Criticality analysis of activity networks under interval uncertainty

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Abstract This paper reconsiders the Project Evaluation and Review Technique (PERT) scheduling problem when information about task duration is incomplete. We model uncertainty on task durations by intervals. With this problem formulation, our goal is to assert possible and necessary criticality of the different tasks and to compute their possible earliest starting dates, latest starting dates, and floats. This paper combines various results and provides a complete solution to the problem. We present the complexity results of all considered subproblems and efficient algorithms to solve them.

Dedicated to the memory of Professor Stefan Chanas.

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1 Introduction

An activity network is a partially ordered set of activities with given duration times, forming a directed acyclic graph. One basic problem when scheduling an activity network representing a project is that of finding critical activities and determining optimal starting times of activities, so as to minimize the makespan. The first step is to determine the earliest ending time of the project. This problem was posed in the 1950s, in the framework of project management, by Malcolm et al. (1959), and the basic underlying graph-theoretic approach, called Project Evaluation and Review Technique, is now popularized under the acronym PERT. The determination of critical activities is carried out via the critical path method (Kelley 1961). The usual assumption in scheduling is that the duration of each task is precisely known, so that solving the PERT problem is rather simple. However, in project management, the durations of tasks are seldom precisely known in advance, at the time when the plan of the project is designed. Detailed specifications of the methods and resources involved for the realization of activities are often not available when the tentative plan is made up. This difficulty was noticed very early by the authors that introduced the PERT approach. They proposed to model the duration of tasks by probability distributions, and tried to estimate the mean value and standard deviation of earliest starting times of activities. Since then, there has been an extensive literature on probabilistic PERT (see Elmaghraby 1977, 2000 and Adlakha and Kulkarni 1989 for a bibliography and recent views). Even if the task duration times are independent random variables, it is admitted that the

problem of finding the distribution of the ending time of a project is intractable, due to the dependencies induced by the topology of the network (Hagstrom 1988; Kleindorfer 1971; Ludwig et al. 2001; Meilijson and Nadas 1979; Shogan 1977). Another difficulty, not always pointed out, is the possible lack of statistical data validating the choice of activity duration distributions. In fact, probability distributions permit us to model the variability of repetitive tasks, but not uncertainty due to a lack of information (Dubois et al. 1996; Ferson and Ginzburg 1996). Even if statistical data are available, they may be partially inadequate because each project takes place in a specific environment, and is not the exact replica of past projects.

The simplest form of noncommittal uncertainty representation for activity duration is the interval. Assigning some time interval I to an activity duration means that the actual duration of this activity will take some value within I, but it is not possible at present to predict which one. In this paper, every precise instantiation of duration times will be called a *configuration* (it is also called a scenario in the optimization literature (Kouvelis and Yu 1997) and a possible world in formal logic). Using intervals for representing uncertainty in scheduling means that the part of uncertainty about task duration due to partial ignorance prevails because the available statistical information is considered too scarce or irrelevant. So our model is more adapted to unique projects involving nonrepetitive tasks. The managerial value of intervals can be questioned. Indeed, experts may find it difficult to provide narrow intervals (because they may be wrong). Moreover using too wide intervals may make further analysis useless because it is noninformative. So, we do not propose the use of intervals as the definite answer to scheduling uncertainty. However, the importance of studying interval PERT is motivated by several considerations:

- Interval uncertainty is present as soon as information is incomplete. So, solving the extreme case of pure intervals is a first step before considering more elaborate representations of uncertainty where both probability and intervals are combined.
- By and large, an expert working on a project may be reluctant to provide point values for the expected duration for each task. Allowing the collection of a minimal and a maximal duration may be felt as more realistic, even if the elicitation process should force the expert to provide as narrow intervals as possible.
- A more elaborate approach could be to collect both intervals and plausible values from experts. Then the interval PERT results could be seen as providing safeguards on the expected behavior of the project, and a precise reference plan based on plausible estimates could still be used.

Note that resorting to subjective probability for eliciting duration times of nonrepetitive tasks is not very convincing.

Eliciting subjective probabilities would be burdensome to experts, and it is well known that subjective probability is neither a faithful nor a stable representation of partial ignorance (Dubois et al. 1996). More precisely, if all that is known of an activity duration time d is that it lies between two bounds $a < b$, some may consider it natural to model this lack of knowledge by a uniform probability on $[a, b]$. However, if the scale were distorted by a monotonic function, say f, whereby $f(a) < f(d) < f(b)$, then it is clear that the resulting probability distribution on $[f(a), f(b)]$ would not be uniform. If one describes ignorance by uniform probabilities, then the rescaling function seems to generate information out of the blue. Moreover, it seems paradoxical to know the probability distribution precisely, while claiming ignorance. The interval representation stands for all probability distributions with support inside $[a, b]$, which can hardly be challenged as representing ignorance. Of course, following the betting procedure of Bayesians, ignorance forces the expert to bet using a uniform probability. The same betting rates would be produced if the expert knew that the values of d are indeed uniformly distributed. So, the subjective probabilities produced have an ambiguous meaning, and, under a cautious approach one should perhaps on the contrary interpret a subjective uniform probability as a mere interval, unless the expert knows that there is a genuine underlying random phenomenon. See Dubois et al. (2008) for a general approach to reinterpret subjective probabilities as expressing partial ignorance.

Strangely enough, the PERT analysis with ill-known processing times modeled by simple intervals does not seem to have received much attention in the literature. Yet, the computation of the minimal completion time of a project, the determination of critical paths and activities, and the determination of activity floats have been considered as important problems and have been widely acknowledged to be pervaded with uncertainty. However, the overwhelming part of the literature devoted to this topic adopts an orthodox stochastic approach, thus leading to a very complex problem that is still partially unsolved to date (Hagstrom 1988). To the best of the authors' knowledge, interval-valued PERT analysis seems to have existed only as a special case of fuzzy PERT studies that have been proposed since the late 1970s (Chanas and Kamburowski 1981; Dubois 1983; Hapke et al. 1994; Hapke and Słowiński 1996; Loostma 1997; McCahon and Lee 1988; McCahon 1993; Nasution 1994; Prade 1979; Rommelfanger 1994). Recently, Conde (2009) has used the *min–max regret criterion* to find a *robust* set of critical tasks in a project with interval task durations. This criterion is discussed in the book Kouvelis and Yu (1997), which is entirely devoted to robust discrete optimization. In Conde (2009), exact and approximation algorithms are proposed to solve the min–max regret version of the longest path problem with interval

weights in acyclic graphs (see Averbakh and Lebedev 2004; Karasan et al. 2001; Kasperski and Zieliński 2006 for a formulation and complexity results). A determined optimal min–max regret path, which minimizes the maximal deviation (regret) from optimum over all configurations, represents a robust set of critical tasks.

A basic result in standard (deterministic) activity network analysis states that a task is *critical* if and only if its earliest and latest starting dates are equal, and that critical tasks form critical paths, that is, longest paths from the initial node (event) to the final one. So finding the critical paths yields the critical tasks. When the tasks have ill-known durations, modeled by intervals, these results are no longer valid. Namely, floats can no longer be recovered from the intervals containing earliest and latest starting dates, and critical paths may no longer exist, even when critical tasks are present. Instead of being critical or not, as in the deterministic activity network analysis, three cases may now be observed for every task or path: it is surely not critical (necessarily not critical) or surely critical (necessarily critical) or possibly critical. A task (a path) is *necessarily critical* if it is critical whatever the actual values of task durations turn out to be. A task (a path) is *possibly critical* if there are values of task durations leading to a configuration in which it is critical. Necessarily critical paths may fail to exist while necessarily critical tasks may be isolated.

The notions of possible and necessary criticality of tasks and paths are closely related to the min–max regret version of the longest path problem in acyclic directed graphs with interval weights (Karasan et al. 2001; Kasperski and Zieliński 2006). Namely, every optimal min–max regret path is possibly critical and every necessary critical path is optimal min–max regret with zero maximal regret. Hence, an optimal min–max regret path can be viewed as a possibly critical one, which minimizes a "distance" to the necessary criticality. Furthermore, necessarily noncritical tasks can be removed from an instance of the min–max regret longest path without violating optimal min–max regret paths and necessarily critical tasks can be automatically added to a constructed optimal min–max regret path (see Kasperski and Zieliński 2010 for a detailed discussion concerning the above relations).

This paper combines various existing partial results about the complexity of finding intervals containing earliest and latest starting times and floats of tasks (Chanas and Zieliński 2002, 2003; Dubois et al. 2003, 2005; Zieliński 2005) and proposes new results, especially about evaluating necessary criticality, computing the maximal float of a task, and computing the minimal latest starting times for all tasks, thus providing a complete solution to the PERT scheduling problem under the representation of interval-based uncertainty. A set of efficient algorithms for determining the criticality of tasks, the optimal intervals containing their earliest starting dates, latest starting dates, and their floats is provided.

It is shown that the only strongly NP-hard problem is the one of finding minimal floats (Chanas and Zieliński 2003), which is closely related to asserting the possible criticality of a task that turned out to be strongly NP-complete (Chanas and Zieliński 2002). All other problems turn out to be polynomial.

The paper is organized as follows. Section 2 motivates the work by showing the collapse of the usual criticality analysis concepts and techniques when duration times of tasks are imprecisely known. Section 3 proposes an exponential but practically efficient algorithm for achieving the complete criticality analysis under incomplete knowledge and a polynomial algorithm for computing the minima of latest starting dates for all task in a network, based on an enumeration of paths. Section 4 presents polynomial algorithms for asserting necessary criticality of tasks, and computing the optimal intervals containing their latest starting dates, as well as the maximum of float values. These algorithms are constructive in the sense that they select appropriate values of task duration times in a step-by-step manner, such that the computation of some bound comes down to computing a standard PERT problem. An illustrative example is provided in Sect. 5. Section 6 contains exhaustive complexity results and experimental evidence of the efficiency of algorithms.

2 Definition and challenge of the interval-valued model

In this section, we briefly recall how to solve the classical PERT model. Then we formally introduce the interval-valued PERT model, and explain why the classical approach no longer works.

2.1 Solving standard PERT problem

An activity network is classically defined as a set of activities (or tasks) with given duration times, related to each other by means of precedence constraints. When there are no resource constraints, it can be represented by a directed, connected, and acyclic graph $G = \langle V, A \rangle$, where V is the set of nodes (events), $|V| = n$, and $A \subseteq V \times V$ is the set of arcs (activities), $|A| = m$. We use the activity-on-arc convention. The set $V = \{1, 2, \ldots, n\}$ is labeled in such a way that $i < j$ for each activity $(i, j) \in A$. Activity duration times d_{ij} (the weights of the arcs) $(i, j) \in A$ are known. Two nodes 1 and n are distinguished as the *initial* node (source) and *final* node (sink), respectively (no activity enters 1 and no activity leaves n). It is of major concern to minimize the ending time of the last task, also called the *makespan* of the network. Of interest to the project manager are *earliest starting dates*, *latest starting dates* and *floats* of activities. The *critical activities* have zero float. The essence of the PERT method

involves two recurrence formulae, *forward* and *backward* recursions. The earliest starting dates of events $k \in V$ and tasks $(k, l) \in A$, denoted by est_k and est_{kl}, respectively, are determined by means of the forward recursion:

$$\text{est}_k = \begin{cases} 0 & \text{if } k = 1, \\ \max_{j \in \text{Pred}(k)}(\text{est}_j + d_{jk}) & \text{otherwise,} \end{cases} \quad (1)$$

$$\text{est}_{kl} = \text{est}_k, \quad (2)$$

where $\text{Pred}(i)$ refers to the set of nodes that immediately precede node $i \in V$. The earliest starting date est_k of event k is the length of a longest path from the beginning of the project, represented by node 1, to node k. We arbitrarily fix the *starting time* of the project to $\text{est}_1 = 0$. Of course, the earliest starting date of a task (k, l) is equal to the earliest starting date of event k. The *earliest ending time* of the project is the earliest starting date of the last event n. In order to ensure minimal duration of the project, the latest date of event n, denoted by lst_n, is equal to its earliest starting date, $\text{lst}_n = \text{est}_n$. The latest starting date lst_k of event k is the latest time to complete tasks that are ended at k without delaying the end of the project, and thus it is the *latest finishing date* of tasks ending at k. Difference $\text{lst}_n - \text{lst}_k$ is the length of a longest path from node k to node n. Hence, the latest starting date of task (k, l), denoted by lst_{kl}, is equal to $\text{lst}_l - d_{kl}$. The latest starting dates of events and tasks can be found by the use of the backward recursion:

$$\text{lst}_k = \begin{cases} \text{est}_n & \text{if } k = n, \\ \min_{l \in \text{Succ}(k)}(\text{lst}_l - d_{kl}) & \text{otherwise,} \end{cases} \quad (3)$$

$$\text{lst}_{kl} = \text{lst}_l - d_{kl}, \quad (4)$$

where $\text{Succ}(i)$ refers to the set of nodes that immediately follow node $i \in V$. The float of task (k, l), which represents the length of the time window for the beginning of the execution of the task, is the difference between the latest starting date and the earliest starting date:

$$f_{kl} = \text{lst}_{kl} - \text{est}_{kl}. \quad (5)$$

The PERT method computes the earliest starting dates, latest starting dates and floats of tasks in $O(m + n)$ time. One can also determine, simultaneously, critical tasks and critical paths and thus a subgraph consisting of all critical tasks and paths.

