Solving multi-criteria decision problems under possibilistic uncertainty using optimistic and pessimistic utilities

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Abstract. This paper proposes a qualitative approach to solve multi-criteria decision making problems under possibilistic uncertainty. Depending on the decision maker attitude with respect to uncertainty (i.e. optimistic or pessimistic) and on her attitude with respect to criteria (i.e. conjunctive or disjunctive), four *ex-ante* and four *ex-post* decision rules are defined and investigated. In particular, their coherence w.r.t. the principle of monotonicity, that allows Dynamic Programming is studied.

1 Introduction

A popular criterion to compare decisions under risk is the expected utility model (EU) axiomatized by Von Neumann and Morgenstern [9]. This model relies on a probabilistic representation of uncertainty: an elementary decision is represented by a probabilistic lottery over the possible outcomes. The preferences of the decision maker are supposed to be captured by a utility function assigning a numerical value to each consequence. The evaluation of a lottery is then performed through the computation of its expected utility (the greater, the better). When several independent criteria are to be taken into account, the utility function is the result of the aggregation of several utility functions u_i (one for each criterion). The expected utility of the additive aggregation can then be used to evaluate the lottery, and it is easy to show that it is equal to the additive aggregation of the mono-criterion expected utilities.

These approaches presuppose that both numerical probability and additive utilities are available. When the information about uncertainty cannot be quantified in a probabilistic way the topic of possibilistic decision theory is often a natural one to consider. Giving up the probabilistic quantification of uncertainty yields to give up the EU criterion as well. The development of possibilistic decision theory has lead to the proposition and the characterization of (mono-criterion) possibilistic counterparts of expected utility: Dubois and Prade [3] propose two criteria based on possibility theory, an optimistic and a pessimistic one, whose definitions only require a finite ordinal scale for evaluating both utility and plausibility. Likewise, qualitative approaches of multi-criteria decision

making have been advocated, leading to the use of Sugeno Integrals (see e.g. [1, 8]) and especially weighted maximum and weighted minimum [2].

In this paper, we consider possibilistic decision problems in the presence of multiple criteria. The difficulty is here to make a double aggregation. Several attitudes are possible: shall we consider the pessimistic/optimistic utility value of a weighted min (or max)? or shall we rather aggregate with a weighted min (or max) the individual pessimistic (or optimistic) utilities provided by the criteria? In short, shall we proceed in an *ex-ante* or *ex-post* way?

The remainder of the paper is organized as follows: Section 2 presents a refresher on possibilistic decision making under uncertainty using Dubois and Prade's pessimistic and optimistic utilities, on one hand, and on the qualitative approaches of MCDM (mainly, weighted min and weighted max), on the other hand. Section 3 develops our proposition, defining four ex-ante and four ex-post aggregations, and shows that when the decision maker attitude is homogeneous, i.e. either fully min-oriented or fully max-oriented, the ex-ante and the ex-post possibilistic aggregations provide the same result. Section 4 studies the monotonicity of these decision rules, in order to determine the applicability of Dynamic Programming to sequential decision making problems.[1]

2 Background on one-stage decision making in a possibilistic framework

2.1 Decision making under possibilistic uncertainty (U^+ and U^-)

Following Dubois and Prade's possibilistic approach of decision making under qualitative uncertainty, a one stage decision can be seen as a possibility distribution over a finite set of outcomes also called a (simple) *possibilistic lottery* [3]. Since we consider a finite setting, we shall write $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$ s.t. $\lambda_i = \pi_f(x_i)$ is the possibility that decision f leads to outcome x_i; this possibility degree can also be denoted by $L[x_i]$. We denote $\mathcal{L}$ the set of all simple possibilistic lotteries.

In this framework, a decision problem is thus fully specified by a set Δ of possibilistic lotteries on a set of consequences X and a utility function $u : X \mapsto [0,1]$. Under the assumption that the utility scale and the possibility scale are commensurate and purely ordinal, Dubois and Prade have proposed the following qualitative degrees for evaluating any simple lottery $L = \langle \lambda_1/x_1, \ldots, \lambda_n/x_n \rangle$:

Optimistic utility (U^+) [3, 15, 16]: $\qquad U^+(L) = \max_{x_i \in X} \min(\lambda_i, u(x_i))$ (1)

Pessimistic utility (U^-) [3, 14]: $\qquad U^-(L) = \min_{x_i \in X} \max((1 - \lambda_i), u(x_i))$ (2)

The value $U^-(L)$ is high only if L gives good consequences in every "rather plausible" state. This criterion generalizes the Wald criterion, which estimates

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the utility of an act by its worst possible consequence. $U^-(L)$ is thus "pessimistic" or "cautious". On the other hand, $U^+(L)$ is a mild version of the maximax criterion which is "optimistic", or "adventurous": act L is good as soon as it is totally plausible that it gives a good consequence.

