A LOGICAL APPROACH TO INTERPOLATION BASED ON SIMILARITY RELATIONS

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Abstract

One of the possible semantics of fuzzy sets is in terms of similarity, namely a grade of membership of an item in a fuzzy set can be viewed as the degree of resemblance between this item and prototypes of the fuzzy set. In such a framework, an interesting question is how to devise a logic of similarity, where inference rules can account for the proximity between interpretations. The aim is to capture the notion of interpolation inside a logical setting. In this paper, we investigate how a logic of similarity dedicated to interpolation can be defined, by considering different natural consequence relations induced by the presence of a similarity relation on the set of interpretations. These consequence relations are axiomatically characterized in a way that parallels the characterization of nonmonotonic consequence relationships. It is shown how to reconstruct the similarity relation underlying a given family of consequence relations that obey the axioms. Our approach strikingly differs from the logics of indiscernibility, such as the rough sets logics, because emphasis is put on interpolation capabilities. Potential applications are fuzzy rule-based systems and fuzzy case-based reasoning, where notions of similarity play a crucial role.

1. Introduction

Commonsense reasoning often involves one of three basic epistemic notions, viz. uncertainty, preference and similarity. Uncertainty has been studied for a long time and is a major topic of Artificial Intelligence research. There exists many formal approaches to uncertain reasoning, based on probability theory (Pearl, 1988), possibility theory (Dubois, Lang and Prade, 1994), belief functions (Smets and Kennes, 1994), or nonmonotonic inference (e.g., Lehman and Magidor, 1992). Preference modeling has received a lot of attention in decision theory for a long time, using numerical approaches such as utility theory (Savage, 1954), or relational approaches (Kitainik, 1993; Fodor and Roubens, 1995). On the contrary the notion of similarity has received less attention in the scope of logical models of reasoning. Yet, a certain number of works do exist on the topic, but are rather scattered in the literature: behavioral studies such as Tversky (1977), mathematical works that consider graded extensions of equivalence relations (Zadeh, 1971; Valverde, 1985; Ovchinnikov, 1991), a logical treatise describing similarities in a qualitative way (Lewis, 1973) and some extensions of it (Williamson, 1988), and a few attempts at modelling approximate reasoning based on a notion of similarity (Ying, 1994; Klawonn and Castro, 1995; Klawonn and Nóvak, 1996)

Similarity is the basic tool in at least three cognitive tasks: classification, case-based reasoning and interpolation. In classification tasks, objects are put in the same class insofar as they are indistinguishable with respect to suitable criteria. Similarity is meant to describe indistinguishability, and an important limiting case is obtained using equivalence relations leading to the partitioning of a set of objects. Classification based on equivalence relations is done in the theory of rough sets (Pawlak, 1991). Case-based reasoning (Kolodner, 1993) exploits the similarity between already solved problems and a new problem to be solved in order to build up a solution to this new problem. When this solution to a new problem is obtained by adapting solutions to already solved problems, the reasoning methodology then comes close to a matter of interpolation, whereby the value of a partially unknown function at a given point of a space is estimated by exploiting the proximity of this point to other points for which the value of the function is known. Although interpolative inference is part of usual commonsense reasoning tasks, it has been seldom considered as amenable to logical settings, because it fundamentally relies on a gradual view of proximities that is absent from classical logic. In contrast, uncertain reasoning, which also involves gradual notions, has received a logical treatment. Results in nonmonotonic reasoning show that some form of uncertain reasoning can be captured by equipping the set of interpretations with an ordering structure expressing plausibility (Shoham, 1988; Lehmann and Magidor, 1992). It is thus tempting to model interpolative reasoning by equipping a set of logical interpretations with a proximity structure.

This kind of investigation has been started by Ruspini (1991) with a view to cast fuzzy patterns of inference such as the generalized modus ponens of Zadeh (1979) into a logical setting. Indeed in the scope of similarity modeling, a basic reasoning pattern can be expressed informally as follows,

$$\frac{\begin{array}{c}\text{p is close to being true}\\ \text{p approximately implies q}\end{array}}{\text{q is not far from being true}} \qquad \text{(ES)}$$

where "close", "approximately", and "not far" refer to a similarity relation, while p and q are classical propositions. This pattern will be referred to as the *extrapolative syllogism* (ES), and is in accordance with the generalized modus ponens of Zadeh.

An example of situation where this type of inference pattern looks natural is the following. Consider the expert advice in finance: "if you have saved more than 10.000 $ (p) then you should invest 50 % of your capital". Suppose you have 9.500 $ (p'). Using classical logic, p' ¡ p, and thus p' $\wedge$ (p $\rightarrow$ q) ¡ q. But in practice people would not wait to reach the 10.000 $ threshold and would start investing some percentage of their savings, all the closer to 50 % as these savings amount to near 10.000 $. In that case the similarity stems from the metric structure equipping the monetary scale.

In this paper an attempt is made to equip the above pattern of inference with logical underpinnings. In the first section the concept of a similarity relation is recalled. Then two types of approximate inference based on similarity are defined. They are based on implication measures introduced by Ruspini (1991). It is shown that they are equivalent. A third type of inference, basically due to the lack of truth-functionality of logics of similarity, is also studied. Similarity based consequence relations are given representation theorems that highlight their characteristic properties. The next section generalizes one of the representation theorems to when the set of interpretations is restricted to a subset of models. Then three distinct approaches to approximate entailment exist. The following section explains how the proposed setting captures the type of interpolative reasoning at work in fuzzy rule-based systems, and suggests gradual rules (Dubois and Prade, 1992a) as a natural notion of fuzzy rules in this context. Some hints are given on the relation between similarity based consequence relations and Lewis' logics of counterfactuals. Proofs of main results are gathered in the Appendix.

2. Similarity and Approximate Satisfiability

One of the possible semantics of fuzzy sets (Zadeh, 1965) is in terms of similarity, namely a grade of membership $\mu_F(\omega)$ of an element ω in a fuzzy set F can be viewed as the degree of resemblance between ω and prototptypes of the fuzzy set F. One way of proceeding is to start with a set equipped with a similarity relation. Let Ω be a finite set and let I be structure of a complete lattice ordered semi-group, a structure proposed by Goguen (1969). A typical example of such a structure is the unit interval [0,1] equipped with maximum and minimum, an order-reversing operation (1 – x), and a semi-group operation $\otimes$ with identity 1, also known as a triangular norm (Schweizer and Sklar, 1983). Operation $\otimes$ is associative, commutative, non-decreasing in each place, $0 \otimes a = 0$ and $1 \otimes a = a$, $\forall a \in I$. For simplicity, we shall assume I = [0,1] in the paper. A fuzzy relation on Ω in the sense of (Zadeh, 1971) is a function S from $\Omega \times \Omega$ to I. Function S defines a similarity relation if and only if it satisfies the following properties (Valverde, 1985):

– reflexivity: $S(\omega,\omega') = 1$
– symmetry: $S(\omega,\omega') = S(\omega',\omega)$
– $\otimes$-transitivity: $S(\omega,\omega'') \geq S(\omega,\omega') \otimes S(\omega',\omega'')$.

When $S(\Omega \times \Omega) \in \{0,1\}$, S is clearly an equivalence relation on Ω. The operation $\otimes$ is supposed to be continuous. Typical instances of operations $\otimes$ are

– $\otimes$ = minimum: then S is a similarity relation in the sense of Zadeh (1971). Especially, each level-cut $S_\alpha = \{(\omega,\omega') \mid S(\omega,\omega') \geq \alpha\}$ is an equivalence relation, and 1 – S defines a pseudo-ultrametric;
– $\otimes$ = product: this type of fuzzy relation goes back to Menger (1951) and has been studied by Ovchinnikov (1991);
– $\otimes$ = Lukasiewicz conjunction, i.e., $a \otimes b = \max(0, a + b - 1)$. This type of fuzzy relation is studied by Ruspini (1977), Bezdek and Harris (1978) who call it a likeness relation. Then 1 – S is a pseudo-metric.

A similarity in thus a notion dual to a distance. In the scope of interpolation it is relevant to reinforce reflexivity as follows: $S(\omega,\omega') = 1$ iff $\omega = \omega'$. Then 1 – S is a generalization of a metric, and the 1 – cut of S (that is, $\{(\omega,\omega') \mid S(\omega,\omega') = 1\}$) is just the equality on Ω. In this paper, only such similarity relations will be used. Another expression for the transitivity is obtained from the concept of residuation. Namely let $a \otimes\!\!\rightarrow b = \sup\{c \mid c \otimes a \leq b\}$, where $\otimes\!\!\rightarrow$ is a multiple-valued implication obtained via residuation. The residuation concept enables the following equivalence to be stated:

$$a \otimes\!\!\rightarrow b \geq c \quad \text{iff} \quad c \otimes a \leq b$$

and it is obvious that the ⊗-transitivity of S is equivalent to stating for all ω, ω', ω":

$$S(\omega, \omega') \otimes\!\!\rightarrow S(\omega', \omega'') \geq S(\omega, \omega'').$$

Similarity relations and fuzzy sets can be closely related. Namely let $A \subseteq \Omega$ be a non-empty subset of Ω. Then a similarity relation S allows us to define the non-empty normalized fuzzy set A* of elements close to A as follows:

$$\mu_{A^*}(\omega) = \max_{\omega' \in A} S(\omega,\omega').$$

A* is the image of A through the relation S and will be denoted A±S as well. Reciprocally, any non-empty fuzzy set F on Ω can be viewed as deriving from a similarity relation S and a subset A such that

$$A = \{\omega \mid \mu_F(\omega) = 1\} \ (\neq \emptyset)$$

$$S(\omega,\omega') = \begin{cases} \min(\mu_F(\omega),\mu_F(\omega')), & \text{if } \mu_F(\omega) \neq \mu_F(\omega') \\ 1, & \text{otherwise} \end{cases}$$

In the above definition the fuzzy relation is min-transitive. However it is always possible to get a general ⊗-transitive relation. This is due to Valverde (1985)'s theorem of representation of similarity relations by fuzzy sets, based on residuation. Then the formula defining S in terms of F generalizes into

$$S(\omega,\omega') = \min(\mu_F(\omega) \otimes\!\!\rightarrow \mu_F(\omega'), \mu_F(\omega') \otimes\!\!\rightarrow \mu_F(\omega)).$$

When ⊗ = product,

$$S(\omega,\omega') = \min\left(\frac{\mu_F(\omega)}{\mu_F(\omega')}, \frac{\mu_F(\omega')}{\mu_F(\omega)}\right)$$

and S is product-transitive; when ⊗ is the Lukasiewicz conjunction, $S(\omega,\omega') = 1 - |\mu_F(\omega) - \mu_F(\omega')|$. This result gives a formal justification to the fact that a degree of membership $\mu_F(\omega)$ in a fuzzy set can be interpreted as a degree of similarity of ω to "prototypes" of F, which form the set A. Moreover it points out that if q is a proposition in a formal propositional language L, of which Ω is the (finite) set of interpretations, then the similarity induces a fuzzy proposition q* whose (fuzzy) set of models is [q*] = [q]* = [q] ± S defined by means of the fuzzy relation S, where [q] is the set of models of q (the set of interpretations where q is true). Intuitively q* means "approximately q", "not far from q", where "approximately", "not far from" is mathematically expressed by the similarity relation S.

Clearly, a logic dealing with propositions of the form q* is a fuzzy logic in the sense of a many-valued logic, whose truth-value set is the range of S(ω,ω'), for instance [0,1]. The satisfaction relation is graded and denoted ¥$_S^\alpha$; namely,

$$\omega \ ¥_S^{\alpha}\ q \text{ iff } \mu_{[q^*]}(\omega) \geq \alpha$$

That is, there exists a model ω' of q which is α-similar to ω. In other words, ω belongs to the α-cut of [q*] that will be denoted by $[q^*]_\alpha$. The degree of approximate satisfaction of q by ω in the sense of S has been introduced by Ruspini (1991), and shall be denoted $I_S(q \mid \omega) = \mu_{[q^*]}(\omega) = \max_{\omega' ¥ q} S(\omega,\omega')$. Note that, identifying each interpretation ω of Ω with the conjunction of literals made true by ω[1], we have that $I_S(\omega' \mid \omega) = S(\omega,\omega')$. Thus, one may have ω ¥$_S^\alpha$ ω' for ω' ≠ ω. Indeed it means that ω and ω' are close enough to each other in the sense that S(ω,ω') ≥ α. Note that ω ¥$_S^\alpha$ ω' is equivalent to ω' ¥$_S^\alpha$ ω since S is symmetric. For the sake of simplicity, the subscript S will be omitted from the symbol ¥$_S^\alpha$ whenever no confusion is possible.