In the next section, we will see that some obvious properties of this scheduling model such as: "there exists at least one critical path, which is a longest path in the network", or, "one can form a critical path with critical activities", or yet "each critical activity belongs to a critical path", do not hold anymore under incomplete information.

2.2 Interval-valued problem definition

In practice, duration times of tasks are not well known for several reasons. Some design choices are not fixed because we design a predictive plan. We can also suppose that there can be some uncertainty about the duration times of tasks. It is clearly the case for subcontracted tasks. To take into account this uncertainty, we now model each task duration by an interval. It is easy for an expert working on a project to give a minimal and a maximal expected duration for each task. So activity duration times d_{ij} (weights of the arcs) $(i, j) \in A$ are only known to belong to time intervals $D_{ij} = [d_{ij}^-, d_{ij}^+]$, $d_{ij}^- \geq 0$. This means that we neither know the exact duration times of tasks, nor can we set them precisely. Now, depending on the effective duration of each task (that we do not precisely know), several earliest starting dates, latest starting dates, and floats can be considered. We assume that the task durations are unrelated to one another. A vector $\Omega = (d_{ij})_{(i,j) \in A}$, $d_{ij} \in D_{ij}$, that represents an assignment of duration times d_{ij} to task $(i, j) \in A$ is called a *configuration* (Buckley 1989). Thus, every configuration expresses a realization of the duration times. We denote by $\mathscr{C}$ the set of all the configurations, i.e., $\mathscr{C} = \times_{(i,j) \in A}[d_{ij}^-, d_{ij}^+]$. The duration of task (i, j) in configuration Ω is denoted by $d_{ij}(\Omega)$, $d_{ij}(\Omega) \in D_{ij}$. Among the configurations of $\mathscr{C}$ a crucial role is played by the *extreme* ones, which belong to $\times_{(i,j) \in A}\{d_{ij}^-, d_{ij}^+\}$. Let $B \subseteq A$ be a given subset of activities. We define the extreme configuration Ω_B^+ as the configuration where all activities $(i, j) \in B$ have duration times d_{ij}^+ and all the remaining activities have duration times d_{ij}^-. Similarly, in configuration Ω_B^- all activities $(i, j) \in B$ have duration times d_{ij}^- and all the remaining activities have duration times d_{ij}^+.

Using configurations, the possible values $\text{EST}_{kl} = [\text{est}_{kl}^-, \text{est}_{kl}^+]$ for the earliest starting date est_{kl}, the possible values $\text{LST}_{kl} = [\text{lst}_{kl}^-, \text{lst}_{kl}^+]$ for the latest starting date lst_{kl}, and the possible values $F_{kl} = [f_{kl}^-, f_{kl}^+]$ for the float f_{kl} can be rigorously defined as follows (Dubois et al. 2003; Fargier et al. 2000):

$$\text{est}_{kl}^- = \min_{\Omega \in \mathscr{C}} \text{est}_{kl}(\Omega), \qquad \text{est}_{kl}^+ = \max_{\Omega \in \mathscr{C}} \text{est}_{kl}(\Omega), \quad (6)$$

$$\text{lst}_{kl}^- = \min_{\Omega \in \mathscr{C}} \text{lst}_{kl}(\Omega), \qquad \text{lst}_{kl}^+ = \max_{\Omega \in \mathscr{C}} \text{lst}_{kl}(\Omega), \quad (7)$$

$$f_{kl}^- = \min_{\Omega \in \mathscr{C}} f_{kl}(\Omega), \qquad f_{kl}^+ = \max_{\Omega \in \mathscr{C}} f_{kl}(\Omega), \quad (8)$$

where $\text{est}_{kl}(\Omega)$, $\text{lst}_{kl}(\Omega)$, and $f_{kl}(\Omega)$ denote the earliest and the latest starting date and the float of activity (k, l) in configuration Ω, respectively.

Criticality in interval-valued problems is defined as follows (Chanas et al. 2002): a task $(k, l) \in A$ (resp. a path p in network G from node 1 to node n) is *possibly critical* if there exists a configuration $\Omega \in \mathscr{C}$ in which (k, l) (resp.

path p) is critical in the usual sense. A task $(k, l) \in A$ (resp. a path p in network G from 1 to n) is *necessarily critical* if (k, l) (resp. path p) is critical in the usual sense in all configurations $\Omega \in \mathscr{C}$.

In order to compute the intervals of possible values of earliest starting dates, latest starting dates, and floats (see (6)–(8)), the same PERT algorithm has been traditionally used (see formulae (1)–(5)), the only difference being the use of interval arithmetic instead of classical arithmetic. For such a straightforward extension of the PERT algorithm, it turns out that the forward recursion correctly computes the interval of possible earliest starting dates (Chanas and Kamburowski 1981; Dubois and Prade 1980; Gazdik 1983). Indeed, one can obtain all the optimal lower and upper bounds of earliest starting dates of tasks by applying the forward recursion (formulae: (1), (2)) for configurations Ω_A^- and Ω_A^+, respectively. But the backward recursion fails to compute the set of possible latest starting dates (McCahon 1993; McCahon and Lee 1988; Prade 1979) and, in consequence, floats can no longer be recovered from this procedure. Let us illustrate the point that the backward recursion cannot be used to compute the latest starting dates and the floats. We first recall definitions of arithmetical operations on intervals (see Moore 1979). Let $* \in \{+, -, \min, \max\}$ and let $U = [u^-, u^+]$ and $W = [w^-, w^+]$ be two intervals such that $u^- \le u^+$ and $w^- \le w^+$. Then the following expression:

$$Z = U *_I W = \{u * w \mid u \in U, w \in W\}$$

defines operation $*_I \in \{+_I, -_I, \min_I, \max_I\}$ on intervals and $Z = [z^-, z^+]$ is a resulting interval such that $z^- \le z^+$. The following formulae are instrumental when performing interval operations as defined above: $U +_I W = [u^- + w^-, u^+ + w^+]$, $U -_I W = [u^- - w^+, u^+ - w^-]$, $\min_I\{U, W\} = [\min\{u^-, w^-\}, \min\{u^+, w^+\}]$ and $\max_I\{U, W\} = [\max\{u^-, w^-\}, \max\{u^+, w^+\}]$.

Consider the one activity network with duration time $d_{12} \in D_{12} = [0, 1]$. Its earliest starting date is $\mathrm{EST}_1 = \mathrm{EST}_{12} = [0, 0]$. The earliest ending time of the project is ill known: $\mathrm{EST}_2 = \mathrm{EST}_1 +_I D_{12} = [0, 1]$. And thus the backward recursion yields the following information: $\mathrm{EST}_2 -_I D_{12} = [0, 1] -_I [0, 1] = [-1, 1]$ on latest starting dates LST_1 and LST_{12}, which is, of course, false since the latest starting dates of event 1 and activity $(1, 2)$ are always null in order to ensure a minimal project duration, $\mathrm{LST}_1 = \mathrm{LST}_{12} = [0, 0] \subset \mathrm{EST}_2 -_I D_{12} = [-1, 1]$. This error comes from the fact that the variable which represents the duration appears two times in the computations: once in the forward recursion, and once in the backward one. Thus, applying interval arithmetic with linked variables always results in an over-imprecise bracketing of the exact result. For example, for all $x \in [0, 1]$ $x - x = 0$, but $[0, 1] -_I [0, 1] = [-1, 1]$. Some authors tried to repair this propagation error in several ways: one classical

approach is the use of a nonstandard interval arithmetic (Hapke and Słowiński 1996; Kaufmann and Gupta 1991; Rommelfanger 1994). For example, some authors use for subtraction an "inverse" of the interval addition (Hapke and Słowiński 1996). These operations are defined for fuzzy intervals, but their interval counterparts can be expressed as follows: $[u^-, u^+] - [w^-, w^+] = [u^- - w^-, u^+ - w^+]$. The use of this particular arithmetic does not lead to correct results for computing of either latest starting dates or floats of tasks. In particular, the results may not be well-formed intervals. For example, the difference between the intervals $[0, 1]$ and $[0, 2]$ would lead to $[0, -1]$, which is not an interval, $0 \not\le -1$. Thus, techniques using nonstandard interval arithmetic will generally fail to produce the correct values. In Nasution (1994), symbolic computations on variable duration times were suggested. However, this technique is unwieldy and highly combinatorial. Accordingly, no complete, correct, and efficient solution to the problem of scheduling under interval uncertainty has been reached up until now.

There are obvious relations between the criticality of a task and the possible values of its floats and its earliest and latest starting dates, namely, task (k, l) is possibly (resp. necessarily) critical if and only if $f_{kl}^- = 0$ (resp. $f_{kl}^+ = 0$). Moreover, if task (k, l) is necessarily critical, then $\mathrm{EST}_{kl} = \mathrm{LST}_{kl}$. As shown in Fig. 1a, the converse statement is false: task $(3, 4)$ has $\mathrm{EST}_{34} = \mathrm{LST}_{34} = [3, 4]$ for intervals of possible values of earliest and latest starting dates, but its interval of possible values of floats is $F_{34} = [0, 1]$. It can also be noted that the float is no longer the difference between the latest starting date and the earliest starting date; for instance, $\mathrm{LST}_{34} -_I \mathrm{EST}_{34} = [-1, 1] \ne F_{34}$. In fact, when $\mathrm{EST}_{kl} = \mathrm{LST}_{kl}$, task (k, l) is only possibly critical. The last example makes it clear that the interval containing the floats of a task cannot be calculated by means of the intervals containing the earliest and latest starting dates of this task. The float, from which the criticality of the task can be assessed, must be computed separately.