2.2 Multi-criteria decision making (MCDM) using Agg^+ and Agg^-

The previous setting assumes a clear ranking of X by a single preference criterion, hence the use of a single utility function u. When several criteria, say a set $C = \{c_1...c_p\}$ of p criteria, have to be taken into account, u shall be replaced by a vector $\boldsymbol{u} = \langle u_1, \ldots, u_p \rangle$ of utility functions $u_j : X \mapsto [0,1]$ and the global (qualitative) utility of each consequence $x \in X$ can be evaluated either in a conjunctive, cautious, way according to the Wald aggregation $(Agg^-(x) = \min_{j=1,p} u_j(x))$, or in an disjunctive way according to its max-oriented counterpart $(Agg^+(x) = \max_{j=1,p} u_j(x))$. When the criteria are not equally important, a weight $w_j \in [0,1]$ can be associated to each c_j. Hence the following definitions relative to multi-criteria utilities [2]:

$$Agg^+(x) = \max_{j=1,p} \min(w_j, u_j(x)). \tag{3}$$

$$Agg^-(x) = \min_{j=1,p} \max((1 - w_j), u_j(x)). \tag{4}$$

These utilities are particular cases of the Sugeno integral $[1, 8, 13]$:

$$Agg_{\gamma,u}(L) = \max_{\lambda \in [0,1]} \min(\lambda, \gamma(F_\lambda)) \tag{5}$$

where $F_\lambda = \{c_j \in C, u_j(x) \geq \lambda\}$, γ is a monotonic set-function that reflects the importance of criteria's set. Agg^+ is recovered when γ is the *possibility measure* based on the weight distribution $(\gamma(E) = \max_{c_j \in E} w_j)$, and Agg^- is recovered when γ corresponds to *necessity measure* $(\gamma(E) = \min_{c_j \notin E}(1 - w_j))$.

3 Multi-criteria decision making under possibilistic uncertainty

Let us now study possibilistic decision making in a multi-criteria context. Given a set X of consequences, a set C of independent criteria we define a multi-criteria decision problem under possibilistic uncertainty as triplet $\langle \Delta, \boldsymbol{w}, \boldsymbol{u} \rangle^2$ where:

– Δ is a set of possibilistic lotteries;

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[2] Classical problems of decision under possibilistic uncertainty are recovered when $|C| = 1$; Classical MCDM problems are recovered when all the lotteries in Δ associate possibility 1 to some x_i and possibility 0 to all the other elements of X: Δ is identified to X, i.e. is a set of "alternatives" for the MCDM decision problem.

- $\boldsymbol{w} \in [0,1]^p$ is a weighting vector: w_j denotes the weight of criterion c_j;
- $\boldsymbol{u} = \langle u_1, \ldots, u_p \rangle$ is a vector of p utility functions on X: $u_j(x_i) \in [0,1]$ is the utility of x_i according to criterion c_j;

Our aim consists in comparing lotteries according to decision maker's preferences relative to their different consequences (captured by the utility functions) and the importance of the criteria (captured by the weighting vector). To do this, we can proceed in two different ways namely *ex-ante* or *ex-post*:

- The *ex-ante* aggregation consists in first determining the aggregated utilities (Agg^+ or Agg^-) relative to each possible consequence x_i of L and then combine them with the possibility degrees.
- The *ex-post* aggregation consists in computing the (optimistic or pessimistic) utilities relative to each criterion c_j, and then perform the aggregation (Agg^+ or Agg^-) using the criteria's weights.

We borrow this terminology from economics and social welfare economics, were agents play the role played by criteria in the present context (see e.g. [7, 10]). Setting the problem in a probabilistic context, these works have shown that the two approaches can lead to different results (the so-called "timing effect") coincide iff the collective utility is affine. As a matter of fact, it is easy to show that the expected utility of the weighted sum is the sum of the expected utilities.

Let us go back to possibilistic framework. The decision maker's attitude with respect to uncertainty can be either optimistic (U^+) or pessimistic (U^-) and her attitude with respect to criteria can be either conjunctive (Agg^+) or disjunctive (Agg^-), hence the definition of four approaches of MCDM under uncertainty, namely U^{++}, U^{+-}, U^{-+} and U^{--}; the first (resp. the second) indices denoting the attitude of the decision maker w.r.t. uncertainty (resp. criteria).

Each of these utilities can be computed either *ex-ante* or *ex-post*. Hence the definition of eight utilities:

