization $(N\text{-}i)_{i=1,3}$. However, in the case of rule (N-1), associativity holds if $\pi_{\mathfrak{C}}$ is defined by means of the product t-norm (Dubois and Prade, 1988b). When associativity does not hold, we have first to combine all the possibility distributions before normalizing the result.

3.4 - Conflict-Respectful Combination Rule

As explained above the renormalization erases the conflict between the sources expressed by the subnormalization. Even if it is good that the result of the combination focuses on the values on which all the sources partially agree (in the sense that none of them gave to these values a possibility equal to 0), it would be better that the result also takes into account the conflict in some way. A natural idea for taking into account a partial conflict is to discount the result given by (N-i) for i=1,3, by a weight corresponding to the lack of normalization, i.e., $1-h(\pi_{\mathfrak{C}})$, thus partial inconsistency is transformed into partial uncertainty. Namely $1-h(\pi_{\mathfrak{C}})$ is viewed as the degree of possibility that both sources are wrong since when $h(\pi_{\mathfrak{C}})=1$ nothing suggests that a source is wrong. This discounting leads to the following modified conjunctive combination rule:

$$\forall\ \omega,\ \pi_{CoR-N_i-\mathfrak{C}}(\omega) = \max(\pi_{N_i-\mathfrak{C}}(\omega),\ 1-h(\pi_{\mathfrak{C}})) \qquad \text{(CoRM)}$$

The amount of conflict $1-h(\pi_{\mathfrak{C}})$ induces a uniform level of possibility for all values outside the ones emerging in the subnormalized intersection of π_i's, i.e., the result of the combination $(N\text{-}i-\mathfrak{C})$ is not fully certain. Note that this rule leaves $\pi_{N\text{-}3\text{-}\mathfrak{C}}$ unchanged since N-3 already incorporates a discounting effect.

3.5 - Prioritized Aggregation of Expert Opinions

When combining information conjunctively or disjunctively, we may give priority to some sources over others, and thus discount the information provided by a less prioritary source if this information is not in agreement with the information provided by a more prioritary source. In the following, we only consider min and max-based combinations. Then the following combination rule has been proposed (for the case of two possibility distributions, π_1 and π_2, where π_1 is more reliable than π_2) (Dubois and Prade, 1988c; Yager, 1991):

$$\forall\omega,\ \pi_{CPM1>2}(\omega) = \min(\pi_1(\omega),\ \max(\pi_2(\omega), 1-h(\pi_{cm}))). \qquad \text{(CPM)}$$

Note that in case of total conflict, i.e., $h(\pi_{cm})=0$, only the information which has priority, represented by π_1, is retained ($\pi_{CPM1>2} = \pi_1$). More generally, if $1-h(\pi_{cm})\geq$

0.5 then for any ω, $\pi_{CPM1>2}(\omega)=\min(\pi_1(\omega),1-h(\pi_{cm}))$.[3] Moreover, if there is no conflict then the usual conjunctive combination is performed: $\pi_{CPM1>2}=\pi_{cm}$. The CPM rule is easily interpreted as the conjunctive combination of the information supplied by source 1 and the information supplied by source 2, the latter being discounted by a certainty coefficient $h(\pi_{cm})$, such that the degree of possibility that source 2 is wrong is $1-h(\pi_{cm})$.

The disjunctive counterpart of (CPM) has also been proposed by Dubois and Prade (1988d) and Yager (1991)

$$\forall\omega,\ \pi_{DPM1>2}(\omega) = \max(\pi_1(\omega),\ \min(\pi_2(\omega),h(\pi_{cm}))). \qquad \text{(DPM)}$$

The effect of this rule is to "truncate" the information supplied by the less prioritary source (by performing $\min(\pi_2,h(\pi_{cm}))$ in the above formula), while it is disjunctively combined with information given by source 1. Again if the two sources disagree ($h(\pi_{cm})=0$) then $\pi_{DPM1>2} = \pi_1$; if $h(\pi_{cm})=1$ then $\pi_{DPM1>2} = \pi_{dm}$.