Many intuitions learned on the deterministic problem fail in the interval version. For example, when duration times are precisely set, critical activities always form a critical path. But in the interval case, possibly critical activities do not always form a possibly critical path. In Fig. 1b, task $(1, 3)$ is critical in the configuration where $d_{12} = d_{23} = 0$ and $d_{34} = d_{45} = 2$, and task $(3, 5)$ is critical in the configuration where $d_{12} = d_{23} = 2$ and $d_{34} = d_{45} = 0$, but the path $(1, 3, 5)$ is never critical. In Fig. 1c, dashed arcs represent possibly critical tasks, and the bold task $(1, 2)$ is the only necessarily critical task. So, it can be seen that some necessarily critical tasks can be isolated. There is no necessarily critical path, but several paths are possibly critical (paths consisting of dashed arcs in Fig. 1c). However, it is easy to check that if there exists a necessarily critical path, then it is unique (under the assumption that all duration times are
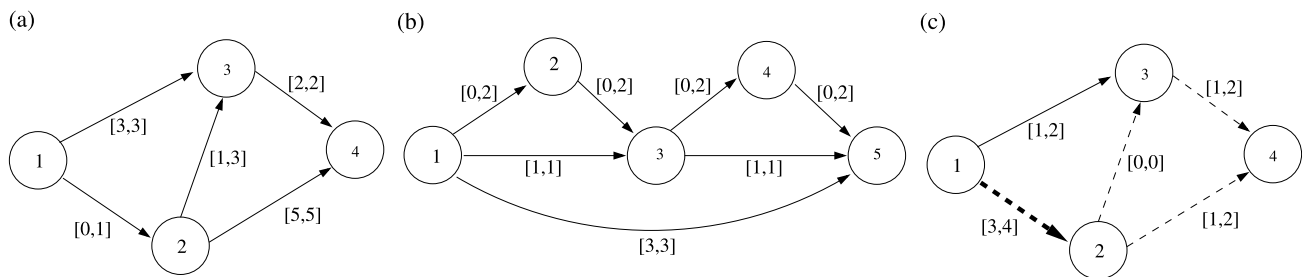
(a)   (b)   (c)

Fig. 1 **a** A network in which the necessary criticality of $(3, 4)$ cannot be deduced from $EST_{34} = LST_{34}$. **b** A network in which possibly critical activities do not form a possibly critical path. **c** Dashed tasks form possibly critical paths, and the bold task is necessarily critical

intervals) and it is then easy to identify (Chanas et al. 2002). But this case seems to appear rarely in practice. Therefore, the classical PERT results and algorithms cannot be directly adapted to the interval-valued problem.

The first important step for the resolution of the problems of computing the optimal bounds on the intervals of earliest starting dates, latest starting dates, and floats (see (6)–(8)) was taken by Dubois et al. (2003). They showed that these bounds are attained on extreme configurations due to the monotonicity of functions that compute the quantities under concern. So a naive algorithm is to perform one PERT analysis on each extreme configuration which implies an $O((m + n)2^m)$ algorithm.

Other approaches have been developed to compute latest starting dates and floats. The first idea is to find a *small* subset of extreme configurations on which the bounds of the quantities of interest are attained. One such subset is defined by all extreme configurations in which all tasks have minimal duration times, but on a path from the initial node to the final one. On this path, task duration times are set to their upper bounds. The optimal upper bound on latest starting dates, and the optimal lower and upper bounds on floats are reached on a configuration of this form. This idea will be presented in Sect. 3.

Another idea, called the *constructive approach*, consists in a direct construction of one configuration (and just one) for which the extrema of the quantity of interest (for instance, the optimal upper bound on latest starting dates of a task), is attained. In practice, we have to consecutively instantiate duration times of all tasks in a network without changing the result of the quantity of concern. The way of instantiation of a task duration to its least or greatest value is not simple. If we can assign precise duration times task by task in order to construct an extreme configuration, without changing the result of the quantity of concern, then this approach will lead to polynomial algorithms (see Sect. 4). Unfortunately, such an approach cannot be applied to compute the optimal lower bound on floats of a task—this problem is known to be strongly NP-hard in a general acyclic network (Chanas and Zieliński 2002) and NP-hard in a planar

acyclic network (Chanas and Zieliński 2003). These results follow from the fact that the problem of deciding possible criticality of a given task is strongly NP-complete in a general acyclic network and remains NP-complete even in an acyclic planar network of node degree three (Chanas and Zieliński 2002, 2003). Moreover, if $P \neq NP$, then the problem of computing the minimal float value of a task is not at all approximable even in acyclic planar networks of node degree three (Zieliński 2005).

Note that all considered interval problems have been completely solved when a network is series-parallel (Dubois et al. 2003; Fargier et al. 2000; Zieliński 2006). A graph is said to be *series-parallel* if it is recursively defined as follows (Valdes et al. 1982). A graph consisting of two nodes joined by a single arc is series-parallel. If G_1 and G_2 are series-parallel, so are the graphs constructed by each of the operations: *parallel composition*—identify the source of G_1 with the source of G_2 and the sink of G_1 with the sink of G_2; *series composition*—identify the sink of G_1 with the source of G_2. It results from this definition that each series-parallel graph is acyclic and planar and has exactly one source and exactly one sink. For example, the graph pictured in Fig. 1b is series-parallel, but the graph shown in Fig. 1c is not. Fargier et al. (2000) proposed an $O(n)$ algorithm for computing bounds on possible values of latest starting dates and bounds on possible values of floats of a task and $O(n)$ algorithms for evaluating possible and necessary criticality of a task in a series-parallel network. Therefore, applying algorithms provided in Fargier et al. (2000) to each task in a network for computing bounds on floats and bounds on latest starting dates of all tasks leads to methods that require $O(n^2)$ time (note that for series-parallel graphs $m = O(n)$). In Zieliński (2006), the time for computing bounds on floats and bounds on latest starting dates and for evaluating the possible and necessary criticality for one task in a series-parallel network has been reduced to $O(\log n)$ by applying a more clever data structure. Therefore, computing bounds on floats and bounds on latest starting dates and evaluating the possible and necessary criticality of all tasks in a series-parallel network requires $O(n \log n)$ time.

In order to deal with the interval PERT problem in the general case, we introduce some additional notation.

- SUCC(i, j) (resp. PRED(i, j)) denotes the set of all arcs that come after (resp. before) $(i, j) \in A$, and SUCC(j) (resp. PRED(j)) stands for the set of all nodes that come after (resp. before) $j \in V$.
- $P_{u,v}$ is the set of paths $p(u, v)$ in G from node u to node v. We denote by P the set of all paths in G from node 1 to node n.
- $P_{1,(k,l),n}$ is the set of paths in G from node 1 to node n traversing task (k, l), and $P_{(k,l),n}$ represents the set of paths in G from node k to node n traversing task (k, l).
- $l_p(\Omega)$ denotes the length of a path $p \in P_{u,v}$ in Ω, $l_p(\Omega) = \sum_{(i,j) \in p} d_{ij}(\Omega)$.
- $G(i, j)$ is the subgraph of G composed of nodes succeeding i and preceding j.
- $G(d_{ij} = d)$ is the graph G in which duration of task (i, j) is replaced by d.

3 Path enumeration approach

In this section, we give two algorithms based on some properties in which paths have particular interest. The first one, proposed in Dubois et al. (2005), computes maximal latest starting dates and minimal and maximal floats of each task of a network in one execution. We call it the *path algorithm*. The second algorithm only computes minimal latest starting dates in polynomial time. It is called the *polynomial path algorithm*.

3.1 Path algorithm

We first recall that maximal latest starting dates and minimal and maximal floats of tasks are attained on extreme configurations in which task duration times on a path from node 1 to node n are set to their maximal values and all the remaining task duration times are set to their minimal ones.

Proposition 1 (Dubois et al. 2005) *Let* $(k, l) \in A$ *be a task of* G. *There exist paths* $p_1, p_2, p_3 \in P$ *such that* $\Omega^+_{p_1}$ *maximizes* $lst_{kl}(.)$, $\Omega^+_{p_2}$ *minimizes* $f_{kl}(.)$, $\Omega^+_{p_3}$ *maximizes* $f_{kl}(.)$ *over the set* $\mathscr{C}$.

The key to constructing the path algorithm (Algorithm 1), proposed in Dubois et al. (2005), is Proposition 1. The idea of the algorithm consists in performing the classical PERT analysis for each configuration Ω^+_p such that p is a path in network G from 1 to n. Clearly, the number of tested configurations is potentially exponential, but in practice the algorithm runs fast on realistic problems (see Sect. 6).

Algorithm 1: (Path Algorithm) A calculation of the maximal latest starting dates and the minimal and maximal floats of each task in network G

Input: A network G, interval duration times D_{uv}, $(u, v) \in A$.
Output: The maximal latest starting dates, the minimal and maximal floats of each task in G.

begin
  foreach *task* $(k, l) \in A$ **do**
    $lst^+_{kl} \leftarrow 0$; $f^+_{kl} \leftarrow 0$; $f^-_{kl} \leftarrow +\infty$;
  foreach *path* $p \in P$ **do**
    Compute $lst_{kl}(\Omega^+_p)$ and $f_{kl}(\Omega^+_p)$ for each $(k, l) \in A$ by the classical PERT;
    foreach *task* $(k, l) \in A$ **do**
      if $lst^+_{kl} < lst_{kl}(\Omega^+_p)$ **then** $lst^+_{kl} \leftarrow lst_{kl}(\Omega^+_p)$;
      if $f^+_{kl} < f_{kl}(\Omega^+_p)$ **then** $f^+_{kl} \leftarrow f_{kl}(\Omega^+_p)$;
      if $f^-_{kl} > f_{kl}(\Omega^+_p)$ **then** $f^-_{kl} \leftarrow f_{kl}(\Omega^+_p)$;
end

3.2 Polynomial path algorithm

We first recall a result given in Dubois et al. (2005) that describes the form of configurations where the minimal latest starting date of a given task (k, l) in a network G is attained.

Proposition 2 (Dubois et al. 2005) *Let* $(k, l) \in A$ *be a task of network* G. *There exists a path* $p_{kl} \in P_{(k,l),n}$ *that induces the extreme configuration such that* $\Omega^+_{p_{kl}} = \arg\min_{\Omega \in \mathscr{C}} lst_{kl}(\Omega)$ *and* $lst^-_{kl} = lst_{kl}(\Omega^+_{p_{kl}})$.

It is easily seen that optimal path $p_{kl} \in P_{(k,l),n}$, which induces configuration $\Omega^+_{p_{kl}}$, is a longest path from k to n traversing l in $\Omega^+_{p_{kl}}$. We will show that this path can be recursively constructed. Suppose that a path $p_{lu} \in P_{(l,u),n}$ which induces configuration $\Omega^+_{p_{lu}}$ such that $\Omega^+_{p_{lu}} = \arg\min_{\Omega \in \mathscr{C}} lst_{lu}(\Omega)$ is known for each node $u \in \text{Succ}(l)$. Then one can construct an optimal path p_{kl} among paths p_{lu}, where $u \in \text{Succ}(l)$.

Proposition 3 *Let* (k, l) *be a task of network* G *and let* $p_{lu} \in P_{(l,u),n}$ *be a path such that* $lst^-_{lu} = lst_{lu}(\Omega^+_{p_{lu}})$, *where* $u \in \text{Succ}(l)$. *Then* $lst^-_{kl} = \min_{u \in \text{Succ}(l)} lst_{kl}(\Omega^+_{\{(k,l)\} \cup p_{lu}})$.