Definition 1. *Given a possibilistic lottery L on X, a set of criteria C defining a vector of utility functions $\boldsymbol{u}$ and weighting vector $\boldsymbol{w}$, let:*

$$U^{++}_{ante}(L) = \max_{x_i \in X} \min(L[x_i], \max_{c_j \in C} \min(u_j(x_i), w_j)). \tag{6}$$

$$U^{--}_{ante}(L) = \min_{x_i \in X} \max((1 - L[x_i]), \min_{c_j \in C} \max(u_j(x_i), (1 - w_j))). \tag{7}$$

$$U^{+-}_{ante}(L) = \max_{x_i \in X} \min(L[x_i], \min_{c_j \in C} \max(u_j(x_i), (1 - w_j))). \tag{8}$$

$$U^{-+}_{ante}(L) = \min_{x_i \in X} \max((1 - L[x_i]), \max_{c_j \in C} \min(u_j(x_i), w_j)). \tag{9}$$

$$U^{++}_{post}(L) = \max_{c_j \in C} \min(w_j, \max_{x_i \in X} \min(u_j(x_i), L[x_i])). \tag{10}$$

$$U^{--}_{post}(L) = \min_{c_j \in C} \max((1 - w_j), \min_{x_i \in X} \max(u_j(x_i), (1 - L[x_i]))). \tag{11}$$

$$U^{+-}_{post}(L) = \min_{c_j \in C} \max((1 - w_j), \max_{x_i \in X} \min(u_j(x_i), L[x_i])). \tag{12}$$

$$U_{post}^{-+}(L) = \max_{c_j \in C} \min(w_j, \min_{x_i \in X} \max(u_j(x_i), (1 - L[x_i]))). \qquad (13)$$

Interestingly, the optimistic aggregations are related to their pessimistic counterparts by duality as stated by the following proposition.

Proposition 1. *Let $P = \langle \Delta, \boldsymbol{w}, \boldsymbol{u} \rangle$ be a qualitative decision problem, let $P^\tau = \langle \Delta, \boldsymbol{w}, \boldsymbol{u}^\tau \rangle$ be the inverse problem, i.e. the problem such that for any $x_i \in X, c_j \in C, u_j^\tau(x_i) = 1 - u_j(x_i)$. Then, for any $L \in \Delta$:*

$$U_{ante}^{++}(L) = 1 - U_{ante}^{\tau--}(L) \qquad\qquad U_{post}^{++}(L) = 1 - U_{post}^{\tau--}(L)$$
$$U_{ante}^{--}(L) = 1 - U_{ante}^{\tau++}(L) \qquad\qquad U_{post}^{--}(L) = 1 - U_{post}^{\tau++}(L)$$
$$U_{ante}^{+-}(L) = 1 - U_{ante}^{\tau-+}(L) \qquad\qquad U_{post}^{+-}(L) = 1 - U_{post}^{\tau-+}(L)$$
$$U_{ante}^{-+}(L) = 1 - U_{ante}^{\tau+-}(L) \qquad\qquad U_{post}^{-+}(L) = 1 - U_{post}^{\tau+-}(L)$$

As previously said the *ex-ante* and the *ex-post* approaches coincide in the probabilistic case. Likewise, the following Proposition 2 shows that when the decision maker attitude is homogeneous, i.e. either fully min-oriented or fully max-oriented, the *ex-ante* and the *ex-post* possibilistic aggregations provide the same result.

Proposition 2. *For any $L \in \mathcal{L}$, $U_{ante}^{++}(L) = U_{post}^{++}(L)$ and $U_{ante}^{--}(L) = U_{post}^{--}(L)$.*

Hence, for any multi-criteria decision problem under possibilistic uncertainty, U_{ante}^{++} (resp. U_{ante}^{--}) is equal to U_{post}^{++} (resp. U_{post}^{--}). Such an equivalence between the *ex-ante* and *ex-post* does not hold for U^{+-} nor for U^{-+}, as shown by the following counter-example.

Counter-example 1 *Consider a set C of two equally important criteria c_1 and c_2, and a lottery L (cf. Figure 1) leading to two equi-possible consequences x_1 and x_2 such that x_1 is good for c_1 and bad for c_2, and x_2 is bad for c_1 and good for c_2; i.e. $L[x_1] = L[x_2] = 1$, $w_1 = w_2 = 1$, $u_1(x_1) = u_2(x_2) = 1$ and $u_2(x_1) = u_1(x_2) = 0$.*
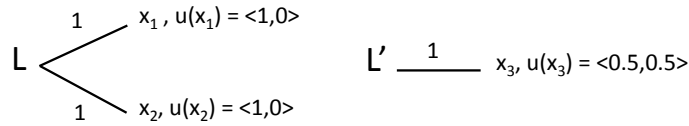
Fig. 1. Lotteries L and L' relative to counter-example 1

We can check that $U_{ante}^{+-}(L) = 0 \neq U_{post}^{+-}(L) = 1$:

$$U_{ante}^{+-}(L) = \max \ (\ \min(L[x_1], \min(\max(u_1(x_1), (1 - w_1)), \max(u_2(x_1), (1 - w_2)))),$$
$$\min(L[x_2], \min(\max(u_1(x_2), (1 - w_1)), \max(u_2(x_2), (1 - w_2)))))).$$
$$= \max \ (\ \min(1, \min(\max(1, (1 - 1)), \max(0, (1 - 1)))) \ ,$$
$$\min(1, \min(\max(0, (1 - 1)), \max(1, (1 - 1))))))$$
$$= 0.$$

$$U_{post}^{+-}(L) = \min \ (\ \max((1-w_1), \max(\min(u_1(x_1), L[x_1]), \min(u_1(x_2), L[x_2]))),$$
$$\max((1-w_2), \max(\min(u_2(x_1), L[x_1]), \min(u_2(x_2), L[x_2]))))).$$
$$= \min \ (\ \max((1-1), \max(\min(1,1), \min(0,1)))\ ,$$
$$\max((1-1), \max(\min(0,1), \min(1,1))))$$
$$= 1\ .$$