One might be tempted to define a multiple-valued logic of similarity, requiring the usual truth-functionality assumptions. Unfortunately such a logic cannot be truth-functional. Namely given S, the truth-value evaluation I(q | ω) associated to the interpretation ω is truth-functional neither for the negation nor for the conjunction. To see it notice that if $A^* = [q]$, $\overline{A} = [\neg q]$

$$\mu_{(\overline{A})^*}(\omega) = \max_{\omega' \notin A} S(\omega,\omega')$$

while, noticing that the membership of the complement of a fuzzy set F is $c(\mu_F)$ where c is a negation function (decreasing, involutive, c(0) = 1, c(1) = 0), here c(x) = 1 – x:

$$\mu_{\overline{(A^*)}}(\omega) = 1 - \max_{\omega \in A} S(\omega,\omega') = \min_{\omega \in A} 1 - S(\omega,\omega').$$

The latter expression is clearly unrelated to the former, and is called lower approximation of äA, denoted $(äA)_*$ (Dubois and Prade, 1992b). It is easy to check that $A_* \subseteq A \subseteq A^*$, so that the fuzzy proposition A_* is the "interior" of A while A* encompasses its neighborhood.

For conjunction notice that generally

[1]From now on, and for the sake of simplicity, we shall use the same symbol ω to denote both an interpretation of the language L and the proposition which is the conjunction of literals made true by ω.

$$\max_{\omega' \in A \cap B} S(\omega',\omega)$$

is not a function of $\max_{\omega' \in A} S(\omega,\omega')$ and $\max_{\omega' \in B} S(\omega,\omega')$). For instance if A = $\{\omega_1\}$, B = $\{\omega_2\}$, then $[A \cap B]^* = Ø$ (empty set) but $A^* \cap B^* \neq Ø$, choosing ω_1 and ω_2 such that $S(\omega_1,\omega_2) > 0$. However for disjunction we do have that $[p \vee q]^* = [p]^* \cup [q]^*$ where $\cup$ translates into maximum of truth-value of p* and truth-value of q* since $\max_{\omega' \in A \cup B} S(\omega',\omega) = \max(\max_{\omega' \in A} S(\omega',\omega), \max_{\omega' \in B} S(\omega',\omega))$. Hence

$$I_S(p \vee q \mid \omega) = \max(I_S(p \mid \omega), I_S(q \mid \omega)).$$

This fact stresses the difference between similarity logic and other logics underlying fuzzy sets such as Lukasiewicz logic or the more recent family of monoidal logics (Höhle, 1994). The reason for the lack of truth-functionality is that here all fuzzy propositions are interpreted in the light of a single similarity relation. This fact puts a severe constraint on the set of fuzzy propositions, so that there are in some sense less fuzzy propositions here than in most standard many-valued calculi. Similarity logic is more constrained, since the set of fuzzy subsets of Ω corresponding to classical propositions $\{[q^*] \mid q \in L\}$ is only a proper subset of the set $[0,1]^\Omega$ of all fuzzy subsets of Ω. This lack of truth-functionality has also been noticed in the theory of rough sets (Pawlak, 1991). Rough sets are a theory of similarity based on equivalence relations that handles upper and lower approximations A_* and A^* of sets. The lack of truth-functionality is thus not due to the fuzziness of similarity.

A more natural logical setting for similarity-based reasoning is the one of modal logics which is tailored to account for relations on the set of interpretations. The similarity relation S can be considered as a family of nested accessibility relations R_α on the set of possible worlds Ω defined as

$$\omega R_\alpha \omega' \text{ iff } S(\omega, \omega') \geq \alpha$$

Therefore, enlarging the logical language, we can define, for each α, a usual pair of dual modal operators ¿$_\alpha$ and ◊$_\alpha$ with the following standard semantics:

ω ¥ ◊$_\alpha$p	iff	there exists ω' such that ω R_α ω' and ω' ¥ p
ω ¥ ¿$_\alpha$p	iff	for every ω' such that ω R_α ω' then it holds ω' ¥ p

If the similarity relation is min-transitive, i.e., $S(\omega,\omega') \geq \min(S(\omega,\omega''), S(\omega'',\omega'))$, then the accessibility relations R_α are equivalence relations, and therefore, for each α, ¿$_\alpha$ and ◊$_\alpha$ are a pair of dual S5 modal operators. These types of modal logics generalize rough set logics (Orlowska, 1984) and have been studied by Nakamura (1992). It is easy to check that the above defined graded satisfaction ¥$^\alpha$ is directly related to the possibility

operator ◊$_\alpha$ in the sense that if q is a non-modal proposition, then the following equivalence holds:

ω ¥$^\alpha$ q   iff   ω ¥ ◊$_\alpha$q.

However, rather than developping a full-fledged multimodal logic (see Rodriguez et al. (1995) for a preliminary investigation), it is also important to study what types of consequence relations make sense in the presence of similarity relations. Especially our purpose differs from the program of rough set logics and their extensions. In the latter the basic concept is indistinguishability, and the approximate description of objects or concepts that results from blurring effects. On the contrary, our emphasis is on interpolation, that is, exploiting the proximity of situations in order to make inference more powerful, and extrapolate about situations not strictly covered by the available knowledge.

3. Approximate Entailment

The previous graded satisfaction relation can be extended over to a graded semantic entailment relation between classical propositions:

Definition 1. A proposition p *approximately entails* a proposition q at degree α, written p ¥$^\alpha$ q, if and only if each p-world makes q* at least α-true:

p ¥$^\alpha$ q iff $[p] \subset [q^*]_\alpha$.

When $\alpha > 0$, p ¥$^\alpha$ q means that when p is true, q is close to being true, or in other words that p entails a proposition approximately equivalent to q. The meaning of this inference is made clear on Figure 1. The condition of this entailment relation can be also expressed using Ruspini's *implication measure* (Ruspini, 1991), when $p \neq \bot$, $q \neq \bot$ as:

p ¥$^\alpha$q iff $I_S(q \mid p) = \text{Min}_{\omega ¥ p} \max_{\omega' ¥ q} S(\omega,\omega') \geq \alpha$

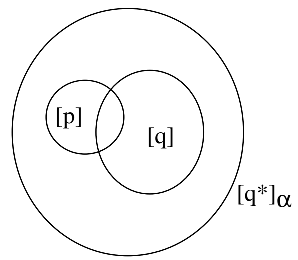

Figure 1: Approximate entailment $p \models^{\alpha} q$.

If $p = \perp$ then $I_S(q \mid \perp) = 1$. If $q = \perp$ and $p \neq \perp$ then $I(\perp \mid p) = 0$, by convention. A companion to this measure is the so-called consistency measure defined as $C_S(q \mid p) = \mathrm{Max}_{\omega \models p} \mathrm{Max}_{\omega' \models q} S(\omega,\omega')$ which will play a minor role in this paper. It is easy to verify that

$$I_S(q \mid p) = \min_{\omega \models p} I_S(q \mid \omega).$$

It is obvious that $\{\models^{\alpha}\}_{\alpha \in [0,1]}$ is a nested family: $p \models^{\alpha} q$ implies $p \models^{\beta} q$ for any $\beta \leq \alpha$. This is due to the nestedness property of level cuts. Besides it is continuous from below: if $p \models^{\beta} q$ for every $\beta < \alpha$, then $p \models^{\alpha} q$. Moreover, $\models^{1}$ is just the classical logic consequence relation. Indeed, $p \models^{1} q$ means that $\mathrm{Min}_{\omega \models p} \mathrm{Max}_{\omega' \models q} S(\omega, \omega') = 1$, and this in turn means that $[p] \subset \{\omega' \models q \mid S(\omega, \omega') = 1\}$. But the similarity relations considered here are such that $S(\omega, \omega') = 1$ implies $\omega = \omega'$. Hence $\{\omega' \models q \mid S(\omega, \omega') = 1\} = [q]$. Of course the inference $\models^{0}$ is the universal one because $I_S(q \mid p) \geq 0$ always holds. Remarkable properties of the approximate entailment relation $\models^{\alpha}$ are pointed out in (Dubois and Prade, 1995; Dubois et al.1995):

$\otimes$-Transitivity:	if $p \models^{\alpha} r$ and $r \models^{\beta} q$ then $p \models^{\alpha \otimes \beta} q$ where $\otimes$ is the t-norm for which the underlying similarity relation is transitive.
Reflexivity:	$\forall \alpha$, $p \models^{\alpha} p$
Right weakening:	if $q \models r$ and $p \models^{\alpha} q$ then $p \models^{\alpha} r$
Left strengthening:	if $p \models r$ and $r \models^{\alpha} q$ then $p \models^{\alpha} q$ (monotonicity)
Left OR:	$p \vee q \models^{\alpha} r$ iff $p \models^{\alpha} r$ and $q \models^{\alpha} r$
Right OR:	If r has a single model, $r \models^{\alpha} p \vee q$ iff $r \models^{\alpha} p$ or $r \models^{\alpha} q$
Consistency preservation:	if $p \neq \perp$ then $p \models^{\alpha} \perp$ only when $\alpha = 0$.

The most remarkable property is the $\otimes$-transitivity, which requires that $\otimes$ be continuous. It follows from the $\otimes$-transitivity of S and has been proved by Ruspini (1991, Theorem 1):

$$I_S(r \mid p) \geq I_S(r \mid q) \otimes I_S(q \mid p).$$

This result entirely bears on the existence of a unique similarity structure on Ω. Generally, *it does not hold* that for any set A and any two fuzzy sets $F = B \circ S$ and $G = C \circ S'$, induced by similarity relations S and S',

$$\min_{\omega \in A} \mu_F(\omega) = \alpha \text{ and } \min_{\omega \in B} \mu_G(\omega) = \beta \text{ imply } \min_{\omega \in A} \mu_G(\omega) = f(\alpha,\beta) > 0$$

Said otherwise, and using the modal notation $p \models \Diamond_{\alpha} q$, notice that $q \models \Diamond_{\alpha} q$ but not the converse $\Diamond_{\alpha} q \models q$, so that formally it is not expected anything informative from $p \models \Diamond_{\alpha} q$ and $q \models \Diamond_{\beta} r$, generally, if nothing relates the modalities. Of course it is possible to infer $p \models \Diamond_{\alpha} \Diamond_{\beta} r$ but no reduction of modalities may take place. In S5, $p \models \Diamond q$ and $q \models \Diamond r$ imply $p \models \Diamond r$, because $\Diamond\Diamond r$ coincides with $\Diamond r$. So if $S = S'$ is a min-transitive relation then $\Diamond_{\alpha} \Diamond_{\alpha}$ is nothing but $\Diamond_{\alpha}$. For other $\otimes$-transitive similarities, the transitivity property is weaker than usual and the graceful degradation of the strength of entailment it expresses is rather natural (Goguen, 1969). The right weakening and left strengthening properties are consequences of the $\otimes$-transitivity property.

The left OR is necessary to handle disjunctive information and is easily derived as follows:

$$\begin{aligned} I_S(q \mid p \vee r) &= \mathrm{Min}_{\omega \models p \vee r} \mathrm{Max}_{\omega' \models q} S(\omega, \omega') \\ &= \min(\mathrm{Min}_{\omega \models p} \mathrm{Max}_{\omega' \models q} S(\omega, \omega'), \mathrm{Min}_{\omega \models r} \mathrm{Max}_{\omega' \models q} S(\omega, \omega')) \\ &= \min(I_S(q \mid p), I_S(q \mid r)). \end{aligned}$$

The right OR is a consequence of the truth-functionality for the disjunction connective in similarity logic. It must be noticed that $\models^{\alpha}$ does not satisfy the *Right And* property, i.e., from $p \models^{\alpha} q$ and $p \models^{\alpha} r$ it does not follow in general that $p \models^{\alpha} q \wedge r$. Hence the set of consequences of p in the sense of $\models^{\alpha}$ will neither be deductively closed nor consistent (it is possible to have both $[p] \subseteq [q^*]_{\alpha}$ and $[p] \subseteq [(\neg q)^*]_{\alpha}$). But what will be exploited for the purpose of interpolative reasoning is not so much the set $\{q \mid p \models^{\alpha} q\}$ of α-consequences of p, but the fuzzy set of approximate conclusions $\{([q^*]_{\alpha}, \alpha) \mid \exists\, \alpha > 0, p \models^{\alpha} q\}$. Note that while $\{q \mid p \models^{\alpha} q\}$ is generally inconsistent, the set $\{[q^*]_{\alpha} \mid \exists\, \alpha > 0, p \models^{\alpha} q\}$ is consistent in the sense that the set of interpretations $\cap\{[q^*]_{\alpha} \mid \exists\, \alpha > 0, p \models^{\alpha} q\}$ is non empty.

