Gradual rules and the approximation of control laws

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Abstract

This paper focuses on the representation of a function f by means of a special kind of fuzzy rules, called gradual rules that express statements of the form "the more X is A_i, the more Z is C_i". This representation agrees with the extension principle since the gradual rules that represent the function can be chosen such that $C_i = f(A_i)$ in the sense of the extension principle. Using a suitable fuzzy partition $\{A_i, i = 1,n\}$ of the unidimensional domain of X, the logic of gradual rules is shown to provide the same interpolation mechanism as Sugeno's method for fuzzy control. More generally, it is shown that any one input-one output function f can be recovered from a set of gradual rules using a suitable partition. In the case of multiple input-one output functions (corresponding, for two inputs, to gradual rules of the form "the more X is A_i and the more Y is B_i, the more Z is C_i"), only a bracketing approximation of f can be recovered. The locus of points where the logic of gradual rules recovers the original function exactly is determined.In the case of two input conditions the interval braketting the exact value of the function is calculated via a simple geometric construction. Besides, interpolation between ill-located points, or parts of incompletely known functions is also discussed.

7.1. Introduction

With the success of fuzzy controllers, a lot of methods have been proposed that turn a set of parallel fuzzy rules relating numerical parameters into a continuous function corresponding to, e.g., a control law. Usually this is achieved by the choice of a multiple-valued logic for rule representation, and an operation that aggregates the rules in order to build a fuzzy relation expressing them ; lastly a defuzzification operation enables a precise output to be obtained. Sugeno [1] (see also Sugeno and Nishida [2], Takagi and Sugeno [3], Uchino et al. [4]) has already noticed that this last step can be obviated by simply considering rules with a precise conclusion part right away. Current practice in fuzzy control is still very much ad hoc regarding the knowledge representation aspect. However, recent results indicate that, despite their lack of formal foundations, several approaches to the handling of fuzzy rules lead to universal approximators of functions (Kosko [5], Wang [6], Buckley and Hayashi [7][8]. Hence it makes sense to investigate how a given function can be practically represented by a set of fuzzy rules.

Representing a (precise) function by means of fuzzy rules clearly requires some interpolation method. However in most fuzzy reasoning methods, the interpolation property is obtained via defuzzification, i.e., is external to the inference process itself. An investigation into the semantics of fuzzy 'if… then…" rules has been carried out by Dubois and Prade [9][10][11]. Special attention has been drawn on so-called "Gradual Rules" (Prade [12], Dubois and Prade [13]) which are good models of rules of the form "the more X is A, the more Z is C". The kind of multiple-valued implication adapted to this type of semantics has been laid bare. In this paper we show that fuzzy inference based on gradual rules genuinely models the process of interpolation, and is capable of bracketting a general class of functions. Gradual rules offer a natural and practical tool for turning a real function into a rule-based system that approximates this function. What is obtained as a result of the inference process is a bracketting of the function f, namely an interval U(X) that contain the value of f(X) for any value of the input vector X. The construction of the set of fuzzy rules is based on the computation of the fuzzy image of the elements of the fuzzy partition of the input space by means of Zadeh [14][15]'s extension principle. The approach proposed in the paper is due to a fundamental coherence between gradual rules and the calculus of fuzzy intervals as defined by the extension principle. This paper revises and expands the contents of two previous notes by the authors (Dubois, Grabisch and Prade [16][17]).

The paper is organized as follows : after recalling the basics of fuzzy arithmetics and gradual rules, we focus on the approximations of one input-one output functions. We show that the logic of gradual rules is such that any continuous function which is monotonic on intervals can be recovered exactly by a fuzzy rule-based system where the rules are computed by the extension principle. This result is similar to but more general than the one of Lee and Chae [18] who moreover use Mamdani [19] defuzzification method (i.e., external interpolation). We show that linear interpolation is a special case of gradual rule interpolation, that coincides with Sugeno's fuzzy control method with fixed outputs. These results are extended in two directions. First we consider interpolation between known parts of a function and between fuzzily located elements of the graph of a function. Next, we consider the case of n-input 1-output functions and how to approximate them by fuzzy rule systems. We suggest that recovering the function exactly is impossible when $n > 1$, but only bracketting approximations can be attained. Some simulation results are given.

7.2. Fuzzy Intervals : A Refresher

The extension principle [14][15] enables the image f(A) of a fuzzy set A via a function f from a set $\mathcal{X}$ to a set $\mathcal{Z}$ to be computed as

$$\forall z \in \mathcal{Z},\ \mu_{f(A)}(z) \begin{aligned}[t] &= \sup_{x:z=f(x)} \mu_A(x) \\ &= 0 \text{ if } \nexists x,\ z = f(x). \end{aligned} \tag{1}$$

When A is a crisp set, (1) reduces to the usual extension of a function to set-valued arguments $f(A) = \{z, \exists\, x \in A, z = f(x)\}$. It is in agreement with possibility theory (see Dubois and Prade [20]). If f is bijective

$$\forall z \in \mathcal{Z},\ \mu_{f(A)}(z) = \mu_A(f^{-1}(z)). \tag{2}$$

When the function f has several arguments $x_1, x_2, \dots, x_n$ an interesting situation is when the fuzzy set A is separable, so that A is a fuzzy Cartesian product $A_1 \times A_2 \times \dots \times A_n$ with membership function $\mu_{A_1} * \mu_{A_2} * \dots * \mu_{A_n}$ where $*$ is a triangular norm which models conjunctions in fuzzy set theory; see Dubois and Prade [20] for instance. Then the extension principle reads:

$$\mu_{f(A)}(z) = \sup\{\mu_{A_1}(x_1) * \mu_{A_2}(x_2) * \dots * \mu_{A_n}(x_n) \mid f(x_1, x_2, \dots, x_n) = z\} \tag{3}$$

which is an interactive form of the extension principle (see, e.g., Dubois and Prade [21]). In this paper, we will often restrict to the non-interactive case, i.e., $* = \min$.

A fuzzy interval M is simply a fuzzy set of the real line such that $\mu_M(x) = 1$ for some x, and whose α-cuts $M_\alpha = \{x, \mu_M(x) \geq \alpha\}$ are closed intervals. The membership function of a fuzzy interval is thus upper semi-continuous (u.s.c.).The calculus of fuzzy intervals is based on the extension principle applied to real-valued real functions. Using u.s.c. fuzzy intervals M we are sure that any level cut $f(M)_\alpha$ of f(M) is equal to the image of the corresponding level cut M_α of M through f when f is continuous. Let the interval $(M_i)_\alpha$ be denoted $[m^-_i(\alpha), m^+_i(\alpha)]$. As all our discussion is based on the extension principle, we will frequently use the following result from Dubois and Prade [21].