3.6 - Grading the Reliability of the Sources

We assume that a relative level of reliability λ_j is attached to each source s_j. These reliability levels can be used for weighting conjunctive or disjunctive combinations. A normalisation condition is supposed to hold for the λ_j's, namely $\max_j\lambda_j=1$, i.e., the most reliable source(s) is graded by 1. We only consider min and max-based combination again. The weighted disjunctive combination mode is defined by (e.g., Dubois et al., 1992):

$$\forall\omega,\ \pi_{DRM}(\omega) = \max_{j=1,n} \min(\pi_j(\omega),\lambda_j) \qquad \text{(DRM)}$$

i.e., π_{DRM} is obtained as a weighted union of the possibility distributions associated with each source. If all the λ_j's are equal to 1 then the disjunction mode (DM) applied to the π_j's is recovered. If $\lambda_j=0$ the information provided by the source s_j is not taken into account. For intermediary λ_j, only the possible worlds whose possibility degree is low (which are rather impossible) are taken into account.

The conjunctive counterpart of (DRM) is defined by (e.g., Dubois et al., 1992):

$$\forall\omega,\ \pi_{CRM}(\omega) = \min_{j=1,n} \max(\pi_j(\omega),\ 1-\lambda_j) \qquad \text{(CRM)}$$

For $\lambda_j=1$, $\forall j$, the min combination mode is recovered. Note that (CRM) and (DRM) are similar to (CPM) and (DPM). One may think of combining them in order to take into account both the conflict of the sources and the reliability degrees attached to the sources. Assuming that $\lambda_1=1>\lambda_2$ (we give the priority to the most reliable source), for the case of two possibility distributions this leads to:

$$\pi_{CPRM1>2} = \min(\pi_1,\ \max(\pi_2,\ 1-\lambda_2,\ 1-h(\pi_{cm})); \qquad \text{(CPRM)}$$

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[3] Since $\pi_{CPM1>2}(\omega)$ can be rewritten as $\max(\min(\pi_1(\omega),\pi_2(\omega)),\ \min(\pi_1(\omega),\ 1-h(\pi_{cm}))$ and $h(\pi_{cm})\leq 0.5 \Rightarrow 1-h(\pi_{cm})\geq 0.5\geq \min(\pi_1,\pi_2)$.

$$\pi_{DPRM1>2} = \max(\pi_1, \min(\pi_2, \lambda_2, h(\pi_{cm}))). \qquad \text{(DPRM)}$$

These types of prioritized combination rules have been proposed by Yager (1991) and studied in detail by Kelman (1996).

4 - Syntactical Combination Modes

In Section 3, several propositions have been made to aggregate possibility distributions. In this section, we are interested in the combination of n possibilistic knowledge bases $\Sigma_{i=1,n}$ provided by n sources. Each possibilistic knowledge base Σ_i is associated with a possibility distribution π_i which is its semantical counterpart (i.e., $\forall\phi,\psi$, $\exists a>Inc(\Sigma_i\cup\{(\phi\ 1)\})$, $\Sigma_i\cup\{(\phi\ 1)\}\vdash(\psi\ a)$ iff $\phi \models_{\pi_i} \psi$). So we need to identify syntactical combination modes C on the Σ_i's which are the counterpart of the combination rules $\mathfrak{C}$ on the π_i's reviewed in the previous section. More formally, given a semantic combination rule $\mathfrak{C}$ we look for a syntactic combination C such that:

$$\mathfrak{C}(\pi_{\Sigma_1}, \ldots,\pi_{\Sigma_n}) = \pi_{\mathsf{C}(\Sigma_1,\ldots\Sigma_n)}.$$

or equivalently,

$$\phi \models_{\pi_{\mathfrak{C}}} \psi \quad \text{iff} \quad \Sigma_{\mathsf{C}} \cup \{(\phi\ 1)\} \vdash (\psi\ a) \text{ with } a>Inc(\Sigma_i\cup\{(\phi\ 1)\}),$$

where $\pi_{\mathfrak{C}}=\mathfrak{C}(\pi_{\Sigma_1}, \ldots,\pi_{\Sigma_n})$ and $\Sigma_{\mathsf{C}}=\mathsf{C}(\Sigma_1,\ldots\Sigma_n)$.