Lastly, as indicated also in (Dubois and Prade, 1995), another property, the so-called CUT, which holds for classical semantical entailment, does not hold either for $\models^{\alpha}$, that is,

$$p \models^{\alpha} r \text{ and } p \wedge r \models^{\beta} q \text{ do not imply } p \models^{f(\alpha,\beta)} q$$

with $\alpha > 0$, $\beta > 0$ and $f(\alpha,\beta) > 0$. Indeed, consider the case where $[p] \cap [r] = \emptyset$, then $I_S(q \mid p \wedge r) = 1$. In general there exists q such that $I_S(q \mid p) = 0$ while there exist p and r such that $I_S(q \mid p \wedge r) \otimes I_S(r \mid p) = 1 \otimes I_S(r \mid p) > 0$, i.e., $\exists \alpha$, $[p] \subseteq [r] \circ S_{\alpha}$ (with $[p] \cap$

$[r] = \emptyset$). Indeed $I_S(r \mid p) > \alpha$ only expresses the inclusion of the set of models of p into the set of neighbours of the models of q, but not necessarily into the subset of the models of q.

A first result towards a characterization of the similarity-based approximate entailment in terms of the above properties is given in the next theorem, where a consequence relation $p\ ƒ^{\alpha}\ q$ means "p entails approximately q", "p entails that q is not far from being true", and α is a level of strength, expressing closeness to truth.

Theorem 1 (Characterization of the approximate entailment). Let L be a finite Boolean algebra of propositions. Suppose we have a family of consequence relations $\{ƒ^{\alpha}\}_{\alpha \in [0,1]}$ (i.e., $ƒ^{\alpha} \subset L \times L$) fulfilling the following properties:

1) $\{ƒ^{\alpha}\}_{\alpha \in [0,1]}$ is a nested family: $p\ ƒ^{\alpha}\ q$ implies $p\ ƒ^{\beta}\ q$ for any $\beta \le \alpha$,
2) $ƒ^{1}$ is exactly the classical logic consequence relation and $ƒ^{0}$ is the universal one,
3) *symmetry*: $\omega\ ƒ^{\alpha}\ \omega'$ iff $\omega'\ ƒ^{\alpha}\ \omega$, for any pair of interpretations.
4) *$\otimes$-transitivity*: $p\ ƒ^{\alpha}\ q$ and $q\ ƒ^{\beta}\ r$ implies $p\ ƒ^{\alpha\otimes\beta}\ r$
5) *Left Or:* $p \vee r\ ƒ^{\alpha}\ q$ iff $p\ ƒ^{\alpha}\ q$ and $r\ ƒ^{\alpha}\ q$
6) *Right OR*: If r has a single model, $r\ ¥^{\alpha}\ p \vee q$ iff $r\ ¥^{\alpha}\ p$ or $r\ ¥^{\alpha}\ q$
7) *Consistency preservation:* if $p \ne \bot$ then $p\ ƒ^{\alpha}\ \bot$ only if $\alpha = 0$.
8) *Continuity from below*: if $p\ ƒ^{\beta}\ q$ for every $\beta < \alpha$, then $p\ ƒ^{\alpha}\ q$

Then, there exists a $\otimes$-similarity relation S on the set Ω such that $p\ ƒ^{\alpha}\ q$ iff $I_S(q \mid p) \ge \alpha$, for each $\alpha \in [0,1]$. And conversely, for any $\otimes$-similarity S on Ω, the consequence relation defined as $p\ ¥^{\alpha}\ q$ iff $I_S(q \mid p) \ge \alpha$ verifies the above set of properties.

The basic step in the proof, given in the Appendix and abusing the notation, is the postulated identity between $S(\omega,\omega') \ge \alpha$ and $\omega\ ƒ^{\alpha}\ \omega'$. So, it is natural to define $S(\omega,\omega') = \sup\{\alpha \mid \omega\ ƒ^{\alpha}\ \omega'\}$, and to check that the axioms make it a similarity relation, the one that generates the family $\{ƒ^{\alpha}\}$. The continuity condition makes it sure that $\omega\ ƒ^{\alpha}\ \omega'$ can be derived from $S(\omega, \omega') \ge \alpha$. This assumption would be not necessary if only a finite set of nested consequence relations $\{ƒ^{\alpha}\}_{\alpha \in G}$ is considered, where G is a finite subset of [0, 1] containing 0 and 1, and (G, $\otimes$) is a finite totally ordered semi-group that is not necessarily the restriction of a t-norm. In that case, a $\otimes$-similarity relation S should be understood as a reflexive, symmetric and $\otimes$-transitive relation $S: \Omega \times \Omega \to G$.

Using again the modal logic setting, an alternative entailment in similarity logic would be: $p\ ¥\ ¿_{\alpha}q$. By definition, we get that $p\ ¥\ ¿_{\alpha}q$ iff $[p^*]_{\alpha} \subset [q]$. It means that not only does p imply q but that q also holds in the vicinity of p. As usual in modal logics,

this notion of entailment is stronger than the classical entailment, both being equivalent only in the case $\alpha = 1$, and thus it does not correspond to the idea of approximate inference. Finally, notice that the implication and consistency measures $I_S(q \mid p)$ and $C_S(q \mid p)$ can also be expressed in terms of the modal entailments (Esteva et al., 1994a):

$$I_S(q \mid p) = \sup\{\alpha \mid p\ ¥\ \Diamond_{\alpha}q\}$$
$$C_S(q \mid p) = \sup\{\alpha \mid p \wedge \Diamond_{\alpha}q \ne \bot\}.$$

4. Proximity Entailment: another view of similarity-based inference.

The considered approximate entailment evaluates to what extent p is close to being true in the extrapolative syllogism, given that another proposition p' is known to hold. Now we turn to the representation of the second premise of the extrapolative syllogism. One way of modeling it is to attach a weight β to this premise and to interpret "p approximately implies q" as $p\ ¥^{\beta}\ q$. However there is another understanding of the second premise, namely: p implies q, and if p is close to being true then q is close to being true as well. In other words the neighborhood of the models of p should lie in the neighborhood of the models of q. This can be formally expressed as $[p^*] \subseteq [q^*]$, where the inclusion is in the sense of Zadeh (1965). This inference is denoted as $p \models\!\equiv q$. Obviously, $p \models\!\equiv q$ iff $\forall\omega$, $I_S(p \mid \omega) \le I_S(q \mid \omega)$, or equivalently,

$$\forall\omega,\ \max_{\omega' ¥ p} S(\omega,\omega') \le \max_{\omega' ¥ q} S(\omega,\omega')$$

The name "proximity entailment" expresses that the implication relates not only the models of p to the models of q, but also involves the interpretations close to the models of p; see Figure 2.

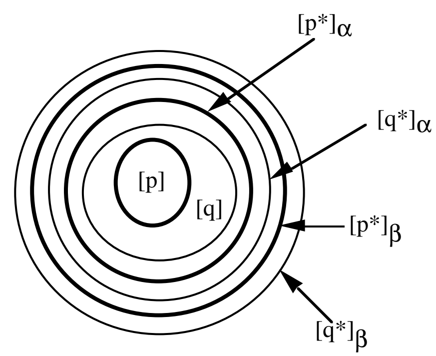

Figure 2: Proximity entailment $p \models\!\equiv^{1} q$

It can also be expressed in terms of a multiple-valued implication $\to$, called Rescher-Gaines implication, such that $a \to b = 1$ if $a \le b$ and 0 otherwise. Then $p^* \to q^*$ is a classical proposition and $p \models q$ iff $\Omega \subseteq [p^* \to q^*]$. However this definition does not look exciting as such. First note that $[\neg p \vee q]$ contains $[p^* \to q^*]$ as a subset. Indeed consider ω ¥ $\neg p \wedge \neg q$ such that $\max_{\omega' ¥ p} S(\omega,\omega') = \alpha > 0$ and $\max_{\omega' ¥ q} S(\omega,\omega') = 0$. Then, ω is in the vicinity of $[p]$ but far from $[q]$. So, $\omega \notin [p^* \to q^*]$ while ω ¥ $\neg p \vee q$. Now if $\omega \in [p^* \to q^*]$, either ω ¥ $\neg p$ and ω ¥ $\neg p \vee q$, or ω ¥ p. In the latter case, $\max_{\omega' ¥ q} S(\omega,\omega') = 1$, hence ω ¥ q since $S(\omega,\omega') = 1$ only when $\omega = \omega'$. As a consequence the entailment $p \models q$ is equivalent to the classical entailment since if $\Omega = [p^* \to q^*]$, then $[\neg p \vee q] = [p^* \to q^*] = \Omega$. This is due to the fact that the same similarity relation is used for exploiting the neighborhood of p and q. However, viewing $p^* \to q^*$ as a constraint, restricting to the interpretations where the formula $p^* \to q^*$ is true, is more demanding than material implication.

The set of models of $[p^* \to q^*]$ can be extended to a fuzzy set of models if we use a residuated implication $\otimes\!\!\to$ and let

$$\mu_{[p^* \otimes\to q^*]}(\omega) = \sup\{\alpha \mid \alpha \otimes \mu_{[p^*]}(\omega) \le \mu_{[q^*]}(\omega)\}.$$

Then $[p^* \to q^*]$ is just the core of $[p^* \otimes\!\!\to q^*]$. In fact $p^* \to q^*$ coincides with a gradual rule (Dubois and Prade, 1992a) of the form the "closer to truth is p, the closer to truth is q". It is easy to figure out that $[p^* \to q^*] = {}_{\alpha\in[0,1]}\ \overline{[p^*]_\alpha} \cup [q^*]_\alpha$. We also get a similar expression when we consider the residuated implication $\otimes\!\!\to$. Namely,

$$[p^* \otimes\to q^*]_\alpha = {}_{\beta\in[0,1]}\ \overline{[p^*]_\beta} \cup [q^*]_{\alpha\otimes\beta}$$

since it holds that $x \otimes\!\!\to y \ge \alpha$ iff $x \ge \beta$ implies $y \ge \alpha\otimes\beta$, for all β. In particular, when $\otimes$ = minimum this expression reduces to

$$[p^* \otimes\to q^*]_\alpha = {}_{\beta\le\alpha}\ \overline{[p^*]_\beta} \cup [q^*]_\beta$$

Hence, the proximity entailment summarizes several classical entailment relations.

Using the fuzzy implication operation, the proximity entailment can be graded as well, and then it differs from the classical one:

Definition 2. The α-proximity entailment $p \models^\alpha q$ is defined by

$$p \models^\alpha q \text{ iff } \Omega \subseteq [p^* \otimes\to q^*]_\alpha.$$

Equivalently,

$$p \models^\alpha q \text{ iff } \forall\omega,\ \alpha \otimes I_S(p \mid \omega) \le I_S(q \mid \omega)$$

since $a \otimes\!\!\to b \ge \alpha$ is equivalent to $a \otimes \alpha \le b$. The corresponding graded satisfiability relation is thus

$$\omega \models^\alpha q \Leftrightarrow S(\omega,\omega') \otimes \alpha \le I_S(q \mid \omega'),\ \forall\, \omega' \in \Omega.$$

Strictly speaking this is not a usual satisfiability relation since it also involves interpretations in the vicinity of ω. It means that for any ω' in the vicinity of ω, there is a model of q, say ω'', that is close to ω' at level $\alpha \otimes S(\omega,\omega')$ at least . The proximity entailment relation, like the approximate entailment can be related to an implication measure $J_S(q \mid p) = \min_{\omega\in\Omega} I_S(p \mid \omega) \otimes\!\!\to I_S(q \mid \omega)$, in the sense that

$$p \models^\alpha q \text{ iff } J_S(q \mid p) \ge \alpha.$$

The implication measure $J_S(q \mid p)$ is strongly related to Ruspini's "conditional necessity distributions" (Ruspini 1991, Definition 6) and it is a particular case of the "conditional implication measure" introduced in (Esteva et al. 1994a). The following proposition shows the proximity entailment is in fact the same as the approximate entailment.