The ex-ante and ex-post approaches may lead to different rankings of lotteries. Consider for instance, a lottery L' leading to the consequence x_3 for sure, i.e. $L'[x_3] = 1$ and $L'[x_i] = 0, \forall i \neq x_3$ (such a lottery is called a constant lottery), with $u_1(x_3) = u_2(x_3) = 0.5$. It is easy to check that $U_{ante}^{+-}(L') = U_{post}^{+-}(L') = 0.5$ i.e. $U_{ante}^{+-}(L) < U_{ante}^{+-}(L')$ while $U_{post}^{+-}(L) > U_{post}^{+-}(L')$.

Using the same lotteries L and L', we can show that:
$U_{ante}^{-+}(L) = 1 \neq U_{post}^{-+}(L) = 0$ *and that* $U_{ante}^{-+}(L') = U_{post}^{-+}(L') = 0.5$; *then* $U_{post}^{-+}(L') > U_{post}^{-+}(L)$ *while* $U_{ante}^{-+}(L') < U_{ante}^{-+}(L)$: *like* U^{+-}, U^{-+} *are subject to the timing effect.*

In summary, U^{-+} and U^{+-} suffer from the timing effect, contrary to U^{--} and U^{++}.

4 Multi-criteria sequential decision making under possibilistic uncertainty

Possibilistic sequential decision making relies on *possibilistic compound lotteries*[3], that is a possibility distributions over (simple or compound) lotteries. Compound lotteries indeed allow the representation of decision policies or "strategies", that associate a decision to each decision point: the execution of the decision may lead to several more or less possible situations, where new decisions are to be made, etc. For instance, in a two stages decision problem, a first decision is made and executed; then, depending on the observed situation, a new, one stage, decision is to be made, that lead to the final consequences. The decisions at the final stage are simple lotteries, and the decision made at the first stage branches on each of them. The global strategy thus defines to a compound lottery .

To evaluate a strategy by U^+, U^- or, in the case of MCDM under uncertainty by any of the eight aggregated utility proposed in Section 3, the idea is to "reduce" its compound lottery into an equivalent simple one. Consider a compound lottery $L = \langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle$; the possibility of getting consequence $x_i \in X$ from one of its sub lotteries L_k is $\pi_{k,i} = \min(\lambda_k, L_k[x_i])$ (for the shake of simplicity, suppose that L'_k are simple lotteries; the principle trivially extends to the general case). Hence, the possibility of getting x_i from L is the *max*, over all the L_k's, of $\pi_{k,i}$. Thus, [3] have proposed to reduce a compound lottery $\langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle$ into a simple lottery, denoted by $Reduction(\langle \lambda_1/L_1, \ldots, \lambda_m/L_m \rangle)$, that is considered as equivalent to the compound one: $Reduction(\langle \lambda_1/L_1, \ldots, \lambda_m/L_k \rangle)$ is the simple lottery that associate to each x_i the possibility degree $\max_{k=1..m} \min(\lambda_k, L_k[x_i])$ (with $L[x_i] = 0$ when none of the L_k's give a positive possibility degree to consequence x_i). See Figure 2 for an example.
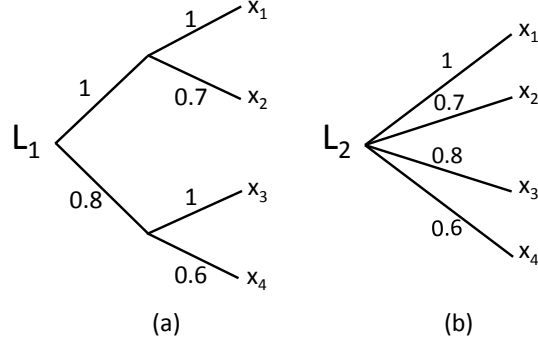
Fig. 2. A compound lottery L_1 (a) and its reduction L_2 (b)

From a practical point of view, sequential decision problems are generally stated through the use of compact representation formalisms, such as possibilistic decision trees [4], possibilistic influence diagrams [5, 6] or possibilistic Markov decision processes [11, 12]. The set of potential strategies to compare, Δ, is generally exponential w.r.t. the input size. So, an explicit evaluation of each strategy in Δ is not realistic. Such problems can nevertheless be solved efficiently, without an explicit evaluation of the strategies, by Dynamic Programming algorithms as soon as the decision rule leads to transitive preferences and satisfies the principle of weak monotonicity. Formally, for any decision rule O (e.g. U^+, U^- or even any of the decision rules proposed in the previous Section) over possibilistic lotteries, $\geq_O$ is said to be weakly monotonic iff whatever L, L' and L'' and whatever (α,β) such that $\max(\alpha, \beta) = 1$:

$$L \succeq_O L' \Rightarrow \langle \alpha/L, \beta/L'' \rangle \succeq_O \langle \alpha/L', \beta/L'' \rangle. \tag{14}$$

Such property ensures that each sub-strategy of an optimal strategy is optimal in its sub-problem. This allows *Dynamic Programming* algorithms to build optimal strategies in an incremental way (e.g. in decision tree, from the last decision to the root of the tree).