Proof From Proposition 2, it follows that there exists a path $p_{kl} \in P_{(k,l),n}$ such that $\Omega^+_{p_{kl}} = \arg\min_{\Omega \in \mathscr{C}} lst_{kl}(\Omega)$. Assume on the contrary that $lst^-_{kl} = lst_{kl}(\Omega^+_{p_{kl}}) < lst_{kl}(\Omega^+_{\{(k,l)\} \cup p_{lu}})$, $\forall u \in \text{Succ}(l)$. Clearly, $lst^-_{kl} = lst_{kl}(\Omega^+_{p_{kl}}) = \max_{p \in P} l_p(\Omega^+_{p_{kl}}) - l_{p_{kl}}(\Omega^+_{p_{kl}})$ and thus

$$lst_{lu}(\Omega^+_{p_{kl}}) = \max_{p \in P} l_p(\Omega^+_{p_{kl}}) - \max_{p \in P_{(l,u),n}} l_p(\Omega^+_{p_{kl}})$$

$$= \max_{p \in P} l_p(\Omega^+_{p_{kl}}) - l_{p_{kl}} + d^+_{kl} = lst^-_{kl} + d^+_{kl}$$

which forces $\text{lst}_{kl}^- = \text{lst}_{lu}(\Omega_{p_{kl}}^+) - d_{kl}^+$. On the other hand, the formula for $\text{lst}_{lu}(\Omega_{\{(k,l)\}\cup p_{lu}}^+)$ can be rewritten: $\text{lst}_{lu}(\Omega_{\{(k,l)\}\cup p_{lu}}^+) = \max_{p\in P} l_p(\Omega_{\{(k,l)\}\cup p_{lu}}^+) - l_{p_{lu}}(\Omega_{\{(k,l)\}\cup p_{lu}}^+) - d_{kl}^+$. From the assumption, we have $\text{lst}_{kl}^- = \text{lst}_{lu}(\Omega_{p_{kl}}^+) - d_{kl}^+ < \text{lst}_{kl}(\Omega_{\{(k,l)\}\cup p_{lu}}^+)$, which yields $\text{lst}_{lu}(\Omega_{p_{kl}}^+) < \max_{p\in P} l_p(\Omega_{\{(k,l)\}\cup p_{lu}}^+) - l_{p_{lu}}(\Omega_{\{(k,l)\}\cup p_{lu}}^+)$. Let $p' \in P$ be a longest path in G in the configuration $\Omega_{\{(k,l)\}\cup p_{lu}}^+$. We consider two cases. (i) If $(k,l) \notin p'$ then $\max_{p\in P} l_p(\Omega_{\{(k,l)\}\cup p_{lu}}^+) = \max_{p\in P} l_p(\Omega_{p_{lu}}^+)$ and so $\text{lst}_{lu}(\Omega_{p_{lu}}^+) < \text{lst}_{lu}^-$, which contradicts the definition of lst_{lu}^-. (ii) If $(k,l) \in p'$ then (k,l) is critical in $\Omega_{\{(k,l)\}\cup p_{lu}}^+$ and consequently $\text{lst}_{kl}(\Omega_{\{(k,l)\}\cup p_{lu}}^+) = \text{est}_{kl}^-$ and $\text{lst}_{kl}(\Omega_{\{(k,l)\}\cup p_{lu}}^+) = \text{lst}_{kl}^-$, which contradicts the fact that $\text{lst}_{kl}(\Omega_{p_{kl}}^+) < \text{lst}_{kl}(\Omega_{\{(k,l)\}\cup p_{lu}}^+)$. Therefore, $\text{lst}_{kl}^- = \text{lst}_{kl}(\Omega_{p_{kl}}^+) \geq \text{lst}_{kl}(\Omega_{\{(k,l)\}\cup p_{lu}}^+)$. □

From Proposition 3, we deduce a polynomial algorithm for computing the minimal starting date of each task. Namely, the algorithm recursively finds a path p_{kl} for which configuration $\Omega_{p_{kl}}^+$ minimizes $\text{lst}_{kl}(\Omega)$ making use of paths p_{lu} such that $\text{lst}_{lu}^- = \text{lst}_{lu}(\Omega_{p_{lu}}^+)$, where $u \in \text{Succ}(l)$, starting from nodes in $\text{Pred}(n)$. Its overall running time is $O(m^2(m+n))$. Fortunately, it is possible to reduce the complexity of the algorithm to $O(m^2)$. It is enough to store the length of a longest path $p^* \in P$ in optimal configuration $\Omega_{p_{lu}}^+$ for (l,u), denoted by l_{lu}^*, and the length of path p_{lu} in $\Omega_{p_{lu}}^+$ denoted by l_{lu}, for every $u \in \text{Succ}(l)$. Obviously, $\text{lst}_{lu}^- = l_{lu}^* - l_{lu}$. Let (l,v) be a task such that $\text{lst}_{kl}^- = \text{lst}_{kl}(\Omega_{\{(k,l)\}\cup p_{lv}}^+)$ (such a task always exists, see Proposition 3). The length l_{kl} of path p_{kl} in optimal configuration $\Omega_{\{(k,l)\}\cup p_{lv}}^+$ for (k,l) equals $d_{kl}^+ + l_{lv}$. Let $p^* \in P$ be a longest path in $\Omega_{\{(k,l)\}\cup p_{lv}}^+$. We consider two cases. (i) If $(k,l) \in p^*$ then the length l_{kl}^* of p^* equals $\text{est}_{kl}^- + d_{kl}^+ + l_{lv}$ (note that $\text{est}_{kl}^- = \text{est}_{kl}(\Omega_A^-) = \text{est}_{kl}(\Omega_{\{(k,l)\}\cup p_{lv}}^+)$). Hence, $\text{lst}_{kl}^- = \text{est}_{kl}^-$. (ii) If $(k,l) \notin p^*$ then $l_{kl}^* = l_{lv}^*$ and $\text{lst}_{kl}^- = l_{kl}^* - l_{kl}$. The preceding remarks allow us to construct an $O(m^2)$ algorithm (see Algorithm 2).

4 Constructive approach

Zieliński (2005) gave polynomial algorithms for computing optimal upper and lower bounds on latest starting times. We do the same for the optimal upper bound of floats. The basic structure is the same for the three algorithms. The first step instantiates the duration times of some tasks in a network without changing the result of the quantity of concern for a distinguished task. The next step instantiates the duration times of the remaining tasks in the network while preserving the global possible or necessary criticality of the task.

Algorithm 2: (Polynomial path algorithm) Calculation of the minima of latest starting dates of all tasks in a network

Input: A network G, interval duration times D_{uv}, $(u,v) \in A$.
Output: The minima of latest starting dates of all the tasks in a network G.

begin
  $V \leftarrow V \cup \{n+1\}$; $A \leftarrow A \cup \{(n, n+1)\}$;
  $D_{nn+1} \leftarrow [0,0]$;
  Use the classical PERT to compute earliest starting date of each task of G in Ω_A^-;
  $\text{lst}_{nn+1}^- \leftarrow \text{est}_{nn+1}(\Omega_A^-)$; $l_{nn+1}^* \leftarrow \text{est}_{nn+1}(\Omega_A^-)$;
  $l_{nn+1} \leftarrow 0$;
  foreach (k,l) such that $k \leftarrow n-1$ **downto** 1 **do**
    Let (l,v) be a task such that $v \in \text{Succ}(l)$;
    $l_{kl} \leftarrow d_{kl}^+ + l_{lv}$;
    if $\text{est}_{kl}(\Omega_A^-) + l_{kl} > l_{lv}^*$ **then**
      $l_{kl}^* \leftarrow \text{est}_{kl}(\Omega_A^-) + l_{kl}$;
      $\text{lst}_{kl}^- \leftarrow \text{est}_{kl}(\Omega_A^-)$;
    else
      $l_{kl}^* \leftarrow l_{lv}^*$;
      $\text{lst}_{kl}^- \leftarrow l_{kl}^* - l_{kl}$;
    foreach (l,u) such that $u \in \text{Succ}(l) \setminus \{v\}$ **do**
      if $\text{est}_{kl}(\Omega_A^-) + d_{kl}^+ + l_{lu} > l_{lu}^*$ **then**
        $l_{kl}^* \leftarrow \text{est}_{kl}(\Omega_A^-) + d_{kl}^+ + l_{lu}$;
        $l_{kl} \leftarrow d_{kl}^+ + l_{lu}$;
        $\text{lst}_{kl}^- \leftarrow \text{est}_{kl}(\Omega_A^-)$;
      else
        if $l_{lu}^* - d_{kl}^+ - l_{lu} < \text{lst}_{kl}^-$ **then**
          $l_{kl}^* \leftarrow l_{lu}^*$;
          $l_{kl} \leftarrow d_{kl}^+ + l_{lu}$;
          $\text{lst}_{kl}^- \leftarrow l_{kl}^* - l_{kl}$;
end

Once duration times of all tasks are precisely set, we need to compute the minimal duration Δ that, added to the duration time of the distinguished task, makes it possibly or necessarily critical in this particular configuration. Duration Δ will easily provide the concerned quantity.

4.1 Evaluating criticality

This section presents methods which can decide in polynomial time if a given task (k,l) is necessarily or possibly critical. First, under the assumption that the duration times of the predecessors of task (k,l) are precisely known, we recall algorithms, proposed in Zieliński (2005), that decide if (k,l) is necessarily or possibly critical. They constitute the basis for computing the minimal and the maximal latest starting dates in polynomial time in Zieliński (2005). We extend some of these results and give a general algorithm that

asserts if (k, l) is necessarily critical in a network G in polynomial time without any consideration of the duration times of tasks preceding (k, l). This result will lead in Sect. 4.3 to a polynomial algorithm which computes the maximal float of a task.

Let us recall characteristic conditions of the nonnecessary criticality of tasks.

Lemma 1 (Dubois et al. 2005) *A task $(k, l) \in A$ is not necessarily critical in G if and only if there exists a path $p \in P$ such that $(k, l) \notin p$, p is critical in configuration Ω_p^+ and no critical path in Ω_p^+ includes (k, l).*

Observation 1 (Zieliński 2005) *A task $(k, l) \in A$ is not necessarily critical in G if and only if (k, l) is not critical in an extreme configuration in which the duration of (k, l) is at its lower bound and all tasks from set $A \setminus (\mathrm{SUCC}(k, l) \cup \mathrm{PRED}(k, l) \cup \{(k, l)\})$ have duration times set to their upper bounds.*

Now under the assumption that tasks preceding (k, l) have precise duration times, we can set the ones of tasks succeeding (k, l) at precise values while maintaining the status of (k, l) in terms of necessary criticality. This yields a configuration where (k, l) is critical if and only if it is necessarily critical in the interval-valued network. These duration times are given by Propositions 4 and 5.

Proposition 4 *Let $(k, l) \in A$ be a distinguished task, and (i, j) be a task such that $(i, j) \in \mathrm{SUCC}(k, l)$. Assume that every task $(u, v) \in \mathrm{PRED}(i, j)$ has precise duration. If (k, l) is critical in $G(1, i)$, then the following conditions are equivalent:*

(i) *(k, l) is necessarily critical in G.*
(ii) *(k, l) is necessarily critical in $G(d_{ij} = d_{ij}^-)$.*

Proof (i) $\implies$ (ii) Obvious.
(i) $\impliedby$ (ii) We use a proof by contraposition. We need to prove that if (k, l) is critical in $G(1, i)$ and (k, l) is not necessarily critical in G, then (k, l) is not necessarily critical in $G(d_{ij} = d_{ij}^-)$. By assumption, (k, l) is not necessarily critical in G. From Lemma 1, it follows that there exists a path $p \in P$ such that $(k, l) \notin p$, p is critical in configuration Ω_p^+ and no critical path in Ω_p^+ includes (k, l). Since (k, l) is critical in $G(1, i)$, $(i, j) \notin p$ in Ω_p^+. Observe that $d_{ij}(\Omega_p^+) = d_{ij}^-$. From this and the fact that (k, l) is not critical in Ω_p^+, we conclude that (k, l) is not necessarily critical in $G(d_{ij} = d_{ij}^-)$. $\square$

Proposition 5 *Let $(k, l) \in A$ be a distinguished task, and (i, j) be a task such that $(i, j) \in \mathrm{SUCC}(k, l)$. Assume that every task $(u, v) \in \mathrm{PRED}(i, j)$ has precise duration. If (k, l)*

is not critical in $G(1, i)$, then the following conditions are equivalent:

(i) *(k, l) is necessarily critical in G.*
(ii) *(k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$.*

Proof A proof follows the definition of necessary criticality. $\square$

Propositions 4 and 5, together with Observation 1, lead us to Algorithm 3 for asserting the necessary criticality of a given task (k, l) in a network in which all tasks that precede (k, l) have precise durations. The algorithm works as follows. First all duration times of tasks neither succeeding nor preceding (k, l) are set to their upper bounds and the earliest starting times of events not succeeding event l are computed by the classical PERT. Then the algorithm tests if (k, l) is critical in the subnetwork $G(1, l)$. We denote by $f_{kl}(u, v)$ the float of (k, l) computed in the network $G(u, v)$. Precise duration times are fixed for tasks immediately succeeding node l. This step is repeated recursively for network $G(1, l + 1), \ldots, G(1, n - 1)$. At the end of this process, all duration times are precisely set, and (k, l) is necessarily critical in the network with the interval durations if and only if (k, l) is critical in the network with the fixed durations. Computing $f_{kl}(1, i)$ and testing if (k, l) is critical in $G(1, i)$ can be done in constant time, since we already know est_j for all $j \in \mathrm{Pred}(i)$, and so Algorithm 3 runs in $O(m)$.

Algorithm 3: Asserting whether a task is necessarily critical when duration times of predecessors are precisely known

Input: A network G, task (k, l), interval duration times D_{uv}, $(u, v) \in A$ and for every task in $\mathrm{PRED}(k, l)$ the duration is precisely given.