Proposition 1. $J_S(q \mid p) = I_S(q \mid p)$

Proof. Let us first show that $J_S(q \mid p) \le I_S(q \mid p)$. Indeed:

$$\begin{aligned} J_S(q \mid p) &= \min_{\omega\in\Omega} I_S(p \mid \omega) \otimes\!\!\to I_S(q \mid \omega) \\ &\le \min_{\omega ¥ p} 1 \otimes\!\!\to I_S(q \mid \omega) && \text{—since } I_S(p \mid \omega) = 1 \text{ when } \omega ¥ p \\ &= \min_{\omega ¥ p} I_S(q \mid \omega) && \text{—since } 1 \otimes\!\!\to a = a \\ &= I_S(q \mid p). \end{aligned}$$

Now for the converse, develop $J_S(q \mid p)$ as

$$\begin{aligned} J_S(q \mid p) &= \min_{\omega\in\Omega} (\max_{\omega' ¥ p} S(\omega', \omega)) \otimes\!\!\to (\max_{\omega'' ¥ q} S(\omega', \omega)) \\ &= \min_{\omega\in\Omega} \min_{\omega' ¥ p} \max_{\omega'' ¥ q} S(\omega', \omega) \otimes\!\!\to S(\omega'', \omega) \end{aligned}$$

(since the implication is decreasing in the wide sense in the first argument and increasing in the second one.)

$$\begin{aligned} &\ge \max_{\omega'' ¥ q} \min_{\omega\in\Omega} \min_{\omega' ¥ p} S(\omega', \omega)) \otimes\!\!\to S(\omega'', \omega) \\ &\ge \max_{\omega'' ¥ q} \min_{\omega' ¥ p} S(\omega', \omega'') && \text{—(using transitivity)} \end{aligned}$$

$= I_S(q \mid p).$ ¿

Consequently, the proximity entailment $|\!\equiv^{\alpha}$ is equivalent to the approximate entailment $¥^{\alpha}$. Note that this equivalence holds due to the transitivity of S. Without this property, there holds that $p \,|\!\equiv^{\alpha} q$ implies $p \,¥^{\alpha}\, q$ only, because $J_S(q \mid p) \le I_S(q \mid p)$ still holds.

5. Yet Another Definition of Approximate Entailment

In classical logic, $p ¥ q$ is equivalent to $p ¥ p \wedge q$, as a consequence of the closure of sets of consequences under the conjunction. However in similarity logic such a closure is not valid. It is thus possible to define an alternative approximate entailment notion that exploits this fact:

$$p \,¥^{\alpha}_{\wedge}\, q \Leftrightarrow p \,¥^{\alpha}\, p \wedge q \Leftrightarrow I_S(p \wedge q \mid p) \ge \alpha.$$

This definition also means that $p \,¥^{\alpha}\, q$ is equivalent to $p \vee q \,¥^{\alpha}_{\wedge}\, q$ since the latter means $I_S(q \mid p \vee q) = \min(I_S(q \mid q), I_S(q \mid p)) = I_S(q \mid p) \ge \alpha$. So, when $q ¥ p$, $p \,¥^{\alpha}_{\wedge}\, q$ is equivalent to $p \,¥^{\alpha}\, q$, and in general $p \,¥^{\alpha}_{\wedge}\, q$ is equivalent to $p \,¥^{\alpha}_{\wedge}\, p \wedge q$. The new concept is however more demanding than the old one.

Proposition 2. $p \,¥^{\alpha}_{\wedge}\, q$ implies $p \,¥^{\alpha}\, q$. For $\alpha \ne 0,1$, the converse is false.

Proof. Just use the right weakening property for $¥^{\alpha}$. For the converse, assume $\alpha \ne 0,1$ (otherwise we get usual kinds of consequence relations). It is obvious that one may have $p \wedge q = \bot$ and $p \,¥^{\alpha}\, q$. But $p \,¥^{\alpha}_{\wedge}\, q$ then means $p \,¥^{\alpha}\, \bot$ which is not valid. ¿

Interestingly, $\omega \,¥^{\alpha}_{\wedge}\, \omega'$ never holds for $\omega \ne \omega'$. However we may have $\omega \vee \omega' \,¥^{\alpha}_{\wedge}\, \omega'$ where for simplicity ω denotes an interpretation as well as the formula whose unique interpretation is ω. It is easy to verify that $\omega \vee \omega' \,¥^{\alpha}_{\wedge}\, \omega'$ is equivalent to $\omega \vee \omega' \,¥^{\alpha}\, \omega'$ using the previously defined approximate entailment. Now, $I_S(\omega' \mid \omega \vee \omega') = \min(S(\omega,\omega'), S(\omega',\omega')) = S(\omega,\omega')$. Hence $\omega \vee \omega' \,¥^{\alpha}_{\wedge}\, \omega'$ is equivalent to $\omega \,¥^{\alpha}\, \omega'$ which is a symmetric notion. Hence $\omega \vee \omega' \,¥^{\alpha}_{\wedge}\, \omega'$ is also equivalent to $\omega \vee \omega' \,¥^{\alpha}_{\wedge}\, \omega$. So $\omega \vee \omega' \,¥^{\alpha}_{\wedge}\, \omega'$ is another way of writing $S(\omega,\omega') \ge \alpha$. Obvious properties of the approximate entailment relation $¥^{\alpha}_{\wedge}$ are reflexivity, right weakening, and cut. The latter writes

$$p \,¥^{\alpha}_{\wedge}\, q \text{ and } p \wedge q \,¥^{\beta}_{\wedge}\, r \quad \text{imply} \quad p \,¥^{\alpha\otimes\beta}_{\wedge}\, r.$$

Indeed this property also reads

$$p \,¥^{\alpha}\, p \wedge q \text{ and } p \wedge q \,¥^{\beta}\, p \wedge q \wedge r \quad \text{imply} \quad p \,¥^{\alpha\otimes\beta}\, p \wedge r$$

which holds by transitivity of $¥^{\alpha}$ and using the Right Weakening. The price paid is the lack of transitivity of $¥^{\alpha}_{\wedge}$. Indeed $p \,¥^{\alpha}_{\wedge}\, q$ and $q \,¥^{\beta}_{\wedge}\, r$ do not imply $p \,¥^{\alpha\otimes\beta}_{\wedge}\, r$ since the two previous conditions do not exclude the case when $p \wedge r = \bot$. The left OR property is restricted into:

Proposition 3. $p \vee q \,¥^{\alpha}_{\wedge}\, r$ is equivalent to $p \,¥^{\alpha}_{\wedge}\, r$ and $q \,¥^{\alpha}_{\wedge}\, r$, if and only if $r \wedge p = r \wedge q \ne \bot$.

Proof. $p \vee q \,¥^{\alpha}_{\wedge}\, r$ means $p \vee q \,¥^{\alpha}\, r \wedge (p \vee q)$ which is equivalent to $p \,¥^{\alpha}\, r \wedge (p \vee q)$ and $q \,¥^{\alpha}\, r \wedge (p \vee q)$. Now $p \,¥^{\alpha}\, r \wedge (p \vee q)$ is equivalent to $p \,¥^{\alpha}\, r \wedge p$ if and only if $(r \wedge p) \vee (r \wedge q) = r \wedge p \ne \bot$. The other condition $q \,¥^{\alpha}_{\wedge}\, r$ equivalent to $q \,¥^{\alpha}\, r \wedge (p \vee q)$ leads to $(r \wedge p) \vee (r \wedge q) = r \wedge q \ne \bot$. The required condition is thus the equivalence between $r \wedge p$ and $r \wedge q$ and their non-contradiction. ¿

Again the left OR property holds if the possibility that $r \wedge p = \bot$ or $r \wedge q = \bot$ are ruled out. However the weaker following property holds unconditionally:

Proposition 4. If $p \,¥^{\alpha}_{\wedge}\, r$ and $q \,¥^{\alpha}_{\wedge}\, r$ then $p \vee q \,¥^{\alpha}_{\wedge}\, r$.

Proof. We have to prove that $p \,¥^{\alpha}\, p \wedge r$ and $q \,¥^{\alpha}\, q \wedge r$ imply $p \vee q \,¥^{\alpha}\, r \wedge (p \vee q)$. Now, this is just a consequence of a more general property of $¥^{\alpha}$, namely: $p \,¥^{\alpha}\, q$ and $p' \,¥^{\alpha}\, q'$ imply $p \vee p' \,¥^{\alpha}\, q \vee q'$. This property is a consequence of the left OR and right weakening properties of $¥^{\alpha}$, applying left OR to $p \,¥^{\alpha}\, q \vee q'$ and $p' \,¥^{\alpha}\, q \vee q'$. ¿

Lastly, it also holds that

$$\forall\omega, (\omega \vee p \,¥^{\alpha}_{\wedge}\, p \text{ or } \omega \vee q \,¥^{\alpha}_{\wedge}\, q) \text{ iff } \omega \vee p \vee q \,¥^{\alpha}_{\wedge}\, p \vee q.$$

This is due to the right OR property of the approximate entailment $¥^{\alpha}$, since $\omega \vee p \,¥^{\alpha}_{\wedge}\, p$ writes $\omega \,¥^{\alpha}\, p$. Of course the AND rule fails for $¥^{\alpha}_{\wedge}$ just like for $¥^{\alpha}$, and for the same reason. Lastly note that $\omega \,¥^{\alpha}_{\wedge}\, p$ either holds trivially with $\alpha = 1$ (when $\omega ¥ p$, since it is equivalent to $\omega \,¥^{\alpha}\, \omega$) or fails ($\alpha = 0$) if $\omega ¥ \neg p$. Let us characterize the strong approximate entailment $¥^{\alpha}_{\wedge}$.

Theorem 2. **Characterization of strong approximate entailment.** Let L be a finite Boolean algebra of proposition and let {ƒ$^{\alpha}$ | α ∈ [0,1]} be a family of consequence relations on L satisfying the following properties

(i) p ƒ$^{\alpha}$ q implies p ƒ$^{\beta}$ q for every β ≤ α

(ii) ƒ^{1} is the classical logical consequence and ƒ^{0} the universal one

(iii) *consistency preservation:* p ƒ$^{\alpha}$ ⊥ only if α = 0 or p = ⊥

(iv) *cut* p ƒ$^{\alpha}$ q and p ∧ q ƒ$^{\beta}$ r imply p ƒ$^{\alpha\otimes\beta}$ r.

(v) p ƒ$^{\alpha}$ q is equivalent to p ƒ$^{\alpha}$ p ∧ q

(vi) *restricted left OR:* p ƒ$^{\alpha}$ r and q ƒ$^{\alpha}$ r implies p ∨ q ƒ$^{\alpha}$ r, and is implied by p ∨ q ƒ$^{\alpha}$ r if and only if r ∧ p = r ∧ q≠ ⊥

(vii) *disjunction on both sides:* ω ∨ p ∨ q ƒ$^{\alpha}$ p ∨ q iff ω ∨ p ƒ$^{\alpha}$ p or ω ∨ q ƒ$^{\alpha}$ q

(viii) *disjunctive symmetry:* ω ∨ ω' ƒ ω ⇔ ω ∨ ω' ƒ ω'

(ix) *Continuity from below*: if p ƒ$^{\beta}$ q for every β < α, then p ƒ$^{\alpha}$ q

Then there exists a strong approximate entailment family {¥$_{\wedge}^{\alpha}$}$_{\alpha\in[0,1]}$ induced by a similarity relation S, such that ƒ$^{\alpha}$ = ¥$_{\wedge}^{\alpha}$, ∀α.