[5, 4] have shown that U^+ and U^- are monotonic. Let us now study whether it is also the case for the *ex-ante* and *ex-post* rules proposed in the previous Section. The *ex-ante* approaches are the easiest to handle: once the vectors of utilities have been aggregated according to Agg^- (resp. Agg^+), these approaches collapse with the classical U^+ and U^- approaches. It is then easy to show that:

Proposition 3. $U_{ante}^{++}, U_{ante}^{--}, U_{ante}^{+-}$ and U_{ante}^{-+} satisfy the weak monotonicity.

Concerning U_{post}^{++} and U_{post}^{--}, recall that the full optimistic and full pessimistic *ex-post* utilities are equivalent to their *ex-ante* counterparts thanks to Proposition 2. This allows us to show that:

Proposition 4. U_{post}^{--} and U_{post}^{++} satisfy the weak monotonicity.

It follows from Propositions 3 and 4 that when the decision is based either on an *ex-ante* approach, or on U_{post}^{++} or U_{post}^{--}, the algorithms proposed by Sabbadin et al. [12, 5] can be used on multi-criteria possibilistic decision trees and influence diagrams after their transformation into single-criterion problems: it is enough to aggregate the vectors of utilities leading to the consequences x into single utilities using Agg^+ (for $U_{ante}^{++}, U_{ante}^{-+}, U_{post}^{++}$) or Agg^- (for $U_{ante}^{--}, U_{ante}^{+-}, U_{post}^{--}$) to get an equivalent single criterion problem where the criterion to optimize is simply U^+ (for $U_{ante}^{++}, U_{ante}^{+-}, U_{post}^{++}$) or U^- (for $U_{ante}^{--}, U_{ante}^{-+}, U_{post}^{--}$).

Such approach cannot be applied when optimizing U_{post}^{-+} or U_{post}^{+-}. First because $U_{post}^{+-}(L) \neq U_{ante}^{+-}(L)$ and $U_{post}^{-+}(L) \neq U_{ante}^{-+}(L)$, i.e. the reduction of the problem to the optimization w.r.t. U^+ (resp. U^-) of a single criterion problem obtained by aggregating the utilities with Agg^- (resp. Agg^+) can lead to a wrong result. Worst, it is not even possible to apply Dynamic Programming, since U_{post}^{-+} and U_{post}^{+-} do not satisfy the weak monotonicity property, as shown by the following counter-example:

Counter-example 2 *Let $X = \{x_1, x_2, x_3\}$ and consider two equally important criteria c_1 and c_2 ($w_1 = w_2 = 1$) with : $u_1(x_1) = 1$, $u_1(x_2) = 0.8$, $u_1(x_3) = 0.5$; $u_2(x_1) = 0.6$, $u_2(x_2) = 0.8$, $u_2(x_3) = 0.8$. Consider the lotteries $L = \langle 1/x_1, 0/x_2, 0/x_3 \rangle$, $L' = \langle 0/x_1, 1/x_2, 0/x_3 \rangle$ and $L'' = \langle 0/x_1, 0/x_2, 1/x_3 \rangle$: L gives consequence x_1 for sure, L' gives consequence x_2 for sure and L'' gives consequence x_3 for sure. It holds that:*

$U_{post}^{-+}(L) = Agg^-(x_1) = \max(1, 0.6) = 1$
$U_{post}^{-+}(L') = Agg^-(x_2) = \max(0.8, 0.8) = 0.8.$

Hence $L >_{U_{post}^{-+}} L'$ with respect to the U_{post}^{-+} rule.
Consider now the compound lotteries $L_1 = \langle 1/L, 1/L'' \rangle$ and $L_2 = \langle 1/L', 1/L'' \rangle$. If the weak monotonicity principle were satisfied, we would get: $L_1 >_{U_{post}^{-+}} L_2$. Since: $Reduction(\langle 1/L, 1/L'' \rangle) = \langle 1/x_1, 0/x_2, 1/x_3 \rangle$ and $Reduction(\langle 1/L', 1/L'' \rangle) = \langle 0/x_1, 1/x_2, 1/x_3 \rangle$. We have:
$U_{post}^{-+}(L_1) = U_{post}^{-+}(Reduction(\langle 1/L, 1/L'' \rangle)) = 0.6.$
$U_{post}^{-+}(L_2) = U_{post}^{-+}(Reduction(\langle 1/L', 1/L'' \rangle)) = 0.8.$
Hence, $L_1 <_{U_{post}^{-+}} L_2$ while $L >_{U_{post}^{-+}} L'$, which contradicts weak monotonicity.