4.1 - t-Norm and t-Conorm-Based Combination Modes

In the framework of substructural logics, Boldrin and Sossai (1995) have proposed an extension of possibilistic logic where a second "and" connective (other than the min-based conjunction), defined from Lukasiewicz t-norm, is introduced for combining information from distinct independent sources. At the syntactic level this new conjunction applied to two one-formula knowledge bases $\Sigma_1=\{(\phi\ a)\}$ and $\Sigma_2=\{(\psi\ b)\}$ results in three possibilistic formulas: $\Sigma_{LM}=\{(\neg\phi\vee\psi\ b), (\phi\vee\neg\psi\ a), (\phi\vee\psi\ \min(1,a+b))\}$ which expresses an adventurous combination mode. Note that this knowledge is equivalent to the following: $\Sigma_{LM}\equiv\{(\psi\ b), (\phi\ a), (\phi\vee\psi\ \min(1,a+b))\}=\Sigma_1\cup\Sigma_2\cup\{(\phi\vee\psi\ \min(1,a+b))\}$, the equivalence being understood as the equality of the associated possibility distributions. This can be easily checked. Indeed, there are two different cases:

- If $\omega\models\phi$ or $\omega\models\psi$ then this trivially implies that $\omega\models\phi\vee\psi$. Hence:

$$\begin{aligned}\pi_{\Sigma_{LM}}(\omega) &=1=\max(0,\pi_1(\omega) + \pi_2(\omega)-1) && \text{if } \omega\models\phi\wedge\psi && (\text{since } \pi_1(\omega)=\pi_2(\omega)=1)\\ &= 1-b=\max(0,\pi_1(\omega) + \pi_2(\omega)-1) && \text{if } \omega\models\phi\wedge\neg\psi && (\text{since } \pi_1(\omega)=1)\\ &= 1-a=\max(0,\pi_1(\omega) + \pi_2(\omega)-1) && \text{if } \omega\models\neg\phi\wedge\psi && (\text{since } \pi_2(\omega)=1)\end{aligned}$$

- If $\omega \not\models \phi$ and $\omega \not\models \psi$ then this implies that $\omega \not\models \phi\vee\psi$. Hence:

$$\begin{aligned}\pi_{\Sigma_{LM}}(\omega) &= 1-\max(a,b,\min(1, a+b))\\ &= 1-\min(1,a+b)\\ &= \max(0,1-a-b)\end{aligned}$$

$= \max(0, 1-(1-\pi_1(\omega)) - (1-\pi_2(\omega)))$ (since $\pi_1(\omega)=1-a$ and $\pi_2(\omega)=1-b$)

$= \max(0, \pi_1(\omega) + \pi_2(\omega) - 1)$.

The previous remark can be generalized to the case of general possibilistic knowledge bases and to any t-norms and t-conorms. Let $\Sigma_1=\{(\phi_i\ a_i) \mid i\in I\}$ and $\Sigma_2=\{(\psi_j\ b_j) \mid j\in J\}$. Namely, we have the following result.

Proposition 1: Let π_{tn} and π_{ct} be the result of the combination based on the t-norm tn and the t-conorm ct. Then, π_{tn} and π_{ct} are respectively associated with the following knowledge bases:

$\Sigma_{tn} = \Sigma_1 \cup \Sigma_2 \cup \{(\phi_i \vee \psi_j\ ct(a_i,b_j)) \mid (\phi_i\ a_i)\in\Sigma_1 \text{ and } (\psi_j\ b_j)\in\Sigma_2\}$ (adventurous combination)
where ct is the t-conorm dual to the t-norm tn.

$\Sigma_{ct} = \{(\phi_i \vee \psi_j\ tn(a_i,b_j)) \mid (\phi_i\ a_i)\in\Sigma_1 \text{ and } (\psi_j\ b_j)\in\Sigma_2\}$ (cautious combination)
where tn is the t-norm dual to the t-conorm ct.