First we need the following lemmas:

Lemma 1. If {ƒ$^{\alpha}$ | α ∈ [0,1]} satisfies (ii) and (iv) then it satistfies the Right Weakening property: p ƒ$^{\alpha}$ q and q ¥ r imply p ƒ$^{\alpha}$ r.

Proof.If q ¥ r then it also holds that p ∧ q ¥ r , and thus, by (ii), we have p ∧ q ƒ^{1} r. Therefore, from p ƒ$^{\alpha}$ q and p ∧ q ƒ^{1} r, by (iv), it comes p ƒ$^{\alpha\otimes 1}$r where α⊗1 = α. ¿

Lemma 2. If {ƒ$^{\alpha}$ | α ∈ [0,1]} satisfies the conditions (i), (ii), (v), (vi) of Theorem 3, then p ƒ$^{\alpha}$ q implies p ∨ r ƒ$^{\alpha}$ q ∨ r.

Proof. Indeed p ∨ r ƒ$^{\alpha}$ q ∨ r iff p ∨ r ƒ$^{\alpha}$ (q ∨ r) ∧ (p ∨ r), by (v), that is, p ∨ r ƒ$^{\alpha}$ r ∨ (p ∧ q). Since r ƒ^{1} r holds r ƒ$^{\alpha}$ r holds by (i) and r ƒ$^{\alpha}$ r ∨ (p ∧ q) by Right weakening; and if p ƒ$^{\alpha}$ q, we can derive p ƒ$^{\alpha}$ (p ∧ q) ∨ r by (v) and Right weakening; hence p ∨ r ƒ$^{\alpha}$ r ∨ (p ∧ q) by left OR (vi). ¿

Proof of Theorem 2. It has been shown above that given S and ¥$_{\wedge}^{\alpha}$ induced by S then all properties (i-ix) hold. Conversely, given {ƒ$^{\alpha}$}$_{\alpha\in[0,1]}$ define S(ω,ω') = sup{α | ω ∨ ω' ƒ$^{\alpha}$ ω}. This relation is symmetric from (viii) and reflexive since ω ƒ^{1} ω always holds. S(ω,ω') = 1 implies ω = ω'. Moreover the continuity condition again makes it sure that ω ∨ ω' ƒ$^{\alpha}$ ω derives from S(ω,ω') ≥ α.Let us prove that S is transitive:

S(ω,ω') ⊗ S(ω',ω") = sup{α | ω ∨ ω' ƒ$^{\alpha}$ ω'} ⊗ sup{β | ω' ∨ ω" ƒ$^{\beta}$ ω"}
= sup{α ⊗ β | ω ∨ ω' ƒ$^{\alpha}$ ω' and ω' ∨ ω" ƒ$^{\beta}$ ω"} (continuity of ⊗)

≤ sup{α ⊗ β | ω ∨ ω' ∨ ω" ƒ$^{\alpha}$ ω' ∨ ω" and ω' ∨ ω" ƒ$^{\beta}$ ω"} (Lemma 2)
≤ sup{α ⊗ β | ω ∨ ω' ∨ ω" ƒ$^{\alpha\otimes\beta}$ ω"} (cut)
≤ sup{γ | ω ∨ ω" ƒ$^{\gamma}$ ω"} (restricted left OR)
= S(ω,ω")

using ω"∧ (ω ∨ ω") = ω" ∧ (ω' ∨ ω") ≠ ⊥ when applying (vi). Now consider the approximate entailment defined by p ¥$_{\wedge}^{\alpha}$ q if and only if I$_{S}$(p ∧ q | p) ≥ α. Assume q ≠ ⊥ and p ≠ ⊥. Then

p ƒ$^{\alpha}$ q	iff	p ƒ$^{\alpha}$ p ∧ q	(v)
	iff	∀ ω ¥ p, ω ∨ (p ∧ q)ƒ$^{\alpha}$ p ∧ q	(vi)
	(since [(ω ∨ (p ∧ q)] ∧ p ∧ q = p ∧ q ≠⊥, and (ω ∨ (p ∧ q) ƒ p))		
	iff	∀ ω ¥ p, ∃ ω' ¥ p ∧ q, ω ∨ ω' ƒ$^{\alpha}$ p ω'	(vii)
	iff	∀ ω ¥ p, ∃ ω' ¥ p ∧ q, S(ω,ω') ≥ α	
	iff	I$_{S}$(p ∧ q | p) ≥ α iff p ¥$_{\wedge}^{\alpha}$ q.	

Consistency preservation ensures that this equivalence holds even when p ≠ ⊥ and q = ⊥. ¿

Actually, the properties of ¥$_{\wedge}^{\alpha}$ are induced by ¥$^{\alpha}$, in the sense that ¥$_{\wedge}^{\alpha}$ is expressible in terms of ¥$^{\alpha}$. The set {(p,q) | p ¥$_{\wedge}^{\alpha}$ q} is indeed a subset of {(p,q) | p ¥$^{\alpha}$ q}. Theorem 2 suggests that the similarity relation underlying ¥$^{\alpha}$ can be characterized by a subpart of it, i.e., {(p,q) | p ¥$^{\alpha}$ q, q ¥ p}using (v).

6. Similarity-based Consequence Relations with Background Knowledge

A natural question about the similarity-based entailment is how to deal with some prior information which is available under the form of a set K of formulas or a subset of worlds E = [K] (the so-called *evidential set* in Ruspini (1991)'s paper). Several extensions of the entailment notions studied above can be envisaged. In particular we shall consider through this section three ways of defining the inference of q by p, restricted to the models of K. These definitions coincide in propositional logic: p ¥$_{K}$ q iff K ∧ p ¥ q iff K ¥ ¬p ∨ q. Moreover although the approximate and proximity entaiments coincide when no background knowledge is considered, they will differ when K is involved.

The first and direct option is just to take the set K as a restriction on the set of p-worlds, and thus considering the extension $¥^{\alpha}_{K}$ of the approximate entailment, defined as follows:

$$p ¥^{\alpha}_{K} q \text{ iff } K \wedge p ¥^{\alpha} q.$$

In other words, we have that $p ¥^{\alpha}_{K} q$ iff $I_S(q \mid K \wedge p) \geq \alpha$. This amounts to expressing that [q] must be stretched to the degree α (at least) in order to encompass the models of K which are models of p. A natural question is whether some form of deduction theorem would still hold, namely by comparing $I(q \mid p \wedge K)$ and $I(\neg p \vee q \mid K)$. Unfortunately, it only holds that

$$I(q \mid p \wedge K) \geq \max(I(q \mid p), I(q \mid K))$$
$$I(\neg p \vee q \mid K) \geq \max(I(\neg p \mid K), I(q \mid K)).$$

Hence there is apparently no way of getting the deduction theorem using approximate entailment. This is in total contrast with possibilistic logic (see Dubois et al. (1994)) where given a set K of necessity-weighted formulas, q is a consequence of $p \wedge K$ with necessity degree α if and only if $\neg p \vee q$ is a consequence of K with necessity degree α.

Although entailment $¥^{\alpha}_{K}$ verifies properties like Reflexivity, Right Weakening or Left Strengthening as $¥^{\alpha}$ does, it does not satisfy the previous transitivity property, even the restricted form of transivity

$$\text{if } ¥^{\alpha}_{K} \text{ r and r } ¥^{\beta}_{K} \text{ q then } ¥^{\alpha\otimes\beta}_{K} \text{ q}$$

does not hold, due to the failure of the inequality $I_S(q \mid p) \geq I_S(q \mid p \wedge r) \otimes I_S(r \mid p)$ which is in turn the responsible of the failure of the CUT property for entailment relation $¥^{\alpha}$ (see Dubois and Prade, 1995). Only the following restricted form of transivity holds:

Restricted Transivity: if $p ¥^{\alpha}_{K} r$ and $r ¥^{\beta}_{K} q$ then $p ¥^{\alpha\otimes\beta}_{K} q$, provided that r ¥ K

This is obvious because then it comes down to checking the transitivity of $¥^{\alpha}$ with propositions $p \wedge K$, r (equivalent to $r \wedge K$) and q respectively. $¥^{\alpha}_{K}$ also satisfies a variant of transitivity in a particular case of formulas with a single model:

$$\omega ¥^{\alpha}_{K} \omega'', \ \omega' ¥^{\beta}_{K} \omega'' \text{ imply } \omega ¥^{\alpha\otimes\beta}_{K} \omega' \text{ when } \omega ¥ K \text{ and } \omega' ¥ K.$$

This is because $\omega ¥^{\alpha}_{K} \omega''$ just expresses $S(\omega'',\omega) \geq \alpha$ and similarly for $\omega' ¥^{\beta}_{K} \omega''$. Symmetry of $\omega ¥^{\alpha}_{K} \omega'$ is also restricted to when ω ¥ K and ω' ¥ K. Other properties of $¥^{\alpha}$ hold. A full characterization of $¥^{\alpha}_{K}$, which is an extension of the previous one for $¥^{\alpha}$, is given in the next theorem.

Theorem 3 . Let L be a finite Boolean algebra of propositions, $\{ƒ^{\alpha}\}_{\alpha\in[0,1]}$ a family of nested binary relations on L, i.e., $ƒ^{\alpha} \subset L \times L$ and $p ƒ^{\alpha} q$ implies $p ƒ^{\beta} q$ for any $\beta \leq \alpha$, and let p_0 be a proposition of L different from ⊥. If $\{ƒ^{\alpha}\}_{\alpha\in[0,1]}$ is a family of consequence relations such that

1) $p ƒ^{1} q$ iff $p \wedge p_0$ ¥ q; $ƒ^{0}$ is the universal consequence relation
2) *restricted symmetry*: $\omega ƒ^{\alpha} \omega'$ iff $\omega' ƒ^{\alpha} \omega$, for ω ¥ p_0 and ω' ¥ p_0
3) *restricted transitivity*:
   (i) if $p ƒ^{\alpha} r$ and $r ƒ^{\beta} q$ then $p ƒ^{\alpha\otimes\beta} q$, provided that r ¥ p_0
   (ii) $\omega ƒ^{\alpha} \omega''$ and $\omega' ƒ^{\beta} \omega''$ implies $\omega ƒ^{\alpha\otimes\beta} \omega'$, for ω ¥ p_0 and ω' ¥ p_0
4) *Left Or*: $p \vee r ƒ^{\alpha} q$ iff $p ƒ^{\alpha} q$ and $r ƒ^{\alpha} q$
5) *Decomposition*: $\omega ƒ^{\alpha} p \vee r$ iff $\omega ƒ^{\alpha} p$ or $\omega ƒ^{\alpha} r$
6) *Coherence:* $p ƒ^{\alpha} q$ iff $p \wedge p_0 ƒ^{\alpha} q$.
7) *Continuity from below*: if $p ƒ^{\beta} q$ for every β < α, then $p ƒ^{\alpha} q$

then there exists a ⊗-similarity relation S on a set of worlds Ω such that $p ƒ^{\alpha} q$ iff for each α ∈ [0,1] it holds $p \wedge p_0 ¥^{\alpha} q$, with K = $\{p_0\}$; and conversely, for any ⊗-similarity S on Ω and any subset of formulas K, the consequence relations $¥^{\alpha}_{K,S}$ verify these properties.

Proof (sketch). When restricting L to $L_K = \{p \mid p ¥ p_0\}$, Theorem 1 says that there exists a similarity S on $[p_0]$ such that $p ƒ^{\alpha} q \Leftrightarrow p ¥^{\alpha}_{S} q$ if p ¥ p_0 and q ¥ p_0, because the conditions of Theorem 4 then reduce to conditions of Theorem 1. Then, if p ¥ p_0, $p ƒ^{\alpha} q$ if and only if $p \wedge p_0 ¥^{\alpha} q$ holds. Now let ω' ¥ $\neg p_0$. Let $S(\omega',\omega) = S(\omega,\omega') = \sup\{\alpha \mid \omega ƒ^{\alpha} \omega'\}$ for ω ¥ p_0, and $S(\omega',\omega'') = \max_{\omega ¥ p_0} S(\omega',\omega) \otimes S(\omega,\omega'')$ otherwise. The lengthiest step is to prove that S is a similarity relation. Symmetry is obvious but transitivity requires some calculation for ω ¥ p_0, ω' ¥ $\neg p_0$, and when $\omega \vee \omega'$ ¥ $\neg p_0$. This is done by using the restricted transitivity conditions (3) as described in Lemma A1 (Appendix).