Using the fact that $U_{post}^{+-} = 1 - U_{post}^{\tau-+}$, this counter-example is modified to show that also U_{post}^{+-} does not satisfy the monotonicity principle. Consider two equally important criteria, c_1^τ and c_2^τ with $w_1 = w_2 = 1$ with: $u_1^\tau(x_1) = 0$, $u_1^\tau(x_2) = 0.2$, $u_1^\tau(x_3) = 0.5$; $u_2^\tau(x_1) = 0.4$, $u_2^\tau(x_2) = 0.2$, $u_2^\tau(x_3) = 0.2$. Consider now the same lotteries L, L' and L'' presented above. It holds that:

$U_{post}^{+-}(L) = Agg^-(x_1) = 0 < U_{post}^{+-}(L') = Agg^-(x_2) = 0.2$, *while*
$U_{post}^{+-}(Reduction(\langle 1/L, 1/L'' \rangle)) = 0.4 > U_{post}^{+-}(Reduction(\langle 1/L', 1/L'' \rangle)) = 0.2.$

The lack of monotonicity of U_{post}^{-+} is not as dramatic as it may seem. When optimizing $U_{post}^{-+}(L)$, the decision maker is looking for a strategy that is good

w.r.t. U^- for at least one criterion. This means that if it is possible to get for each criterion c_j a strategy that optimizes U^- according to this criterion (and this can be done by Dynamic Programming, since U^- do satisfy the principle of monotonicity), the one with the higher U^- is optimal w.r.t. $U_{post}^{-+}(L)$. Formally:

Proposition 5. $U_{post}^{-+}(L) = \max_{j=1,p} \min(w_j, U_j^-(L))$

where $U_j^-(L) = \min_{x_i \in X} \max((1 - L[x_i]), u_j(x_i))$ is the pessimistic utility of L according to the sole criterion j.

Proposition 6. Let $\mathcal{L}$ be the set of lotteries that can be built on X and let:

- $\Delta^* = \{L_1^*, \ldots, L_p^*\}$ s.t. $\forall L \in \mathcal{L}, j \in 1 \ldots p, U_j^-(L_j^*) \geq U_j^-(L)$;
- $L^* \in \Delta^*$ s.t. $\forall L_j^* \in \Delta^*$: $\max_{j=1,p} \min(w_j, U_j^-(L^*)) \geq \max_{j=1,p} \min(w_j, U_j^-(L_i^*))$.

It holds that, for any $L \in \mathcal{L}, U_{post}^{-+}(L^*) \geq U_{post}^{-+}(L)$.

Hence, it is enough to optimize w.r.t. each criterion separately and to compare the results to get an optimal strategy w.r.t. $U_{post}^{-+}(L)$.

Let us finally study U_{post}^{+-}; an analog of Proposition 6 exists:

Proposition 7. $U_{post}^{+-}(L) = \min_{j=1,p} \max((1 - w_j), U_j^+(L))$

where $U_j^+(L) = \max_{x_i \in X} \min(L[x_i], u_j(x_i))$ is the optimistic utility of L according to the sole criterion j.

But this proposition is helpless, since the lottery L maximizing this quantity is not necessarily among those maximizing the U_j^+'s: one lottery optimal for U_1^+ w.r.t. criterion c_1 can be very bad for U_2^+ and thus bad for U_{post}^{+-}.

5   Conclusion

This paper has provided a first decision theoretical approach for evaluating multi-criteria decision problems under possibilistic uncertainty. The combination of the multi-criteria dimension, namely the conjunctive aggregation with a weighted min (Agg^-) or the disjunctive aggregation with a weighted max (Agg^-) and the decision maker's attitude with respect to uncertainty (i.e. optimistic utility U^+ or pessimistic utility U^-) leads to four approaches of MCDM under possibilistic uncertainty. Considering that each of these utilities can be computed either *ex-ante* or *ex-post*, we have proposed the definition of eight aggregations, that eventually reduce to six: U_{ante}^{++} (resp. U_{ante}^{--}) has been shown to coincide with U_{post}^{++} (resp. U_{post}^{--}); such a coincidence does not happen for U^{+-} and U^{-+}, that suffer from timing effect.

Then, in order to use these decision rules in sequential decision problems, we have proven that all *ex-ante* utilities (i.e. $U_{ante}^{++}, U_{ante}^{--}, U_{ante}^{+-}, U_{ante}^{-+}$) satisfy the weak monotonicity while for the *ex-post* utilities, only U_{post}^{++} and U_{post}^{--} satisfy this property. This result means that Dynamic Programming algorithms can

be used to compute strategies that are optimal w.r.t. the rules . We have also shown that the optimization of U_{post}^{+-} can be handled thanks to a call of a series of optimization of pessimistic utilities (one for each criterion). The question of the optimization of U_{post}^{-+} still remains open.