The syntactic counterpart of idempotent conjunctive and disjunctive modes is thus respectively the two following possibilistic knowledge base (letting tn=min and ct=max),

$$\Sigma_{min} = \Sigma_1 \cup \Sigma_2 \cup \{(\phi_i \vee \psi_j\ \max(a_i,b_j)) \mid (\phi_i\ a_i)\in\Sigma_1 \text{ and } (\psi_j\ b_j)\in\Sigma_2\}$$
$$= \Sigma_1 \cup \Sigma_2$$
$$\Sigma_{max} = \{(\phi_i \vee \psi_j\ \min(a_i,b_j)) \mid (\phi_i\ a_i)\in\Sigma_1 \text{ and } (\psi_j\ b_j)\in\Sigma_2\}.$$

Of course, Σ_{min} may be inconsistent and the handling of inconsistency is simply achieved by using the possibilistic entailment; see Section 2. Note that Σ_{max} is always consistent (provided that Σ_1 or Σ_2 is consistent), and that a formula ψ is considered as a plausible consequence of Σ_{max} if ψ is inferred both from Σ_1 and Σ_2. In the fuzzy case, the fuzzy set of possibilistic consequences of Σ_{max} can also be viewed as the fuzzy intersection of the fuzzy set of possibilistic consequences of Σ_1 and the fuzzy set of those of Σ_2.

4.2 - Normalization

This section discusses the effect of the normalization at the syntactical level. Let $\pi_{\oplus}$ be a sub-normalized possibility distribution obtained by combining π_1 and π_2 using a conjunction operator $\oplus$. Let $\Sigma_{\oplus}$ be the possibilistic knowledge base associated with $\pi_{\oplus}$ and built using the result of the previous section. Let $h(\pi_{\oplus})=\max_\omega\{\pi_{\oplus}(\omega)\}$. Then we have:

Proposition 2: 1. (N-1) $\pi_{N1-\oplus}(\omega)=\dfrac{\pi_{\oplus}(\omega)}{h(\pi_{\oplus})}$ is associated with:

$$\Sigma_{N1-\oplus} = \left\{\left(\phi_i\ 1-\frac{1-a_i}{h(\pi_{\oplus})}\right) \mid (\phi_i\ a_i)\in\Sigma_{\oplus} \text{ and } a_i > 1-h(\pi_{\oplus})\right\}.$$

2. (N-2) $\pi_{N2-\oplus}(\omega) = 1$ if $\pi_{\oplus}(\omega) = h(\pi_{\oplus})$
$= \pi_{\oplus}(\omega)$ otherwise

is associated with: $\Sigma_{N2-\oplus}=\{(\phi_i\ a_i) \mid (\phi_i\ a_i)\in\Sigma_{\oplus} \text{ and } a_i>1-h(\pi_{\oplus})\}$.

3. (N-3) $\pi_{N3-\mathcal{C}}(\omega) = \pi_{\mathcal{C}}(\omega) + (1-h(\pi_{\mathcal{C}}))$ is associated with:
$\Sigma_{N3-\mathcal{C}} = \{(\phi_i\ a_i - (1-h(\pi_{\mathcal{C}}))) \mid (\phi_i\ a_i) \in \Sigma_{\mathcal{C}}$ and $a_i > 1-h(\pi_{\mathcal{C}})\}$.

Note that $\Sigma_{N1-\mathcal{C}}$ can be defined from $\Sigma_{N3-\mathcal{C}}$ as follows:

$$\Sigma_{N1-\mathcal{C}} = \left\{\left(\phi_i\ \frac{b_i}{h(\pi_{\mathcal{C}})}\right) \mid (\phi_i\ b_i) \in \Sigma_{N3-\mathcal{C}}\right\}.$$

All the normalization procedures maintain all the formulas of $\Sigma_{\mathcal{C}}$ whose certainty degrees are higher than the inconsistency degree of $\Sigma_{\mathcal{C}}$. The normalization based on (N-2) just forgets the presence of conflicts and does not modify the certainty degrees of the pieces of information encoded by $\pi_{\mathcal{C}}$. The two other normalization procedures modify the certainty degrees of the formulas retained in $\Sigma_{\mathcal{C}}$. Let us observe that all the normalization modes diminishes the certainty levels of formulas in the wide sense. This diminution is more important with (N3) than with (N1).

4.3 - Conflict-Respectful, Prioritized and Reliability-Based Combination Modes

Let us first define two parametered functions which operate on a possibilistic knowledge base Σ:

- *Truncate(Σ,a),* which consists in forcing the inconsistency level of Σ to be at least equal to a (which inhibits the formulas in Σ having certainty degrees strictly less than *a)*, and
- *Discount(Σ,a)* which basically consists in decreasing to level *a* the certainty degree of the formulas of Σ whose certainty is higher.