Output: If $f_{kl}(1, n) = 0$, (k, l) is necessarily critical in G; and if $f_{kl}(1, n) > 0$, (k, l) is not necessarily critical in G.

/* Initialization */
foreach $(u, v) \in A \setminus (\mathrm{SUCC}(k, l) \cup \mathrm{PRED}(k, l) \cup \{(k, l)\})$ **do**
$d_{uv} \leftarrow d_{uv}^+$;
$d_{kl} \leftarrow d_{kl}^-$; /* the partially instantiated configuration */
/* Computation */
Compute est_i of events $i \in V \setminus (\mathrm{SUCC}(l) \cup \{l\})$ by the classical PERT in the partial configuration;
for $i \leftarrow l$ **to** $n - 1$ **such that** $i \in \mathrm{SUCC}(l) \cup \{l\}$ **do**
$\quad$ Compute $f_{kl}(1, i)$;
$\quad$ **if** $f_{kl}(1, i) = 0$ /* (k, l) critical in $G(1, i)$*/ **then**
$\quad\quad$ **foreach** $j \in \mathrm{Succ}(i)$ **do** $d_{ij} \leftarrow d_{ij}^-$
$\quad$ **else**
$\quad\quad$ **foreach** $j \in \mathrm{Succ}(i)$ **do** $d_{ij} \leftarrow d_{ij}^+$

Compute $f_{kl}(1, n)$;
return $f_{kl}(1, n)$;

Algorithm 4: Asserting whether a task is possibly critical when duration times of its predecessors are precisely known

Input: A network G, task (k, l), interval duration times D_{uv}, $(u, v) \in A$ and for every task in $\text{PRED}(k, l)$ the duration is precisely given.

Output: If $f_{kl}(1, n) = 0$, (k, l) is possibly critical in G; and if $f_{kl}(1, n) > 0$, (k, l) is not possibly critical in G.

/* Initialization */

foreach $(u, v) \in A \setminus (\text{SUCC}(k, l) \cup \text{PRED}(k, l) \cup \{(k, l)\})$ **do**

$d_{uv} \leftarrow d_{uv}^-$;

$d_{kl} \leftarrow d_{kl}^+$; /* the partially instantiated configuration */

/* Computation */

Compute est_i of events $i \in V \setminus (\text{SUCC}(l) \cup \{l\})$ by the classical PERT in the partial configuration;

for $i \leftarrow l$ **to** $n - 1$ **such that** $i \in \text{SUCC}(l) \cup \{l\}$ **do**

    Compute $f_{kl}(1, i)$;

    if $f_{kl}(1, i) = 0$ /* (k, l) is critical in $G(1, i)$ */ **then**

        foreach $j \in \text{Succ}(i)$ **do** $d_{ij} \leftarrow d_{ij}^+$

    else

        foreach $j \in \text{Succ}(i)$ **do** $d_{ij} \leftarrow d_{ij}^-$

Compute $f_{kl}(1, n)$;
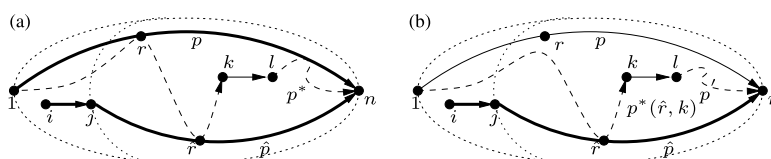
return $f_{kl}(1, n)$

Lemma 1, Propositions 4 and 5, and Observation 1 have counterparts for asserting possible criticality. It suffices to swap duration times d_{uv}^- and d_{uv}^+. This leads to an $O(m)$ algorithm (see Algorithm 4) for asserting the possible criticality of tasks whose predecessors have deterministic durations (Zieliński [2005]).

We now present an algorithm for evaluating the necessary criticality of a fixed task $(k, l) \in A$ in network G with interval duration times, without any restriction. The key to the algorithm lies in Propositions 6 and 7, which enable a network with interval duration times to be replaced by another network with precise durations for tasks preceding a fixed (k, l), in such a way that (k, l) is necessarily critical in the former if and only if it is necessarily critical in the latter.

Proposition 6 *Let $(k, l) \in A$ be a distinguished task, and (i, j) be a task such that $(i, j) \in \text{PRED}(k, l)$. If (k, l) is necessarily critical in $G(j, n)$, then the following conditions are equivalent*:

(i) *(k, l) is necessarily critical in G.*

(ii) *(k, l) is necessarily critical in $G(d_{ij} = d_{ij}^-)$.*

Proof The proof proceeds in a similar manner to the one of Proposition 4. □

Proposition 7 *Let $(k, l) \in A$ be a distinguished task, and (i, j) be a task such that $(i, j) \in \text{PRED}(k, l)$. If (k, l) is not necessarily critical in $G(j, n)$, then the following conditions are equivalent*:

(i) *(k, l) is necessarily critical in G.*

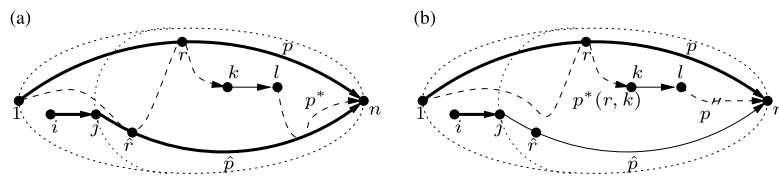(ii) *(k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$.*

Proof (i) $\Longrightarrow$ (ii) Straightforward.

(i) $\Longleftarrow$ (ii) We need to show that if (k, l) is not necessarily critical in $G(j, n)$ and (k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$, then (k, l) is necessarily critical in G. To prove this, assume on the contrary that (k, l) is not necessarily critical in G and (k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$. From Lemma 1, it follows that there exists a path $p \in P$ such that $(k, l) \notin p$, p is critical in configuration Ω_p^+, and no critical path in Ω_p^+ includes (k, l) or equivalently (k, l) is not critical in Ω_p^+. We will show that for each such configuration, where (k, l) is not critical, the other assumptions lead to constructing a critical path that traverses (k, l), which results in a contradiction. By assumption, (k, l) is not necessarily critical in $G(j, n)$. Then there exists a path $\hat{p} \in P(j, n)$ such that $(k, l) \notin \hat{p}$, $\hat{p}$ is critical in configuration $\Omega_{\hat{p}}^+$, and no critical path of $G(j, n)$ traverses (k, l) in $\Omega_{\hat{p}}^+$ (see Lemma 1).

Consider the extreme configuration induced by $p \cup \hat{p} \cup \{(i, j)\}$ and denote it by Ω^* ($\Omega^* = \Omega_{p \cup \hat{p} \cup \{(i, j)\}}^+$). Note that (k, l) is critical in Ω^*, because (k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$. Thus there exists a critical path $p^* \in P$ using (k, l) in Ω^*. We define r to be the last common node of $p^*(1, k)$ and p ($r = \max\{v \mid v \in V, v \in p^*(1, k), v \in p\}$) and define $\hat{r}$ to be the last common node of $p^*(1, k)$ and $\hat{p}$ ($\hat{r} = \max\{v \mid v \in V, v \in p^*(1, k), v \in \hat{p}\}$).

We claim that if node $\hat{r}$ exists, then $\hat{r} = r$ or node $\hat{r}$ lies on p^* before node r. Suppose, contrary to our claim, that $\hat{r}$ lies on p^* after r (see Fig. 2a). Then subpath $p^*(\hat{r}, n)$, $p^*(\hat{r}, n) = p^*(\hat{r}, k) \cup p^*(k, n)$, is at least as long as subpath $\hat{p}(\hat{r}, n)$ in configuration Ω^*. Notice that $p^*(\hat{r}, k)$ is one of the longest paths from $\hat{r}$ to k in Ω^*. We may now decrease some task duration times to their lower bounds in configuration Ω^* in the following way (see Fig. 2b): for every $(u, v) \in A$, $d_{uv}(\Omega') = d_{uv}^+$ if $(u, v) \in \hat{p}$ or $(u, v) = (i, j)$; d_{uv}^- otherwise. Duration $d_{ij}(\Omega') = d_{ij}^+$, and, by assumption, (k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$. Consequently,

Fig. 2 a Configuration $\Omega^* - \hat{r}$ on p^* after r. **b** Configuration Ω' (tasks with the maximal duration times are in *bold*)

Fig. 3 **a** Configuration $\Omega^* - \hat{r}$ on p^* before r.
b Configuration Ω''

(k, l) is critical in this new configuration Ω'. Hence, there exists a critical path $p' \in P$ traversing (k, l). Since node $\hat{r}$ lies on p^* after node r, $l_{p^*(\hat{r},k)}(\Omega^*) = l_{p^*(\hat{r},k)}(\Omega')$. Therefore, path $p^*(\hat{r}, k) \cup p'(k, n)$ is at least as long as subpath $\hat{p}(\hat{r}, n)$ in configuration Ω'. Decreasing $d_{ij}(\Omega')$ to its lower bound gives configuration $\Omega_{\hat{p}}^+$. Observe that the lengths of paths $p^*(\hat{r}, k) \cup p'(k, n)$ and $\hat{p}(\hat{r}, n)$ remain unchanged. Hence, there exists a path in $G(j, n)$ composed of two subpaths $\hat{p}(j, \hat{r})$ and $p^*(\hat{r}, k) \cup p'(k, n)$ that is at least as long as $\hat{p}$, which is impossible because we have assumed that no critical path goes through (k, l) in $G(j, n)$ in configuration $\Omega_{\hat{p}}^+$.

We can now return to the main proof. Consider configuration Ω^*. The previous claim shows that if node $\hat{r}$ exists, then $\hat{r} = r$ or node $\hat{r}$ lies on p^* before node r (see Fig. 3a). In the case that $\hat{r}$ does not exist, the proof proceeds in the same manner. From the above and the criticality of p^* in Ω^*, it follows that subpath $p^*(r, n) = p^*(r, k) \cup p^*(k, n)$ is at least as long as subpath $p(r, n)$ in this configuration. Notice that $p^*(r, k)$ is one of the longest paths from r to k in Ω^*. Decreasing some of the duration times in Ω^* to their lower bounds, we obtain configuration Ω'' in the following form (see Fig. 3b): for every $(u, v) \in A$, $d_{uv}(\Omega'') = d_{uv}^+$ if $(u, v) \in p$ or $(u, v) = (i, j)$; $d_{uv}(\Omega'') = d_{uv}^-$ otherwise. Duration $d_{ij}(\Omega'') = d_{ij}^+$, by assumption, (k, l) is necessarily critical in $G(d_{ij} = d_{ij}^+)$, which implies the criticality of (k, l) in this new configuration Ω''. Hence, there exists a critical path $p'' \in P$ using (k, l). By the claim, $l_{p^*(r,k)}(\Omega^*) = l_{p^*(r,k)}(\Omega'')$. It follows that path $p^*(r, k) \cup p''(k, n)$ is at least as long as subpath $p(r, n)$ in configuration Ω''. If $(i, j) \notin p$, we may decrease duration $d_{ij}(\Omega'')$ to its lower bound. Again by the claim, the lengths of subpaths $p^*(r, k) \cup p''(k, n)$ and $p(r, n)$ remain unchanged in this new configuration, and so $p^*(r, k) \cup p''(k, n)$ is still at least as long as $p(r, n)$. It is easily seen that this new configuration is equal to Ω_p^+. If $(i, j) \in p$, configurations Ω'' and Ω_p^+ are equal. Consequently, path $p(1, r) \cup p^*(r, k) \cup p''(k, n)$ is at least as long as p and, moreover, $p(1, r) \cup p^*(r, k) \cup p''(k, n)$ uses (k, l). This contradicts our assumption that no critical path in Ω_p^+ includes (k, l). □

We are now in a position to give an algorithm (Algorithm 5) for asserting the necessary criticality of a prescribed task in a general acyclic network. At each step of the algorithm, tasks between j and k have precise duration times (so Algorithm 3 can be invoked), and Algorithm 5 assigns

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Algorithm 5: Asserting necessary criticality of task (k, l)

Input: A network $G = \langle V, A \rangle$, activity (k, l), interval duration times D_{uv}, $(u, v) \in A$.