This preliminary work call for several developments. First of all we intend, as future work, to propose and test suitable algorithms to solve sequential qualitative multi-criteria decision problems, e.g. influence diagrams and decision trees. From a more theoretical point of view we have to propose a general axiomatic characterization of our six decision rules. Finally, considering that the possibilistic aggregations used here are basically specializations of the Sugeno integral, we aim at generalizing the study of MCDM decision making under uncertainty through the development of double Sugeno Integrals.

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Appendix: Proofs

Proof (Proposition 1). The proofs relative to these results are quite similar, so we show that $U_{post}^{+-}(L) = 1 - U_{post}^{\tau-+}(L)$, (i.e. $1 - U_{post}^{+-}(L) = U_{post}^{\tau-+}(L)$) and we can apply the same principle to prove other utilities.

$$1 - U_{post}^{+-}(L) = 1 - [\min_{c_j \in C} \max((1-w_j), \max_{x_i \in X} \min(u_j(x_i), L[x_i]))]$$
$$= \max_{c_j \in C} 1 - [\max((1-w_j), \max_{x_i \in X} \min(u_j(x_i), L[x_i]))]$$
$$= \max_{c_j \in C} \min 1 - [((1-w_j), \max_{x_i \in X} \min(u_j(x_i), L[x_i]))]$$
$$= \max_{c_j \in C} \min(w_j, 1 - [\max_{x_i \in X} \min(u_j(x_i), L[x_i])])$$
$$= \max_{c_j \in C} \min(w_j, \min_{x_i \in X} 1 - [\min(u_j(x_i), L[x_i])])$$
$$= \max_{c_j \in C} \min(w_j, \min_{x_i \in X} \max 1 - [(u_j(x_i), L[x_i])])$$
$$= \max_{c_j \in C} \min(w_j, \min_{x_i \in X} \max(1-u_j(x_i), (1-L[x_i]) = \max_{c_j \in C} \min(w_j, \min_{x_i \in X} \max(u_j^\tau(x_i), (1-$$
$$L[x_i])$$
$$= U_{post}^{\tau-+}(L). \quad \square$$

Proof (Propositon 2). In the following, we illustrate the proof relative to equivalence between $U_{ante}^{++}(L)$ *and* $U_{post}^{++}(L)$. The proof relative to U^{--} is similar by replacing max by min, $L[x_i]$ by $(1 - L[x_i])$ and w_j by $(1 - w_j)$.

$$U_{ante}^{++}(L) = \max_{x_i \in X} \min(L[x_i], \max_{c_j \in C} \min(u_j(x_i), w_j))$$
$$= \max_{x_i \in X} \min(\max(\min(u_1(x_i), w_1), \min(u_2(x_i), w_2), \dots, \min(u_p(x_i), w_p)), L[x_i])$$
$$= \max_{x_i \in X} \max(\min(\min(u_1(x_i), L[x_i]), w_1), \min(\min(u_2(x_i), L[x_i]), w_2), \dots, \min(\min(u_p(x_i), L[x_i]), w_p))$$
$$= \max_{c_j \in C} \max_{x_i \in X}(\min(\min(L[x_i], u_j(x_i))), w_j)$$
$$= \max_{c_j \in C} \max(\min(\min(L[x_1], u_j(x_1)), w_j), \min(\min(L[x_2], u_j(x_2)), w_j), \dots, \min(\min(L[x_n], u_j(x_n)), w_j))$$
$$= \max_{c_j \in C} \min(\max(\min(L[x_1], u_j(x_1)), \min(L[x_2], u_j(x_2)), \dots, \min(L[x_n], u_j(x_n))), w_j)$$
$$= \max_{c_j \in C} \min(\max_{x_i \in X}(\min(L[x_i], u_j x_i)), w_j)$$
$$= U_{post}^{++}(L). \quad \square$$

Proof (Propositon 3). To prove that U_{ante} utilities satisfy the weak monotonicity property, we proceed by proving that U_{ante}^{++} (resp. U_{ante}^{--}) and U_{ante}^{+-} (resp. U_{ante}^{-+}) are equivalent to their equivalent in context of mono-criterion decision making i.e. U^+ (resp.U^-) that are known as monotonic. While the principle is always the same, we will limit to the proof relative to the reduction of U_{ante}^{++} to U^+.
$$U^{++}(\langle \alpha \backslash L_1, \beta \backslash L_2, \dots \gamma L_m \rangle).$$
$$= U^{++}(red(\langle \alpha \backslash L_1, \beta \backslash L_2, \dots \gamma L_m \rangle)).$$
$$= U^+(Agg^+(red(\langle \alpha \backslash L_1, \beta \backslash L_2, \dots \gamma L_m \rangle))).$$