Let p and q ∈ L then

$$\begin{aligned} p \text{ ƒ}^{\alpha} q &\Leftrightarrow p \wedge p_0 \text{ ƒ}^{\alpha} q \text{ (property 6)} \\ &\Leftrightarrow \omega \text{ ƒ}^{\alpha} q, \forall \omega \text{ ¥ } p \wedge p_0 \text{ (left OR)} \\ &\Leftrightarrow \forall \omega \text{ ¥ } p \wedge p_0 \ \exists \omega' \text{ ¥ } q, \omega \text{ ƒ}^{\alpha} \omega' \end{aligned}$$

Whether ω' ¥ p_0 or not, and using continuity from below, ω ƒ$^{\alpha}$ ω' means S(ω',ω) ≥ α hence is equivalent to ω ¥$^{\alpha}$ ω'. So p ƒ$^{\alpha}$ q ⇔ p ∧ p_0 ¥$^{\alpha}$ q. ¿

So far we have an extension of the approximate entailment ¥$^{\alpha}$ to deal with a background knowledge K. Now, we turn our attention to the proximity entailment $|\equiv^{\alpha}$ and consider how it can be extended in order to cope with the same problem. From the definition of $|\equiv^{\alpha}$, it seems a natural definition, in order to restrict the worlds to those satisfying K, the following one:

$$p \mid\equiv^{\alpha}_{K} q \text{ iff } [K] \subset [p^* \otimes\!\!\rightarrow q^*]_{\alpha}.$$

In terms of measures, this entailment relation is related to the conditional implication measure $J_{K,S}(q\,|p) = \min_{\omega ¥ K} I_S(p\,|\omega) \otimes\!\!\rightarrow I_S(q|\omega)$, introduced in (Esteva et al. 94a), in the sense that

$$p \mid\equiv^{\alpha}_{K} q \text{ iff } J_{K,S}(q \mid p) \geq \alpha.$$

If p = T then $J_{K,S}(q\,|p) = I_S(p \mid K)$, so that T $|\equiv^{\alpha}_{K}$ q iff K ¥$^{\alpha}$ q; this also means that K ⊂ {q | T $|\equiv^{\alpha}_{K}$ q}, and more precisely, the set of models of K is the set of models of {q | T $|\equiv^{1}_{K}$ q}. For α = 1, it can be seen that p $|\equiv^{\alpha}_{K}$ q iff [K] ⊆ [p* → q*]. Notice that obviously p $|\equiv^{\alpha}_{K}$ q is now a stronger notion than K∧p ¥$^{\alpha}$ q. Indeed:

p $|\equiv^{\alpha}_{K}$ q iff for all ω¥K, $I_S(p\,|\omega) \otimes\!\!\rightarrow I_S(q|\omega) \geq \alpha$
iff for all ω¥K, $I_S(q|\omega) \geq \alpha \otimes I_S(p\,|\omega)$
which implies for all ω¥K∧p, $I_S(q|\omega) \geq \alpha$, that is, K∧p ¥$^{\alpha}$ q.

Hence $J_{K,S}(q \mid p) \leq I_S(q \mid K\wedge p)$. But he converse does not hold in general except for α = 1, or when K = [Ω], since p |»$^{\alpha}$ q iff p ¥$^{\alpha}$ q as noticed in section 4 . Indeed if we rewrite the second part of the proof of proposition 1 for $J_{K,S}(q\,|p)$, we end up with $J_{K,S}(q\,|p) \geq I_S(q|p)$ only. So the restricted forms of the proximity and approximate entailments no longer coincide.

The underlying reason in (Ruspini, 1991) for considering such a conditional measure as $J_{K,S}$, was to model, in its simplest form, the so-called generalized modus ponens in fuzzy logic (Zadeh, 1979) that can be expressed using $|\equiv^{\alpha}_{K}$ as follows:

$$\text{"From } |\equiv^{\alpha}_{K} r \text{ and } r \mid\equiv^{\beta}_{K} q \text{ infer } |\equiv^{\alpha\otimes\beta}_{K} q\text{".}$$

However this is rather misleading since in the generalized modus ponens written as the transitivity pattern K ¥$^{\alpha}$ r, r $|\equiv^{\beta}_{K}$ q, implies K ¥$^{\alpha\otimes\beta}$ q, K is the input observation, and is not the background knowledge, as the notation $|\equiv^{\alpha}_{K}$ suggests. Generalizations of such an inference pattern have been considered in (Esteva et al. 94b). Indeed, a stronger form of transitivity holds, namely:

Transitivity: "From p $|\equiv^{\alpha}_{K}$ r and r $|\equiv^{\beta}_{K}$ q infer p $|\equiv^{\alpha\otimes\beta}_{K}$ q"

This property is a direct consequence of the transivity of the conditional implication measure, i.e.,

Lemma 3. $J_{K,S}(q \mid p) \geq J_{K,S}(r \mid p) \otimes J_{K,S}(q \mid r)$

Proof: $J_{K,S}(r \mid p) \otimes J_{K,S}(q \mid r)$ is of the form

$$\begin{aligned} &\min_{\omega' ¥ K}\{I(p \mid \omega') \otimes\!\!\rightarrow I(r \mid \omega')\} \otimes \min_{\omega ¥ K}\{I(r \mid \omega) \otimes\!\!\rightarrow I(q \mid \omega)\} \\ &\leq \min_{\omega ¥ K}\{(I(p \mid \omega) \otimes\!\!\rightarrow I(r \mid \omega)) \otimes (I(r \mid \omega) \otimes\!\!\rightarrow I(q \mid \omega))\} \\ &\leq \min_{\omega ¥ K}\{I(p \mid \omega) \otimes\!\!\rightarrow I(q \mid \omega)\} = J_{K,S}(q \mid p). \end{aligned}$$

The latter inequality is due to the ⊗-transitivity of residuated implications. ¿

However the relation ω $|\equiv^{\alpha}_{K}$ ω' is generally not symmetric. The consequence operators $|\equiv^{\alpha}_{K}$ also form a nested family since p $|\equiv^{\alpha}_{K}$ q implies p $|\equiv^{\beta}_{K}$ q for every β ≤ α. The operator p $|\equiv^{1}_{K}$ q coincides with the classical logical consequence K ∧ p ¥ q and $|\equiv^{0}_{K}$ is the universal one. Lastly $|\equiv^{\alpha}_{K}$ satisfies the Left-OR property:

Proposition 5. p ∨ r $|\equiv^{\alpha}_{K}$ q iff p $|\equiv^{\alpha}_{K}$ q and r $|\equiv^{\alpha}_{K}$ q.

Proof. It follows from the fact that $I_S(p \vee r \mid \omega) = \max(I_S(p \mid \omega), I_S(r \mid \omega))$, and $\min(a \otimes\!\!\rightarrow b, c \otimes\!\!\rightarrow b) = \max(a,c) \otimes\!\!\rightarrow b$ for residuated implications. Then it comes:

$$\begin{aligned} J_{K,S}(q \mid p \vee r) &= \min_{\omega ¥ K} I_S(p \vee r \mid \omega) \otimes\!\!\rightarrow I_S(q \mid \omega) \\ &= \min_{\omega ¥ K} \min((I_S(p \mid \omega) \otimes\!\!\rightarrow I_S(q \mid \omega)), (I_S(r \mid \omega) \otimes\!\!\rightarrow I_S(q \mid \omega))) \end{aligned}$$

$= \min([\min_{\omega \yen K} I_S(p \mid \omega) \otimes\!\to I_S(q \mid \omega)], [\min_{\omega \yen K} I_S(r \mid \omega) \otimes\!\to I_S(q \mid \omega)])$
$= \min(J_{K,S}(q \mid p), J_{K,S}(q \mid r))$.

Now the left OR just says that $\min(a,b) \geq c$ iff $a \geq c$ and $b \geq c$ ¿

However, $\models^{\alpha}_{K}$ so defined does not verify the CUT property. The decomposition does not hold either, that is, it may be that $\omega \models^{\alpha}_{K} p \vee r$ without $\omega \models^{\alpha}_{K} p$ nor $\omega \models^{\alpha}_{K} r$ being true. Indeed, $\omega \models^{\alpha}_{K} p \vee r$ means

$$\forall \omega' \yen K,\ S(\omega,\omega') \otimes \alpha \leq I_S(p \vee r \mid \omega') = \max(I_S(p \mid \omega'), I_S(q \mid \omega'))$$

So, $\forall \omega' \yen K$, $S(\omega,\omega') \otimes \alpha \leq I_S(p \mid \omega')$ or $S(\omega,\omega') \otimes \alpha \leq I_S(q \mid \omega')$ which does not entail $\forall \omega' \yen K$, $S(\omega,\omega') \otimes \alpha \leq I_S(p \mid \omega')$ or $\forall \omega' \yen K$, $S(\omega,\omega') \otimes \alpha \leq I_S(q \mid \omega')$.

Starting from a family of inference relations $\{\text{ƒ}^{\alpha} \mid \alpha \in [0,1]\}$ on L satisfying the following properties

(i) $p\ \text{ƒ}^{\alpha}\ q$ implies $p\ \text{ƒ}^{\beta}\ q$ for every $\beta \leq \alpha$.
(ii) $p\ \text{ƒ}^{1}\ q$ iff $p \wedge K \yen q$, where $K = \{r, \top\ \text{ƒ}^{1}\ r\}$; ƒ^{0} is the universal consequence relation
(iii) *$\otimes$-transitivity*: $p\ \text{ƒ}^{\alpha}\ q$ and $q\ \text{ƒ}^{\beta}\ r$ implies $p\ \text{ƒ}^{\alpha\otimes\beta}\ r$.
(v) *Left-OR:* $p \vee r\ \text{ƒ}^{\alpha}\ q$ iff $p\text{ƒ}^{\alpha}\ q$ and $r\ \text{ƒ}^{\alpha}\ q$.
(vi) *Weak Right OR*: $\omega\ \text{ƒ}^{\alpha}\ p$ or $\omega\ \text{ƒ}^{\alpha}\ r$ imply $\omega\ \text{ƒ}^{\alpha}\ p \vee r$

it is difficult to to find a similarity relation S such that ƒ^{α} coincides with $\models^{\alpha}_{K}$ generated by S. Of course one can define S by symetrizing ƒ^{α} as follows:

$$S(\omega,\omega') = \mathrm{Min}(\ \mathrm{Sup}\{\alpha \mid \omega\ \text{ƒ}^{\alpha}\ \omega'\}, \mathrm{Sup}\{\beta \mid \omega'\ \text{ƒ}^{\beta}\ \omega\})$$

Then S is $\otimes$–transitive, just like ƒ^{α}. However it looks hopeless to obtain a result stating that whenever for all $\omega \yen K$, $S(\omega',\omega) \geq \alpha \otimes S(\omega'', \omega)$, it does follows that $\omega''\ \text{ƒ}^{\alpha}\ \omega'$, or the converse. A result was only obtained by assuming that $\omega_o\ \text{ƒ}^{\alpha}\ q$ iff for all $\omega \in [K]$, there exists $\omega_o' \yen q$ such that $S(\omega, \omega_o') \geq S(\omega, \omega_o) \otimes \alpha$ for the above defined S (see Dubois et al., 1995). But this requirement sounds somewhat artificial.

Similarly, we can consider extensions of $\yen^{\alpha}_{\wedge}$ using background knowledge K, as well as of $\models^{\alpha}_{\wedge}$. The latter can be found in (Esteva et al., 1994c) where a modified version of the conditional measure $J_{K,S}(p \mid q)$ is proposed by defining

$$J^{\wedge}_{K,S}(q \mid r) = J_{K,S}(q \wedge r \mid r) = \min_{\omega \yen K} I_S(r \mid \omega) \otimes\!\to I_S(q \wedge r \mid \omega),$$

which is close to what a conditional possibility is. Actually when $\otimes = \min$, $J^{\wedge}_{K,S}(q \mid r)$ can be interpreted as the infimum of the family of conditional possibility measures $\{\Pi_{\omega}(q \mid r)\}_{\omega \yen K}$, where $\Pi_{\omega}(p) = I_S(p \mid \omega)$ (Esteva et al., 1994c). Notice that when r is a tautology $\top$, we also have $J^{\wedge}_{K,S}(q \mid \top) = I_S(q \mid K)$. Moreover, $J^{\wedge}_{K,S}$ verifies the inequalities corresponding to *Reflexivity*, *Right weakening*, *Left And*, *Right Or*, and although it does not verify the transitivity property in general, the following *restricted* form of *transitivity* holds.