Output: If $f_{kl}(1, n) = 0$, (k, l) is necessarily critical in G; and if $f_{kl}(1, n) > 0$, (k, l) is not necessarily critical in G.

/* Initialization */
foreach $(u, v) \in A \setminus (\text{SUCC}(k, l) \cup \text{PRED}(k, l) \cup \{(k, l)\})$ **do**
$\quad d_{uv} \leftarrow d_{uv}^+$;
$d_{kl} \leftarrow d_{kl}^-$;
/* Computation */
for $j \leftarrow k$ **downto** 2 **such that** $j \in \text{PRED}(k) \cup \{k\}$ **do**
$\quad f_{kl}(j, n) \leftarrow$ Algorithm 3 with $G(j, n)$ and without the initialization phase;
$\quad$ **if** $f_{kl}(j, n) = 0$ **then**
$\quad\quad$ **foreach** $i \in \text{Pred}(j)$ **do** $d_{ij} \leftarrow d_{ij}^-$
$\quad$ **else**
$\quad\quad$ **foreach** $i \in \text{Pred}(j)$ **do** $d_{ij} \leftarrow d_{ij}^+$

$f_{kl}(j, n) \leftarrow$ Algorithm 3 with $G(1, n)$ and without the initialization phase;
return $f_{kl}(j, n)$;

---

precise duration times to tasks preceding j, while preserving the criticality of task (k, l). Since Algorithm 3 runs in $O(m)$, Algorithm 5 requires $O(mn)$ time.

Unfortunately, Propositions 6 and 7 cannot be adapted to the study of possible criticality, and asserting if a task is possibly critical, in general, is strongly NP-complete (Chanas and Zieliński 2002). So, while these results are instrumental for computing maximal floats, the same approach cannot be applied to compute minimal floats.

4.2 Computing tightest bounds on latest starting dates

Let us present a polynomial algorithm, proposed in Zieliński (2005), that computes maximal latest starting dates. Let us recall the following simple but important result that allows us to reduce the set of configurations $\mathcal{C}$.

Proposition 8 (Dubois et al. 2003) *The minimal upper bound on latest starting dates* lst_{kl}^+ *of task* (k, l) *in* G *is attained on an extreme configuration in which the duration of* (k, l) *is at its lower bound and all tasks that do not belong to set* $\text{SUCC}(k, l)$ *have duration times at their upper bounds.*

Algorithm 6: Computation of maximal latest starting dates of a given task

> **Input**: A network G, task (k, l), interval duration times D_{uv}, $(u, v) \in A$.
> **Output**: The maximal latest starting date lst_{kl}^+.
> /* Initialization */
> **foreach** $(u, v) \in A \setminus (\mathrm{SUCC}(k, l) \cup \{(k, l)\})$ **do** $d_{uv} \leftarrow d_{uv}^+$;
> $d_{kl} \leftarrow d_{kl}^-$;
> Compute est_{kl}^+ by the classical PERT;
> $\Delta \leftarrow 0$;
> /* Computation */
> $f_{kl}(1, n) \leftarrow$ Algorithm 3 without the initialization;
> /* we can retain all the values $f_{kl}(1, i)$ computed by
> Algorithm 3 */
> **while** $f_{kl}(1, n) > 0$ **do**
> > $\delta \leftarrow \min_{i \in \mathrm{SUCC}(l)} \{ f_{kl}(1, i) | f_{kl}(1, i) \neq 0 \}$;
> > $\Delta \leftarrow \Delta + \delta$;
> > $d_{kl} \leftarrow d_{kl}^- + \Delta$;
> > $f_{kl}(1, n) \leftarrow$ Algorithm 3 without the initialization;
> > /* we can retain all the values $f_{kl}(1, i)$ computed by
> > Algorithm 3 */
>
> **return** $\mathrm{est}_{kl}^+ + \Delta$

Algorithm 7: Computation of the minimal latest starting date of a given task

> **Input**: A network G, activity (k, l), interval duration times D_{uv}, $(u, v) \in A$.
> **Output**: Minimal latest starting date lst_{kl}^-.
> /* Initialization */
> **foreach** $(u, v) \in A \setminus (\mathrm{SUCC}(k, l) \cup \{(k, l)\})$ **do** $d_{uv} \leftarrow d_{uv}^-$;
> $d_{kl} \leftarrow d_{kl}^+$;
> Compute est_{kl}^- by the classical PERT;
> $\Delta \leftarrow 0$;
> /* Computation */
> $f_{kl}(1, n) \leftarrow$ Algorithm 4;
> /* we can retain all the values $f_{kl}(1, i)$ computed by
> Algorithm 4 */
> **while** $f_{kl}(1, n) > 0$ **do**
> > $\delta \leftarrow \min_{i \in \mathrm{SUCC}(l)} \{ f_{kl}(1, i) | f_{kl}(1, i) \neq 0 \}$;
> > $\Delta \leftarrow \Delta + \delta$;
> > $d_{kl} \leftarrow d_{kl}^+ + \Delta$;
> > $f_{kl}(1, n) \leftarrow$ Algorithm 4;
> > /* we can retain all the values $f_{kl}(1, i)$ computed by
> > Algorithm 4 */
>
> **return** $\mathrm{est}_{kl}^- + \Delta$

The main idea of the algorithm for determining lst_{kl}^+ of a given task $(k, l) \in A$ is based on Lemma 2. It consists in determining the minimal nonnegative real number f_{kl}^* that, added to the lower bound of the duration interval of a specified task (k, l), makes it necessarily critical.

Lemma 2 (Zieliński 2005) *Let f_{kl}^* be the least nonnegative real number such that (k, l) is necessarily critical with a duration $d_{kl}^- + f_{kl}^*$. Then $\mathrm{lst}_{kl}^+ = \mathrm{est}_{kl}^+ + f_{kl}^*$.*

Thus, Proposition 8 and Lemma 2 suggest an algorithm for determining maximal latest starting dates (Algorithm 6). The algorithm proceeds as follows. First the duration times of tasks not succeeding (k, l) are set to their maximal values, and the duration of (k, l) is set to d_{kl}^- (see Proposition 8). From this point on, it computes maximal earliest starting dates of (k, l). Then it tests if (k, l) is necessarily critical in the partially instantiated configuration by calling Algorithm 3. During this call, the algorithm retains all values $f_{kl}(1, i)$ determined by Algorithm 3. If (k, l) is not necessarily critical, the smallest $f_{kl}(1, i)$, previously computed, is the least duration δ to be added to d_{kl}. Then, the algorithm adds this δ to d_{kl} and repeats the previous steps. Since Algorithm 6 makes at most n calls to Algorithm 3, its complexity is $O(nm)$.

There exist counterparts to Proposition 8, Lemma 2, and Algorithm 6 for minimal latest starting dates. They are exactly symmetrical. Hence, the sketch of the algorithm for computing minimal latest starting dates (Algorithm 7), proposed in Zieliński (2005), is the same as Algorithm 6 and it also runs in $O(nm)$ time.

4.3 Computing maximal floats

In order to compute the maximal float of task $(k, l) \in A$, we first set the duration times of the tasks neither preceding nor succeeding (k, l) according to the following Lemma 3. The maximal float of (k, l) after this partial instantiation is the same as in the original network G.

Lemma 3 (Dubois et al. 2003) *The maximal float f_{kl}^+ of task (k, l) in G is attained on an extreme configuration in which the duration of (k, l) is at its lower bound and all tasks from set $A \setminus (\mathrm{SUCC}(k, l) \cup \mathrm{PRED}(k, l) \cup \{(k, l)\})$ have duration times at their upper bounds.*

Then we increase step by step the duration of (k, l) from $d_{kl} = d_{kl}^-$ until (k, l) becomes necessarily critical. Testing can be done in $O(mn)$ time (see Algorithm 5). Lemmas 4 and 6 give the hint to find the increment of d_{kl} at each step of the algorithm. According to Proposition 9, this incremental technique eventually yields f_{kl}^+.

Lemma 4 *Let tasks $(i, j) \in \mathrm{PRED}(k, l)$ have precise durations in G. Then (k, l) is necessarily critical in G if and only if there exists a path $p \in P(1, k)$ such that for every node $j \in p$, (k, l) is necessarily critical in $G(j, n)$.*

Proof ($\Longrightarrow$) Let us denote by p a longest path from 1 to k. Note that tasks $(i, j) \in \mathrm{PRED}(k, l)$ have precise durations. From the necessary criticality of (k, l) in G, it follows that path p is part of a longest path from 1 to n and this path uses (k, l) for each configuration. Thus, for every node $j \in p$,

the subpath $p(j, n)$ is a critical path in $G(j, n)$. Since this holds true for each configuration, (k, l) is necessarily critical in $G(j, n)$.

($\Longleftarrow$) Just take $j = 1$. $G(1, n) = G$ and so (k, l) is necessarily critical in G. $\qquad\square$

There is a link between the maximal latest starting date lst_{kl}^+ and the maximal float value f_{kl}^+ of task (k, l) under the assumption that tasks $(i, j) \in \mathrm{PRED}(k, l)$ have precise duration times.

Lemma 5 (Zieliński [2005]) *Let tasks $(i, j) \in \mathrm{PRED}(k, l)$ have precise durations in G. Then $f_{kl}^+ = \mathrm{lst}_{kl}^+ - \mathrm{est}_{kl}^+$.*

Thus, the maximal float value f_{kl}^+ can be computed by means of the algorithms for determining lst_{kl}^+ presented previously.

Lemma 6 *Let $\Delta = \min_j \{ f_{kl}^+(j, n) \mid (k, l)$ is not necessarily critical in $G(j, n) \}$ where $f_{kl}^+(j, n)$ is the maximal float of (k, l) in $G(j, n)$, $j \in \mathrm{PRED}(k)$. Then for all $\epsilon < \Delta$, task (k, l) is not necessarily critical in $G(j, n)$, $j \in \mathrm{PRED}(k)$, with duration $d_{kl} = d_{kl}^- + \epsilon$. Moreover, there exists $j^* \in \mathrm{PRED}(k)$ such that (k, l) is not necessarily critical in $G(j^*, n)$ with duration $d_{kl} = d_{kl}^-$, and (k, l) becomes necessarily critical in $G(j^*, n)$ with duration $d_{kl} = d_{kl}^- + \Delta$.*

Proof Consider a node j such that (k, l) is not necessarily critical in $G(j, n)$. By Observation [1], one can assume that task (k, l) has duration time of the form $d_{kl} = d_{kl}^-$. Let Ω be an extreme configuration, where the float of (k, l) attains its maximal value $f_{kl}^+(j, n) > 0$ in $G(j, n)$, and $\epsilon < \Delta \le f_{kl}^+(j, n)$ $(d_{kl}(\Omega) = d_{kl}^-)$. p' denotes a longest path in $G(j, n)$ in Ω, while p'' stands for a longest path in $G(j, n)$ in Ω using (k, l). Therefore, $f_{kl}^+(j, n) = l_{p'}(\Omega) - l_{p''}(\Omega) > \epsilon$. Let us define the configuration Ω' such that $d_{uv}(\Omega') = d_{kl}^- + \epsilon$ if $(u, v) = (k, l)$; $d_{uv}(\Omega') = d_{uv}(\Omega)$ otherwise. Path p' remains a longest path in Ω' with the same length, and p'' is still a longest path traversing (k, l) with length $l_{p''}(\Omega') = l_{p''}(\Omega) + \epsilon$. Since the float of (k, l) in Ω' is $l_{p'}(\Omega') - l_{p''}(\Omega') > 0$, (k, l) is not necessarily critical in $G(j, n)$, $j \in \mathrm{PRED}(k)$, with duration $d_{kl} = d_{kl}^- + \epsilon$.