$= U^+(Agg^+(\max(\min(\alpha, L_1[x_1]), \min(\beta, L_2[x_1]), \dots \min(\gamma, L_m[x_1]))\backslash \boldsymbol{u_1},$
$\max(\min(\alpha, L_1[x_2]), \min(\beta, L_2[x_2]), \dots \min(\gamma, L_m[x_2]))\backslash \boldsymbol{u_2},$
$\dots \max(\min(\alpha, L_1[x_q]), \min(\beta, L_2[x_q]), \dots \min(\gamma, L_m[x_q]))\backslash \boldsymbol{u_q}.$
$= U^+(\max(\min(\alpha, L_1[x_1]), \min(\beta, L_2[x_1]), \dots \min(\gamma, L_m[x_1]))\backslash \max(\min(u_{11}, w_1), \min(u_{12}, w_2), \dots \min(u_{1p}, w_p))),$
$\max(\min(\alpha, L_1[x_2]), \min(\beta, L_2[x_2]), \dots \min(\gamma, L_m[x_2]))\backslash \max(\min(u_{21}, w_1), \min(u_{22}, w_2), \dots \min(u_{2p}, w_p))).$
$\dots \max(\min(\alpha, L_1[x_q]), \min(\beta, L_2[x_q]), \dots \min(\gamma, L_m[x_q]))\backslash \max(\min(u_{q1}, w_1), \min(u_{q2}, w_2), \dots \min(u_{qp}, w_p))).$
$= U^+(red(\alpha\backslash\langle L_1[x_1]\backslash \max(\min(u_{11}, w_1), \dots \min(u_{1p}, w_p)), \dots L_1[x_q]\backslash \max(\min(u_{q1}, w_1), \dots \min(u_{qp}, w_p))\rangle,$
$\dots \gamma\backslash\langle L_m[x_1]\backslash \max(\min(u_{11}, w_1), \dots \min(u_{1p}, w_p)), \dots L_m[x_q]\backslash \max(\min(u_{q1}, w_1), \dots \min(u_{qp}, w_p))\rangle.))$
$= U^+(red(\alpha\backslash Agg^+(L_1), \beta\backslash Agg^+(L_2)\dots\gamma\backslash Agg^+(L_m))).$ □

Proof (Proposition 4). We have proven that U_{ante}^{++} (resp. U_{ante}^{--}) equals to U_{post}^{++}(resp. U_{post}^{--}), also we have shown that U_{ante}^{++} and U_{ante}^{--} are monotonic. So, we can conclude that U_{post}^{++} and U_{post}^{++} satisfy the weak monotonicity property.

Proof (Proposition 5). we have defined $U_{post}^{-+}(L) = \max\limits_{j=1,p} \min(w_j, \min\limits_{x_i \in X} \max((1 - L[x_i]), u_j(x_i)))$ and $U_j^-(L) = \min\limits_{x_i \in X} \max((1 - L[x_i]), u_j(x_i))$. So, $U_{post}^{-+}(L)$ can be expressed as follows: $U_{post}^{-+}(L) = \max\limits_{j=1,p} \min(w_j, U_j^-(L))$

Proof (Proposition 6). We suppose that $\Delta^* = \{L_1^*, \dots, L_p^*\}$ and $\forall L \in \mathcal{L}, U_j^-(L_k^*) \geq U_j^-(L_k)$, we have to prove that: for any $L \in \mathcal{L}, U_{post}^{-+}(L^*) \geq U_{post}^{-+}(L)$.
We start by verifying if $\min(w_j, U_j^-(L_k^*)) \geq \min(w_j, U_j^-(L_k))$

- If $(w_j \leq U_j^-(L_k))$ then:
  $\min(w_j, U_j^-(L_k^*)) = \min(w_j, U_j^-(L_k)) = w_j$
- Else if $(w_j \geq U_j^-(L_k))$ then:

  - If $(w_j \leq U_j^-(L_k^*))$ then:
    $\min(w_j, U_j^-(L_k^*)) = w_j \leq \min(w_j, U_j^-(L_k)) = U_j^-(L_k)$
  - If $(w_j \geq U_j^-(L_k^*))$ then:
    $\min(w_j, U_j^-(L_k^*)) = U_j^-(L_k^*) \geq \min(w_j, U_j^-(L_k)) = U_j^-(L_k)$

Then, $\min(w_j, U_j^-(L_k^*)) \geq \min(w_j, U_j^-(L_k))$
So, $\max\limits_j \min(w_j, U_j^-(L_k^*)) \geq \max\limits_j \min(w_j, U_j^-(L_k))$

Proof (Proposition 7). We have defined $U_{post}^{+-}(L) = \min\limits_{j=1,p} \max((1-w_j), \max\limits_{x_i \in X} \min(L[x_i], u_j(x_i)))$ and $U_j^+(L) = \max\limits_{x_i \in X} \min(L[x_i], u_j(x_i))$ So, $U_{post}^{+-}(L)$ can be expressed as follows: $U_{post}^{+-}(L) = \min\limits_{j=1,p} \max((1 - w_j), U_j^+(L)).$