Proposition 6. $I_S(q \mid K) \geq I_S(p \mid K) \otimes J^{\wedge}_{K,S}(q \mid p)$

Proof.

$I_S(p \mid K) \otimes J^{\wedge}_{K,S}(q \mid p) = I_S(p \mid K) \otimes \min_{\omega \yen K} I_S(p \mid \omega) \otimes\!\to I_S(q \wedge p \mid \omega)$
$\leq I_S(p \mid K) \otimes \min_{\omega \yen K} I_S(p \mid K) \otimes\!\to I_S(q \wedge p \mid \omega)$
(since $I_S(p \mid \omega) \geq I_S(p \mid K)$)
$\leq \min_{\omega \yen K} I_S(p \mid K) \otimes (I_S(p \mid K) \otimes\!\to I_S(q \wedge p \mid \omega))$
$\leq \min_{\omega \yen K} I_S(q \wedge p \mid \omega)$ (since $\alpha \otimes (\alpha \otimes\!\to \beta) \leq \beta$)
$= I_S(q \wedge p \mid K) \leq I_S(q \mid K)$. ¿

For a given similarity relation S and a subset of formulas K, it is natural to consider a new entailment:

$p \models^{\alpha}_{\wedge K} q$ iff $[K] \subset [p^* \Rightarrow (p \wedge q)^*]_{\alpha}$, or equivalently,
iff $J^{\wedge}_{K,S}(q \mid p) \geq \alpha$.

Entailment $\models^{\alpha}_{\wedge K}$ based on $J^{\wedge}_{K,S}(q \mid r)$ has been characterized in (Dubois et al., 1995), with the same kind of construct as for $\models^{\alpha}_{K}$.

7. Application to Interpolative Reasoning

Let us consider a simple interpolation problem, which is paradigmatically tackled with the techniques of fuzzy control. Suppose a two-dimensional domain $U \times V$, with two variables X (input) and Y(output) over U and V respectively. Assume that all it is known about the relationship between the two variables are these two pieces of knowledge:

If X is in A_1 then Y is in B_1
If X is in A_2 then Y is in B_2

where A_1 and A_2 are subsets of U and B_1 and B_2 subsets of V. This is obviously an incomplete description of a mapping. The problem is how to guess a value for variable Y if we know for instance that variable X has got a value $X = x_0$, where x_0 does not belong to A_1 nor to A_2. The intuition says that if x_0 is close to A_1, then the value of Y will be close to B_1, and if it is close to A_2, the value of Y will be close to B_2. But this is easy to model if we equip $U \times V$ with a similarity relation. One can consider, for instance, a $\otimes$-transitive similarity S_U on U and a $\otimes$-transitive similarity S_V on V, and then combine them into the product similarity $S = S_U \times S_V$ on the product space $U \times V$, defined as $S_U \times S_V((x,y), (x',y')) = \min(S_U(x,x'), S_V(y,y'))$.

Let p = "$X = x_0$", q_i = "$X \in A_i$", r_i = "$Y \in B_i$", i = 1,2. Assume we have a knowledge base K consisting of two "extrapolative" or "gradual" rules:

"if q_1 then r_1" modelled by $q_1^* \to r_1^*$

"if q_2 then r_2" modelled by $q_2^* \to r_2^*$

expressing that "the closer X is to A_i, the closer Y is to B_i", for i=1, 2, in the sense of section 4. Actually, the exact meaning of these rules is if X=x, then Y is allowed to take any value y fulfilling $\mu_{A_i^*}(x) \le \mu_{B_i^*}(y)$, i.e. the similarity of X to A_i, acting as a lower bound, constrains the similarity of Y to B_i. Then one can compute the highest degrees in which the proposition "X is x_0" entails the proposition "X is in A_1" and the proposition "X is in A_2". So, we will have in the domain U:

"p ¥$^{\alpha}$ q_1", with $\alpha = I_{S_U}(q_1 \mid p)$;
"p ¥$^{\beta}$ q_2", with $\beta = I_{S_U}(q_2 \mid p)$.

What we are looking for is the most specific proposition of type "Y is S" derivable from the above information. Now what we expect is the following type of reasoning:

Since p is close to q_i, given the "extrapolative" knowledge in K, we can infer, from p, approximately r_i, that is, Y has to be in the vicinity of B_i

Then, the *interpolation* process comes from considering that Y has to be in the vicinity of both B_1 and B_2. The key point in the interpolation procedure is to properly encode the interpolative information, under the form $q_1^* \to r_1^*$. The modelling of *"given the "extrapolative" knowledge in K, we can infer, from p, approximately r_i"* is achieved by the following pattern of inference justified by the previous results:

p ¥$^{\alpha}$ q_1 ; p ¥$^{\beta}$ q_2
$K = \{q_1^* \to r_1^* \;;\; q_2^* \to r_2^*\}$
---
p ¥$^{\alpha}_{K}$ r_1 and p ¥$^{\beta}_{K}$ r_2

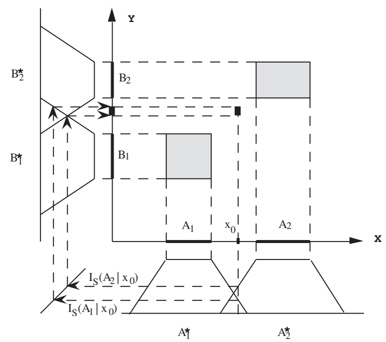

Figure 4.

that is, the knowledge or context K explicitly appears. The above pattern can be called an interpolation syllogism (IS). This reasoning method exactly corresponds to fuzzy inference based on gradual rules (Dubois and Prade, 1992a; Dubois, Grabisch and Prade, 1995) and is of the kind at work in fuzzy control. The fuzzy sets of values close to B_1 and close to B_2 (B^*_1 and B^*_2) plays the role of fuzzy conclusions of fuzzy rules. Namely we could write the above pieces of knowledge in the style of fuzzy sets as follows

$X = x_0$
the more X is F_i, the more Y is G_i where $F_i = A^*_i$, $G_i = B^*_i$

with $\alpha = \mu_{A^*_1}(x_0)$, $\beta = \mu_{A^*_2}(x_0)$. Then we conclude $Y \in (G_1)_\alpha \cap (G_2)_\beta$ (see Figure 4). This example is only for the sake of illustration. Indeed the interpolative syllogism makes sense on non-numerical universes, where similarity is graded on an abstract scale (e.g., ordinal if $\otimes$ = min).

We may be more flexible by only requiring that the background knowledge K has a fuzzy set of models induced by fuzzy rules modelled using the multiple-valued implication $\otimes\!\!\rightarrow$. Then assume that $q_1 \mid\!\approx^{\gamma}_{K} r_1$ and $q_2 \mid\!\approx^{\delta}_{K} r_2$, and use the following fact that provides a formulation of the *extrapolative syllogism* introduced in Section 1.

Proposition 7. $p \models^{\alpha} q$ and $q \mid\!\approx^{\beta}_{K} r$ imply $p \models^{\alpha\otimes\beta}_{K} r$

Proof. On the one hand, $p \models^{\alpha} q$ expresses that $I(q \mid \omega) \geq \alpha$ for each $\omega \models p$. On the other hand, $q \mid\!\approx^{\beta}_{K} r$ expresses that $I(q \mid \omega) \otimes\!\!\rightarrow I(r \mid \omega) \geq \beta$, for each $\omega \models K$. Therefore, for each $\omega \models p \wedge K$ we have that $I(r \mid \omega) \geq \alpha \otimes \beta$, that is, it holds that $p \wedge K \models^{\alpha\otimes\beta} r$, in other words, it holds that $p \models^{\alpha\otimes\beta}_{K} r$.

On the whole we can infer from the following set of statements, by means of $\otimes$-transitivity

$$\frac{\begin{array}{c} p \models^{\alpha} q_1 \;\;;\;\; p \models^{\beta} q_2 \\ q_1 \mid\!\approx^{\gamma}_{K} r_1 \;\;;\;\; q_2 \mid\!\approx^{\delta}_{K} r_2 \end{array}}{p \models^{\alpha\otimes\gamma}_{K} r_1 \;\;;\;\; p \models^{\beta\otimes\delta}_{K} r_2} \qquad \text{(IS)}$$

We conclude that $[p \wedge K] \subset [r_1{}^*]_{\alpha\otimes\gamma}$ and $[p \wedge K] \subset [r_2{}^*]_{\beta\otimes\delta}$, which yields

$$y \in [B_1{}^*]_{\alpha\otimes\gamma} \cap [B_2{}^*]_{\beta\otimes\delta} .$$

Note that in this example, q_i and r_i are logically independent, so that $(q_i \wedge r_i)^* = q^*_i \wedge r^*_i$. As a consequence $q_i \mid\!\approx^{\gamma}_{K} r_i$ is equivalent to $q_i \mid\!\approx^{\gamma}_{K} r_i \wedge q_i$ in that particular situation. This phenomenon always appears when $[p \wedge q]$ can be written as a Cartesian product of interpretations on disjoint languages.

Due to the definition of the approximate enatilment $\models^{\alpha}_{K}$, Proposition 7 admits an even more attractive formulation, allowing to deal with different background knowledges in the approximate and the proximity entailments.

Corollary 1. $p \models^{\alpha}_{K} q$ and $q \mid\!\approx^{\beta}_{K'} r$ imply $p \models^{\alpha\otimes\beta}_{K\cap K'} r$

The above results are a first step towards a logical representation of interpolative reasoning and contrasts with Klawonn and Kruse (1993)'s similarity-based justification of another fuzzy reasoning method (the one of Mamdani (1977)) that does not involve interpolation at all since in the case of Figure 4, the fuzzy conclusion is a weighted *disjunction* of B^*_1 and B^*_2. In this fuzzy control method, interpolation is carried out extra-logically, as a final "defuzzification" step. Klawonn and Kruse (1993)'s justification is based on acknowledging A^*_1, A^*_2, B^*_1, B^*_2 as generalized equivalence classes of some similarity relations, without reference to interpolation as such. See (Dubois and Prade, 1994) for a discussion.

8. Semantics of Similarity Logic in Terms of Sphere Systems

A fuzzy relation R on $\Omega \times \Omega$, in particular a similarity relation, can also be viewed as a ternary relation (on Ω^3), i.e., a collection $\{\geq_\omega \mid \omega \in \Omega\}$ of binary relations that are complete pre-orderings. Then $\mu_R(\omega,\omega') \geq \mu_R(\omega,\omega'')$ can represent a situation where $\omega' \geq_\omega \omega''$, which reads: ω' is closer to ω than ω''. These structures are common in conditional logics of counterfactuals (Lewis, 1973). It strongly suggests that conditional logics might be an even more suitable framework for similarity reasoning than modal logics are. In this context, the similarity relation is viewed as a "system of spheres". However, it is worth noticing that the relationship between similarity relations and systems of spheres only works in one direction, i.e., while a fuzzy similarity relation R on $\Omega \times \Omega$ induces a collection $\{\geq_\omega \mid \omega \in \Omega\}$ of classical binary relations, the converse is not true in general. Indeed similarity relations make it possible to express that ω_1 is closer to ω_2 than ω_3 is to ω_4 as $S(\omega_1,\omega_2) > S(\omega_3,\omega_4)$, with $\omega_1 \neq \omega_3$. This is not possible within a set of relations $\{<_\omega \mid \omega \in \Omega\}$.