Consider a node j^* such that (k, l) is not necessarily critical in $G(j^*, n)$, and set $\Delta = f_{kl}^+(j^*, n)$. Then for all configurations, the difference between a longest path in $G(j^*, n)$ and a longest path using (k, l) is less than or equal to Δ. If we increase the duration of (k, l) to $d_{kl}^- + \Delta$, a longest path, in this new configuration, will traverse (k, l) and thus (k, l) will be necessarily critical in $G(j^*, n)$ with duration $d_{kl} = d_{kl}^- + \Delta$. $\qquad\square$

Proposition 9 *Let f_{kl}^* be the least nonnegative real number such that (k, l) is necessarily critical in $G(d_{kl} = d_{kl}^- + f_{kl}^*)$. Then $f_{kl}^+ = f_{kl}^*$.*

Proof Consider any configuration Ω. Let p' be a longest path in Ω and p'' be a longest path including (k, l) in Ω. Note that $f_{kl}(\Omega) = l_{p'}(\Omega) - l_{p''}(\Omega)$. Let us now modify configuration Ω and denote it by Ω^x. Configuration Ω^x is defined as follows: for every $(u, v) \in A$ $d_{uv}(\Omega^x) = d_{uv}(\Omega) + x$ if $(u, v) = (k, l)$; $d_{uv}(\Omega^x) = d_{uv}(\Omega)$ otherwise, where x is a nonnegative real number. It is clear that $l_{p''}(\Omega^x) = l_{p''}(\Omega) + x$ and p'' remains a longest path including (k, l) in new configuration Ω^x. Consider the following two cases: $(x < f_{kl}(\Omega))$. Then $l_{p''}(\Omega^x) \le l_{p'}(\Omega)$, and so p' is still a critical path in Ω^x and has the same length as in Ω. This gives $f_{kl}(\Omega^x) = f_{kl}(\Omega) - x$. $(x \ge f_{kl}(\Omega))$. Then p'' becomes a critical path and so $f_{kl}(\Omega^x) = 0$. Thus, for all Ω and x, equation $f_{kl}(\Omega^x) = \max(f_{kl}(\Omega) - x, 0)$ holds, in particular, for $x = f_{kl}^*$ and configuration $\Omega = \Omega^*$ such that Ω^* maximizes the float of task (k, l) in G. From the definition of f_{kl}^* we get $f_{kl}(\Omega^{f_{kl}^*}) = 0$, hence $f_{kl}(\Omega^*) - f_{kl}^* \le 0$, and finally $f_{kl}^+ \le f_{kl}^*$.

Suppose that $f_{kl}^+ < f_{kl}^*$. Set $y = (f_{kl}^* + f_{kl}^+)/2$. Then, for every Ω, $f_{kl}(\Omega^y) = \max(f_{kl}(\Omega) - y, 0) = 0$. Note that y is a nonnegative real number, smaller than f_{kl}^*, such that (k, l) is necessarily critical in $G(d_{kl} = d_{kl}^- + y)$, which contradicts the definition of f_{kl}^*. Hence, we conclude that $f_{kl}^+ = f_{kl}^*$. $\qquad\square$

From the above results, we have an algorithm that computes the maximal float of task (k, l) in polynomial time (Algorithm [8]). The algorithm first tests the necessary criticality of (k, l) in G by calling Algorithm [5]. If so, then its float is null. Otherwise, it determines the minimal duration δ that must be added to the duration of (k, l) to change the instantiation made by Algorithm [5] for at least one task preceding (k, l). Moreover, the instantiation made by Algorithm [5] will change if and only if (k, l) becomes necessar-

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Algorithm 8: Computation of the maximal float of a given task

Input: A network G, task (k, l), interval duration times D_{uv}, $(u, v) \in A$.
Output: The maximal float f_{kl}^+.
$\Delta \leftarrow 0$;
$f_{kl}(1, n) \leftarrow$ Algorithm [5];
while $f_{kl}(1, n) > 0$ **do**
    $\delta \leftarrow \infty$;
    foreach $i \in \mathrm{PRED}(k)$ **do**
        $[\mathrm{est}_{kl}^*, \mathrm{lst}_{kl}^*] \leftarrow$ Algorithm [6] in
        $G(i, n)$;/*Algorithm [6] returns additionally est_{kl}^+*/
        $f_{kl}^* \leftarrow \mathrm{lst}_{kl}^* - \mathrm{est}_{kl}^*$;
        if $f_{kl}^* > 0$ **then** $\delta \leftarrow \min\{\delta, f_{kl}^*\}$;
    $\Delta \leftarrow \Delta + \delta$;
    $d_{kl} \leftarrow d_{kl}^- + \Delta$;
    $f_{kl}(1, n) \leftarrow$ Algorithm [5] without the initialization;
return Δ

ily critical for at least one new subnetwork $G(i, n)$. This δ is the least positive value of the maximal float of (k, l) in networks $G(i, n)$, where $i \in \text{PRED}(k)$. Since the duration times of predecessors were previously fixed, these values of the maxima of floats are differences between the maxima of latest starting dates and the minima of earliest starting dates (see Lemma 5). Algorithm 8 executes the `while` loop at most n times. At each iteration of the `while` loop, (k, l) becomes necessarily critical in at least one new subnetwork $G(j, n)$. Thus, the loop is executed at most n times, and thus Algorithm 8 requires $O(n^3 m)$ time.
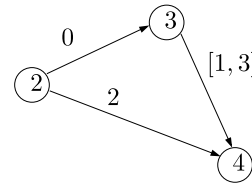
5 Illustrative example

In order to illustrate the presented algorithms, let us consider the execution of the algorithms on the project network given in Fig. 4.

We wish to compute the maximal float of task $(2, 3)$ by means of Algorithm 8. In this section, Ω_{par} represents a partial configuration, that is, the precise duration times of instantiated tasks and the interval duration times of the noninstantiated tasks. In our example, the knowledge about duration times is represented by the partial configuration Ω_{par} (initially the duration times are ill known): $D_{12} = [1, 2]$, $D_{23} = [0, 3]$, $D_{34} = [1, 3]$, $D_{24} = [0, 2]$, $D_{14} = [0, 1]$. We now analyze a complete execution of Algorithm 8 in order to compute the maximal float of task $(2, 3)$.

Algorithm 8 After initializing Δ to 0, Algorithm 8 calls Algorithm 5.

Algorithm 5 The initialization of Algorithm 5 assigns precise duration times to tasks $(2, 3)$, $(2, 4)$, and $(1, 4)$, obtaining the partial configuration Ω_{par}: $D_{12} = [1, 2]$, $D_{23} = 0$, $D_{34} = [1, 3]$, $D_{24} = 2$, $D_{14} = 1$. Then the `for` loop is executed. At the first iteration, Algorithm 3 is called for the subnetwork $G(2, 4)$ without the initialization. It means that Algorithm 3 only considers the network composed of tasks $(2, 3)$ with duration 0, $(3, 4)$ with duration $[1, 3]$, and task

Fig. 4 A project network

(2, 4) with duration two (see the next network).

Algorithm 3 The classical PERT is first performed for determining the earliest starting time of event 2, which is equal to null. Then the `for` loop of Algorithm 3 is executed and the float of $(2, 3)$ is computed in the network constituted only by this task $(2, 3)$. Its float is obviously null and then Algorithm 3 assigns the minimal duration to task $(3, 4)$. The partial configuration Ω_{par} is then $D_{12} = [1, 2]$, $D_{23} = 0$, $D_{34} = 1$, $D_{24} = 2$, $D_{14} = 1$.

At the second iteration of the `for` loop of Algorithm 3, the float of $(2, 3)$ is computed in the subnetwork $G(2, 4)$ with durations $D_{23} = 0$, $D_{34} = 1$, $D_{24} = 2$. The float is one in this subnetwork. Thus, Algorithm 3 returns one to Algorithm 5.

Task $(2, 4)$ is not critical in $G(2, 4)$; in consequence, Algorithm 5 sets the duration times of the tasks preceding node two to their upper bounds. This leads to the partial configuration Ω_{par}: $D_{12} = 2$, $D_{23} = 0$, $D_{34} = [1, 3]$, $D_{24} = 2$, $D_{14} = 1$. The `for` loop terminates. Before the end of Algorithm 5, Algorithm 3 is invoked for $G(1, 4)$. This call is the same as an execution of the `for` loop with $j = 1$.

Algorithm 3 The classical PERT is first called for determining the earliest starting times of events 1 and 2, which are equal to zero and two, respectively. Then the `for` loop of Algorithm 3 is executed and the float of task $(2, 3)$ is computed in $G(1, 3)$ (the network is constituted only by tasks $(1, 2)$ and $(2, 3)$). Its float is obviously null and then Algorithm 3 assigns the minimal duration to task $(3, 4)$. The partial configuration Ω_{par} is then $D_{12} = 2$, $D_{23} = 0$, $D_{34} = 1$, $D_{24} = 2$, $D_{14} = 1$.

At the second iteration of the `for` loop of Algorithm 3, the float of $(2, 3)$ is computed in the subnetwork $G(1, 4)$ with the partial configuration. The float of $(2, 3)$ is one in this subnetwork and Algorithm 3 returns one to Algorithm 5.

Algorithm 5 returns one to Algorithm 8.

$(2, 3)$ is not necessarily critical in G, so Algorithm 8 runs the `while` loop. In this loop, maximal latest starting dates are computed in all subnetworks $G(i, 4)$, $i \in \text{PRED}(2)$ (`for` loop). Algorithm 6 is first called in $G(2, 4)$.

Algorithm 6 lst_{23}^+ is determined in $G(2, 4)$ as follows: the duration times of tasks not succeeding $(2, 3)$ are set to their upper bounds and the duration of $(2, 3)$ is set to its lower
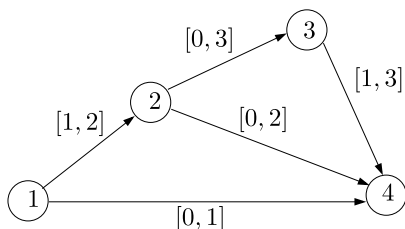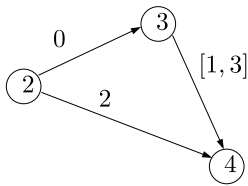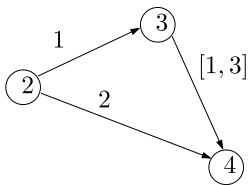
bound. Thus, the partial configuration Ω_{par} is $D_{23} = 0$, $D_{34} = [1, 3]$, $D_{24} = 2$ (see the network below).

In the considered network, the maximal earliest starting date of $(2, 3)$ is 0. Then, Algorithm 6 calls Algorithm 3 without the initialization.

Algorithm 3 f_{23} is null in $G(2, 3)$, since there is only task $(2, 3)$ in the network (this float is retained by Algorithm 6). Then the duration of task $(3, 4)$ is set to $d_{23}^- = 1$, and finally f_{23} in $G(2, 4)$ is computed in configuration Ω_{par}: $D_{23} = 0$, $D_{34} = 1$, $D_{24} = 2$. This float is equal to one and is returned to Algorithm 6.

The minimal float computed by the previous algorithm is $\Delta = 1$, and thus the duration of $(2, 3)$ is increased by one. Then Algorithm 3 is called one more time in $G(2, 4)$, with duration times Ω_{par}: $D_{23} = 1$, $D_{34} = [1, 3]$, $D_{24} = 2$ (see the network below).

Algorithm 3 During the execution of the algorithm, the partial configuration Ω_{par}: $D_{23} = 1$, $D_{34} = 1$, $D_{24} = 2$ is constructed. This allows us to deduce that $(2, 3)$ is necessarily critical in the network $G(2, 4)$ (with the possible duration times set before the call of the algorithm).

$(2, 3)$ is necessarily critical in $G(2, 4)$, for a duration of $(2, 4)$ of $d_{23} = d_{23}^- + 1$, so the latest starting date of $(2, 3)$ is one in $G(2, 4)$.

In the configuration constructed by the previous Algorithm 6, the earliest starting date of $(2, 3)$ is null. Thus, the value f_{kl}^* is equal to one, and δ is set to one.

In the second iteration of the `for` loop, Algorithm 8 calls Algorithm 6 in $G(1, 4)$ to compute the upper bound on the latest starting dates of $(2, 3)$ when its duration is $d_{23} = 0$.