More formally, we have:

Truncate $(\Sigma,a)=\{(\phi\ b) \mid (\phi\ b) \in \Sigma \text{ and } b\geq a\}\cup\{(\perp\ a)\}$
Discount $(\Sigma,a)=\{(\phi\ a) \mid (\phi\ b) \in \Sigma \text{ and } b\geq a\}\cup\{(\phi\ b) \mid (\phi\ b)\in \Sigma \text{ and } b<a\}$.

It is easy to check that each possibilistic conclusion of Truncate(Σ,a) or of Discount(Σ,a) is also a possibilistic conclusion of Σ. The converse is false. The following proposition lays bare the semantical counterparts of these two functions.

Proposition 3: Let Σ be a possibilistic knowledge base, and π its associated possibility distribution. Then:
– Truncate (Σ,a) is associated with $\pi'=\min(\pi,1-a)$,[4] and
– Discount (Σ,a) is associated with $\pi'=\max(\pi,1-a)$.

4 Indeed, $h(\pi')\leq 1-a$ and thus the level of inconsistency is at least equal to a. Moreover, using equation (1) of Section 2.2., it is clear that the formulas $(\phi_i\ b_i)$ with $b_i<a$ have no influence in π' since $1-a<1-b_i$.

The corresponding syntactic counterparts of conflict-respectful, prioritized and reliability-based combination modes are immediate using the discussion of the previous sub-sections, we get the corresponding possibilistic knowledge bases:

$$\Sigma_{CoR-Ni-\mathcal{C}} = \mathrm{Discount}(\Sigma_{Ni-\mathcal{C}}, h(\pi_{\mathcal{C}}))$$
$$C_{CPM1>2}(\Sigma_1,\Sigma_2) = \Sigma_1 \cup \mathrm{Discount}(\Sigma_2, h(\pi_{cm})),$$
$$C_{DPM1>2}(\Sigma_1,\Sigma_2) = C_{dm}(\Sigma_1, \mathrm{Truncate}(\Sigma_2, 1-h(\pi_{cm}))),$$
$$C_{DRM}(\Sigma_i)_{i=1,n} = C_{dm}\,(\mathrm{Truncate}(\Sigma_i, 1-\lambda_i))_{i=1,n},$$
$$C_{CRM}(\Sigma_i)_{i=1,n} = \bigcup_i \mathrm{Discount}(\Sigma_i,\lambda_i).$$

where C_{dm} denotes the combination operator yielding the possibilistic knowledge base Σ_{dm}, and $C_X(\Sigma_i)_{i=1,n}$ is the syntactic combination operator whose results is associated with π_X for X=CPM1>2, PM1>2, DRM, CRM.

Note that the addition of reliability degrees and priorities aims either to decrease the certainty degrees of highest formulas, or simply to remove formulas which are not sufficiently entrenched.

5 - Concluding Discussions

This paper has proposed a preliminary investigation of syntactic combination modes which can be applied to layered propositional logic knowledge bases, and which are the counterparts of possibilistic combination rules applied to the possibility distributions encoding the semantics of the layered bases. This can be viewed as a logical analog of the fusion of belief networks in agreement with the combination laws of probability distributions, see (Matzkevick and Abramson, 1992).

This work belongs to a larger research trend which also encompasses belief revision where syntactic encoding of possibilistic revision methods have been devised (e.g., Dubois and Prade, 1996; Williams, 1996). However, in belief revision the input information does not play a symmetrical role with respect to the knowledge we start with, while in this paper all the sources play symmetrical roles. Moreover the syntactic combination methods are exact counterparts of semantic ones. As a consequence they are drastic since they use truncation effects in the presence of inconsistency. All formulas in layers below the inconsistency level are lost. This is called the drowning effect (Benferhat et al., 1993). One might thing of taking into account the syntax of the knowledge bases when combining, and especially when restoring consistency, in order to delete formulas only when necessary.

Lastly, in this paper the weights attached to formulas were understood as uncertainty levels and the combination laws which have been considered are compatible with this view. They might be also interpreted in terms of priority levels, thus expressing preferences rather than uncertainty. In such a case, we would have to look for counterparts of fuzzy set aggregation operations which are meaningful in multiple criteria evaluation when preference profiles can be viewed as fuzzy sets.

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