Let S be a $\otimes$-similarity relation on Ω. The sphere system associated to S is defined in the obvious way: For every ω, Σ_ω is the set of open and closed α-cuts of the fuzzy set ω^* of models close to ω, whose membership function is defined by $\mu_{\omega^*}(\omega') = S(\omega,\omega')$. Namely

$$\Sigma_\omega = \{A^\alpha_\omega \mid \alpha \in (0,1]\} \cup \{\underline{A}^\alpha_\omega \mid \alpha \in [0,1)\}$$
$$A^\alpha_\omega = \{\omega' \mid S(\omega,\omega') \geq \alpha\}$$
$$\underline{A}^\alpha_\omega = \{\omega' \mid S(\omega,\omega') > \alpha\}.$$

In this case the set of accessible worlds from ω is $\underline{A}^0{}_\omega = \{\omega' \mid S(\omega,\omega') > 0\}$. This system is obviously centered. We show next how Lewis' concepts of "counterfactual possibility/ necessity" can be formulated in such a sphere system and related to similarity-based indices. In the following $Poss_\omega$ will denote the possibility measure induced by the fuzzy set ω^*, i.e., $Poss_{\omega}(p) = \max_{\omega' ¥ p} S(\omega,\omega') = I_S(p \mid \omega)$. Therefore we can write $I_S(q \mid p) = \min_{\omega' ¥ q} Poss_{\omega}(p)$.

The definition of Counterfactual necessity, i.e.

ω ¥ p ¿→ q iff - either there does not exist any world $\omega' \in \underline{A}^0{}_{\omega}$ such that ω' ¥ p,
- or there exists a sphere $SP \in \Sigma_{\omega}$ such that [p] overlaps SP (i.e., $\exists\ \omega'' \in$ SP s.t. ω'' ¥ p) and ω' ¥ $\neg p \vee q$ for every $\omega' \in$ SP.

can be now equivalently expressed in the framework of similarity logic, for non modal propositions, as

ω ¥ p ¿→ q iff: - either $Poss_{\omega}(p) = 0$, or
- $Poss_{\omega}(p \wedge q) > Poss_{\omega}(p \wedge \neg q)$, that is,
$I_S(p \wedge q \mid \omega) > I_S(p \wedge \neg q \mid \omega)$

If $Poss_{\omega}(q \mid p)$ denotes the conditional possibility

$$Poss_{\omega}(q \mid p) = 1, \text{ if } I_S(p \wedge q \mid \omega) = I_S(p \mid \omega)$$
$$= I_S(p \wedge q \mid \omega), \text{ otherwise}$$

and $Nec_{\omega}(q \mid p) = 1 - Poss_{\omega}(\neg q \mid p)$ is the conditional necessity, then

ω ¥ p ¿→ q $\Leftrightarrow Nec_{\omega}(q \mid p) > 0$.

whenever $Poss_{\omega}(p) > 0$. It means that when ω ¥ p, ω is closer to q than to $\neg q$. Hence more generally, K ¥ p ¿→ q iff $\forall\ \alpha > 0$, K $¥^{\alpha}$ $\neg q \wedge p$ implies $\exists\ \beta > \alpha$, K $¥^{\beta}$ $q \wedge p$. Analogously, for the counterfactual possibility we have that

ω ¥ p ◊→ q if: - $Poss_{\omega}(p) > 0$, and
- $Poss_{\omega}(q \mid p) = 1$

Therefore, it is straightforward to notice that ω ¥ p ◊→ q iff $I_S(p \wedge q \mid \omega) = I_S(p \mid \omega) > 0$. More generally, the following result relating conditional logics and similarity logic holds:

K ¥ p ◊→ q iff $J_{K,S}(p \wedge q \mid p) = 1$ and $I_S(p \mid K) > 0$, i.e.
iff $p \models^{1}_{\wedge K} q$ and K $¥^{\alpha}$ p, for some $\alpha > 0$.

Conditional logic as such is thus not expressive enough to describe similarity logic because it does not handle spheres explicitly. One should have a multi-level conditional logic.

9. Discussion

The approximate entailments based on similarity are quite different from the preferential entailment $¥_{\pi}$ of possibilistic logic (Dubois et al., 1994). In the latter case, Ω is equipped with a complete partial ordering that expresses how plausible interpretations are, and is encoded as a possibility distribution π. p $¥_{\pi}$ q means that q is true in all the best models of p, which correspond to shrinking p (instead of stretching q as here). The inference relation $¥_{\pi}$ satisfies properties different from those of $¥^{\alpha}$ (see Dubois and Prade, 1995). However as indicated by Esteva et al. (1994c), π can always be constructed from a given subset A, taken as background information, and the similarity relation S, such that $\pi = \mu_{A^*}$. Then p $¥_{\pi}$ q can be interpreted as: all models of p that are "as close as possible" to A are models of q where "as close as possible to A" means "close to normal". Moreover, from the sphere semantics perspective it is very clear what possibilistic and similarity logics have both in common and what discriminates them. From the model construction point of view, possibilistic logic is built upon a unique complete partial ordering on the worlds which determines an absolute system of spheres (see Farinas and Herzig, 1991), whereas a similarity relation corresponds to a complete partial ordering attached to each world, and thus it leads to a centered system of spheres. From the inferential point of view, possibilistic inference is related to the counterfactual necessity, whereas a form of proximity entailment is more directly related to the counterfactual possibility.

Also of interest is the study of the links between similarity-based entailments and various graded extensions of consequence relations in multiple-valued logic, as studied by Chakraborty (1988), Castro et al. (1994). This is a topic for further research.

The long term perspective of this work could be to provide logical foundations to some forms of "fuzzy logic", and also case-based reasoning where similarity plays a basic role. The next step would be to start from a set of conditional statements of the form "p is not far from implying q", that forms a conditional similarity-oriented knowledge base, and to reconstruct the underlying similarity measure. This has been done in the characterisation theorems, albeit with a complete set of such statements, sufficient to define a single similarity relation. It would be worthwile doing the same with an incomplete such set, given by a domain expert, using the characteristic axioms as inference rules so as to help constructing a "least committed" similarity relation, by analogy to the treatment of conditional knowledge bases in nonmonotonic reasoning.

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Appendix

Proof of Theorem 1. We only prove the first claim, the second is proved in the main text. Take Ω as the set of all interpretations of L and let define $S(\omega,\omega') = \sup\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega'\}$. This supremum always exists because the set $\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega'\}$ is never empty since 0 always belongs to it. Notice that with this definition, and due to the continuity assumption, we have $S(\omega,\omega') \geq \alpha$ iff ω ƒ$^{\alpha}$ ω'. Moreover, S so defined is a $\otimes$-similarity relation on the set Ω:

1) S is reflexive: $S(\omega,\omega) = \sup\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega\} = 1$, since ω ƒ^{1} ω

2) S is symmetric: $S(\omega,\omega') = \sup\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega'\} = \sup\{\alpha \mid \omega'$ ƒ$^{\alpha}$ $\omega\} = S(\omega',\omega)$, since ω ƒ$^{\alpha}$ ω' iff ω' ƒ$^{\alpha}$ ω

3) S is $\otimes$-transitive: $S(\omega,\omega') \otimes S(\omega',\omega'') =$
$= \sup\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega'\} \otimes \sup\{\beta \mid \omega'$ ƒ$^{\beta}$ $\omega''\}$
$= \sup\{\alpha \otimes \beta \mid \omega$ ƒ$^{\alpha}$ ω' and ω' ƒ$^{\beta}$ $\omega''\}$
$\leq \sup\{\alpha \otimes \beta \mid \omega$ ƒ$^{\alpha\otimes\beta}$ $\omega''\} = S(\omega,\omega'')$.

Moreover,

4) $S(\omega,\omega') = 1$ iff $\omega = \omega'$: $\sup\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega'\} = 1$ iff ω ƒ^{1} ω'
iff ω ¥ ω' iff $\omega = \omega'$.

Now we have to check that p ƒ$^{\alpha}$ q iff $I_S(q \mid p) \geq \alpha$. This is proved as follows. Suppose $I_S(q \mid p) \geq \alpha$. By definition, this means that for every interpretation ω such that ω ¥ p, there exists an interpretation ω' such that ω' ¥ q and $S(\omega,\omega') \geq \alpha$ (or equivalently ω ƒ$^{\alpha}$ ω'). Since any proposition is equivalent to a unique disjunction of interpretations, by the Decomposition property, the former statement is equivalent to ω ƒ$^{\alpha}$ q for all interpretations proving p, and finally, by the Left Or property, equivalent to p ƒ$^{\alpha}$ q. Consistency preservation makes it such that this equivalence holds even if $q = \perp$. ¿

Lemma A1. Let E be a subset of Ω. Assume that $\{$ƒ$^{\alpha}\}_{\alpha\in[0,1]}$ is a nested family such that conditions of Theorem 4 hold, and define a fuzzy relation S such that

$S(\omega,\omega') = \sup\{\alpha \mid \omega$ ƒ$^{\alpha}$ $\omega'\}$ for $\omega \in E$
$= \sup\{\alpha \mid \omega'$ ƒ$^{\alpha}$ $\omega\}$ for $\omega' \in E$, $\omega \notin E$
$= 1$ if $\omega = \omega' \notin E$
$= \max_{x\in E} S(\omega,x) \otimes S(x,\omega')$ otherwise

then S is a similarity relation on Ω.

Proof. S is clearly reflexive and symmetric. For transitivity we shall consider three cases according to the location of ω and ω' in or out of E, and due to symmetry .

1) $\omega \in$ E, $\omega' \in$ E

$\max_x S(\omega,x) \otimes S(x,\omega')$
$= \max(\max_{x\notin E} \sup\{\alpha \otimes \beta \mid \omega$ ƒ$^{\alpha}$ x and ω' ƒ$^{\beta}$ x$\}$,
$\max_{x\in E} \sup\{\alpha \otimes \beta \mid \omega$ ƒ$^{\alpha}$ x and x ƒ$^{\beta}$ $\omega'\})$
$\leq \sup\{\gamma \mid \omega$ ƒ$^{\gamma}$ $\omega'\} = S(\omega,\omega')$ using 3ii) and 3i) of Theorem 4 respectively.

2) $\omega \in$ E, $\omega' \notin$ E

$\max_x S(\omega,x) \otimes S(x,\omega') =$

$$\max(\max_{x\in E} S(\omega,x) \otimes S(x,\omega'), \max_{x\notin E} S(\omega,x) \otimes S(x,\omega'))$$

Now as above $\max_{x\in E} S(\omega,x) \otimes S(x,\omega') = S(\omega,\omega')$ using 3i) of Theorem 4. Besides

$\max_{x\notin E} S(\omega,x) \otimes S(x,\omega') = \max_{x\notin E} \max_{y\in E} S(\omega,x) \otimes S(x,y) \otimes S(y,\omega')$
$= \max_{x\notin E, y\in E} \max(\alpha \otimes \beta \otimes \gamma$, ω ƒ$^{\alpha}$ x and y ƒ$^{\beta}$ x and y ƒ$^{\gamma}$ $\omega'\}$
$\leq \max_{y\in E} \max(\delta \otimes \gamma$, ω ƒ$^{\gamma}$ y and y ƒ$^{\gamma}$ ω') (using 3ii) of Theorem 4)
$= S(\omega,\omega')$ (using 3i) of Theorem 4)

3) $\omega \notin$ E, $\omega' \notin$ E

We proceed as in 2). If $x \in E$ then $\max_{x\in E} S(\omega,x) \otimes S(x,\omega') \leq S(\omega,\omega')$ using 3i) of Theorem 4, since ω ƒ$^{\alpha}$ x and x ƒ$^{\beta}$ ω' imply ω ƒ$^{\alpha\otimes\beta}$ ω'.

Now if $x \notin E$ then $S(\omega,x) \otimes S(x,\omega') = \max_{y,z\in E} S(\omega,y) \otimes S(y,x) \otimes S(x,z), S(z,\omega')$. It leads to considering inferences such as y ƒ$^{\alpha}$ ω, y ƒ$^{\beta}$ x, z ƒ$^{\gamma}$ x, z ƒ$^{\delta}$ ω' which reduce as above into y ƒ$^{\alpha}$ ω, y ƒ$^{\beta\otimes\gamma}$ z, z ƒ$^{\delta}$ ω' by 3ii) of Theorem 4 and then y ƒ$^{\alpha}$ ω, y ƒ$^{\beta\otimes\gamma\otimes\delta}$ ω' by 3i) of Theorem 4 and finally ω ƒ$^{\alpha\otimes\beta\otimes\gamma\otimes\delta}$ ω' by 3ii) of Theorem 4. ¿