Algorithm 6 After the initialization, the partial configuration Ω_{par} is $D_{12} = 2$, $D_{23} = 0$, $D_{34} = [1, 3]$, $D_{24} = 2$, $D_{14} = 1$. The algorithm is executed as done previously and returns 3 for the latest starting date of $(2, 3)$ in $G(1, 4)$.

Table 1 The intervals of values of earliest and latest starting times and of floats and criticality evaluations for all tasks of the network given in Fig. 4

Task (i, j)	EST_{ij}	LST_{ij}	F_{ij}	Possibly critical	Necessarily critical
$(1, 2)$	$[0, 0]$	$[0, 0]$	$[0, 0]$	yes	yes
$(1, 4)$	$[0, 0]$	$[1, 8]$	$[1, 8]$	no	no
$(2, 3)$	$[1, 2]$	$[1, 3]$	$[0, 1]$	yes	no
$(2, 4)$	$[1, 2]$	$[1, 8]$	$[0, 6]$	yes	no
$(3, 4)$	$[1, 5]$	$[1, 6]$	$[0, 1]$	yes	no

In the constructed configuration, the earliest starting date of $(2, 3)$ is 2, $f_{kl}^* = 1$, and δ still equals one. Now Algorithm 8 sets the duration of $(2, 3)$ to $d_{23}^- + \delta$. Algorithm 5 is called with the possible durations Ω_{par}: $D_{12} = [1, 2]$, $D_{23} = 1$, $D_{34} = [1, 3]$, $D_{24} = [0, 2]$, $D_{14} = [0, 1]$.

Algorithm 5 The algorithm works as previously. This time task $(2, 3)$ is necessarily critical and Algorithm 5 returns 0.

Task $(2, 3)$ is necessarily critical in the constructed network, so the `while` loop terminates. Algorithm 8 returns one. This value is the total duration added to the lower bound of $(2, 3)$ to make it necessarily critical. Therefore, the maximal float of $(2, 3)$ is one. Task $(2, 3)$ is not necessarily critical.

Table 1 reports all the computational results on the intervals of possible values of earliest and latest starting times and floats and on criticality evaluations for all tasks of the project network given in Fig. 4.

6 Complexity and experimental results

Table 2 summarizes the complexity of the different problems of the PERT on intervals. Moreover, Table 2 gives the running times of the best-known algorithms that compute the quantities of interest. In particular, the path algorithm that computes minimal and maximal floats and maximal latest starting dates requires $O((n + m)|P|)$ time, where $|P|$ is the number of paths of a network. This running time depends on the topology of a network. Note that some algorithms need only one execution to compute a quantity (for example, minimal starting dates) for all tasks of a network. Others need to be executed for each task. This means that the comparison of their running times can be difficult. Table 3 gives the running time of each of the proposed algorithms for computing the quantity of interest for all tasks in a network and recalls the quantities computed by the algorithms. MAXLST, MINLST, MAXF, and MINF denote maximal latest starting dates, minimal latest starting dates, maximal floats, and minimal floats, respectively.

Table 2 The complexity of the interval problems and the running times of the algorithms

Earliest starting date (all tasks)	MINEST	P	$O(n+m)$	(Chanas and Kamburowski 1981)
	MAXEST	P	$O(n+m)$	(Chanas and Kamburowski 1981)
Latest starting date (one task)	MINLST	P	$O(mn)$	Algorithm 6 (Zieliński 2005)
	MAXLST	P	$O(mn)$	Algorithm 7 (Zieliński 2005)
Latest starting date (all tasks)	MINLST	P	$O(m^2)$	Algorithm 2
Float (all tasks)	MINF	NP-hard	$O((m+n)|P|)$	Algorithm 1 (Dubois et al. 2005)
Float (one task)	MAXF	P	$O(n^3m)$	Algorithm 8

Table 3 The running times of the algorithms for computing quantities for all tasks in a network

Algorithm 1 (Path Algorithm MAXLST, MINF, MAXF)	$O((m+n)|P|)$
Algorithm 2 (Polynomial Path Algorithm MINLST)	$O(m^2)$
Algorithm 6 (MAXLST)	$O(m^2n)$
Algorithm 7 (MINLST)	$O(m^2n)$
Algorithm 8 (MAXF)	$O(n^3m^2)$

Table 4 Minimal, maximal, and average execution times (in milliseconds) of five algorithms tested on 600 problems from PSPLIB

	Min	Max	Avg.
Algorithm 1 (Path Algorithm MAXLST, MINF, MAXF)	6	138	36
Algorithm 2 (Polynomial Path Algorithm MINLST)	1.4	2.2	1.6
Algorithm 6 (MAXLST)	4	16	8.1
Algorithm 7 (MINLST)	41	9	19
Algorithm 8 (MAXF)	210	2860	1090

We now present some computational results in order to evaluate the performance of all the considered algorithms on a scheduling problem library of 600 networks having 120 tasks. Those instances of problems can be found in the Project Scheduling Problem Library (PSPLIB). They have been generated by ProGen, the program for activity network generation (Kolisch and Sprecher 1996), which can be downloaded from the PSPLIB web site.[1] We have added a range of 20% to task duration times to obtain intervals. Table 4 reports minimal, maximal, and average execution times (in milliseconds) of five algorithms tested on those 600 problems. All the algorithms were written in C language and run on a PC computer equipped with a 2.5 GHz CPU. As seen from the experimental results, the path algorithm (potentially exponential) for simultaneously computing the maxima of latest starting dates and the minima and maxima of floats is very efficient in practice. This comes from the fact that the number of paths in these networks is not huge, i.e., between 408 and 6559 different paths. Of course, one can construct more complex networks, but PSPLIB is known to be relevant to realistic scheduling problems. On the other hand, the polynomial algorithm that computes maximal floats is not as efficient for these scheduling

problems. These results show that the proposed algorithms are very efficient for problems involving about 100 tasks.

We have made a second batch of experiments in order to evaluate the algorithms on networks with different complexities. For this purpose we have used the RanGen 1 project generator, which can be downloaded from the RanGen web page.[2] RanGen 1 is particularly useful because it can generate networks with various order strengths, denoted by OS. This coefficient measures the hardness of a problem instance. The order strength is the number of precedence relations divided by the theoretical maximal number of precedence relations in a network. More details on RanGen can be found in Demeulemeester et al. (2003). We have generated by RanGen nine sets of 100 networks with 100 tasks in which the values of OS are equal to $i \cdot 0.1$, $i = 1, \dots, 9$, one network with OS equal to 0 (all tasks of this network can be executed in parallel) and one network in which OS equals one (all tasks are series). Again, we have ignored resource constraints of these problems and added a range of 20% to task duration times to obtain intervals. Table 5 reports execution times of the algorithms on the generated problems by RanGen. It is easy to observe that the computational time of most of the algorithms depends on values of OS. Figure 5

Table 5 Average execution times (in milliseconds) of the algorithms tested on problems from RanGen

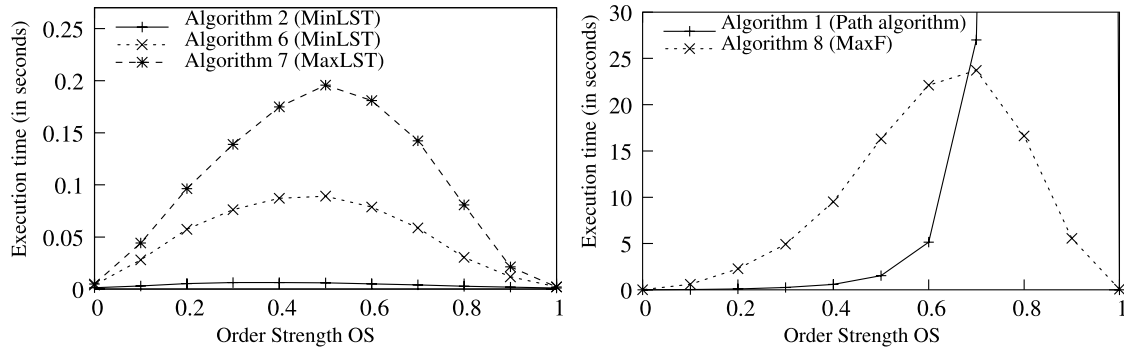|	Order Strength (OS)										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Algorithm 1	4	35	111	249	583	1534	5161	27 000	303 117		2
Algorithm 2	1	5	6	6	6	6	5	4	3	2	1
Algorithm 6	4	28	57	76	87	89	79	58	30	12	2
Algorithm 7	5	44	96	138	174	195	181	142	80	21	2
Algorithm 8	16	600	2280	4926	9520	16 321	22 100	23 712	16 625	5562	78

Fig. 5 Execution times of the algorithms on RanGen benchmark

presents execution times of the algorithms. The computational time of the path algorithm drastically increases when the value of OS increases. This is due to the fact that the number of potential paths in the networks increases when OS is increased (if OS is very close to one then the networks become parallel ones). The other algorithms are quite efficient. In particular, the polynomial algorithm for computing maximal floats is more efficient than the path algorithm when OS is greater than 0.7. Each polynomial algorithm achieves its maximal execution time on networks with the OS parameter between 0.5 and 0.7.

7 Conclusion

This paper provides a complete solution to the criticality analysis of activity networks when the duration times of tasks are modeled by means of intervals. It is shown that moving from precise to imprecise durations radically changes the nature and complexity of the problem, ruining the classical critical path method. The part of the problem pertaining to computing minimal float values becomes strongly NP-hard (Chanas and Zieliński 2002) and remains NP-hard, even in an acyclic planar network of node degree three (Chanas and Zieliński 2003). Moreover, it is not at all approximable, unless P = NP (Zieliński 2005). The other questions remain polynomially solvable, although not straightforwardly so.

An interesting question is how to exploit the results of an interval criticality analysis. The end-user may not be accustomed to obtaining imprecise floats or latest starting times. On a real configuration, a task that has a large maximal float is less likely to become critical in practice than a task that has a small maximal float. On the other hand, all tasks having positive minimal float are ensured not to be critical whatever the actual configuration turns out to be. So any interval PERT software would present results to the user by eliminating tasks with positive minimal floats first, and drawing the user's attention to necessary critical tasks first and then to tasks having small maximal floats. It is clear that, insofar as obtained results are sufficiently precise, the information derived from an interval criticality analysis is richer than that provided by a standard criticality analysis, because it is supposed to cover many possible scenarios at the same time.

The presented results can also be applied to the criticality analysis of activity networks with imprecise duration times of tasks modeled by fuzzy intervals, whose membership functions are regarded as possibility distributions for the values of the unknown duration times called fuzzy-valued PERT. Fuzzy intervals are more informative than crisp ones, as they also account for plausible values of task duration times. They address the dilemma between safe but uninformative intervals and narrow but possibly wrong ones. This gradual extension of interval PERT can be computed via the standard decomposition in α-cuts that consists in decomposing the fuzzy-valued PERT into a family of interval-valued PERT problems, which can be solved by

algorithms presented here (for details see Dubois et al. 2003; Zieliński 2005). Another method for addressing fuzzy interval PERT exploits the notion of *gradual number* (Fortin et al. 2008). Gradual numbers provide a new outlook on fuzzy intervals that can be viewed as classical intervals of gradual numbers. This allows us to directly apply the interval algorithms proposed in this paper to fuzzy-valued PERT problems (see Fortin and Dubois 2006 for an example of algorithms for computing upper bounds on floats).

Note that the presented results can be useful tools for min–max regret versions of the longest (shortest) path problem with interval weights in acyclic graphs (see Averbakh and Lebedev 2004; Karasan et al. 2001; Kasperski and Zieliński 2006). In particular, the algorithms for evaluating the necessary criticality and for computing the maximal float of an activity (arc) are useful in preprocessing an instance of the problem, since a necessarily critical activity (arc) can be automatically added to a constructed min–max regret path. In consequence, one can decompose an instance with a necessarily critical arc into two separate instances in smaller subgraphs (Kasperski and Zieliński 2010). An activity (arc) with a small maximal float can be considered as a candidate for belonging to a min–max regret path.

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