Belief, knowledge and common knowledge about a proposition

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(contains link to the slides)

WoLLIC 2021, October 5, 2021

Background

- standard modalities of epistemic logic since [Hintikka, 1962]:

$$\mathbf{K}_i\varphi \quad = \quad \text{“agent } i \text{ knows that } \varphi\text{”}$$
$$\mathbf{B}_i\,\varphi \quad = \quad \text{“agent } i \text{ believes that } \varphi\text{”}$$

- . . . but there is more: cf. Yanjing Wang's "beyond knowing-that" research program
  - know whether [Fan et al., 2013, Fan et al., 2015]
  - know what [Wang and Fan, 2014]
  - know value [van Eijck et al., 2017]
  - know how [Fervari et al., 2017, Wang, 2018]
  - know why [Xu et al., 2021]
  - . . .

'Know wh' logics

- two possibilities:
  1. reduce to 'know that' $\Rightarrow$ quantification [Hintikka, 1962]
  2. new modality [Wang, 2016]
     - either studied in isolation, or together with 'know that'
- logics are typically exotic
  - non-normal modalities
  - non-trivial completeness proofs
- interesting for philosophical logic
  - which primitive concepts?
  - which interplay with logics of action?
  - ...
- impact on computer science and AI?
  - knowledge representation, planning,...

This talk: modalities of the ‘know whether’ kind

- motivation: ‘know whether’ more primitive than ‘know that’
  - knowing the truth value of a proposition more basic than knowing that the truth value equals 1

    “To know is to know the value of a variable” [Baltag, 2016]
- related to:
  - non-contingency logics [Montgomery and Routley, 1966, Humberstone et al., 1995]
  - logic of ignorance [Kubyshkina and Petrolo, 2019]

Knowledge and belief about a proposition

- ‘know whether’ has no belief-counterpart in natural language (just as the other ‘know wh’ modalities) [Egré, 2008]
- therefore:

$$\begin{aligned} \mathbf{KA}_i\varphi &= \text{“agent } i \text{ has knowledge about } \varphi\text{”} \\ \mathbf{BA}_i\varphi &= \text{“agent } i \text{ has belief about } \varphi\text{”} \end{aligned}$$

  alternatively: “i is opinionated about φ”

‘About’ modalities: expressivity

1. ‘belief about’: weaker [Fan et al., 2015]

$$\mathbf{BA}_i\varphi \leftrightarrow \mathbf{B}_i\,\varphi \vee \mathbf{B}_i\,\neg\varphi$$
$$\mathbf{B}_i\,\varphi \leftrightarrow\ ?$$

2. ‘knowledge about’: equi-expressive

$$\mathbf{KA}_i\varphi \leftrightarrow \mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi$$
$$\mathbf{K}_i\varphi \leftrightarrow \varphi \wedge \mathbf{KA}_i\varphi$$

but:

- ‘knowledge about’ can express things more succinctly [van Ditmarsch et al., 2014]
- equivalent presentations may lead to new insights
  - cf. Kosta Došen: “Had Gentzen used Tarski’s consequence operator $Cn(\Gamma)$, he wouldn’t have found the cut rule”

This talk

1. new axiom relating individual and common knowledge
   - more intelligible
   - based on: AH & E. Perrotin "On the axiomatisation of common knowledge", Proc. AiML 2020
2. interesting lightweight fragments
   - same complexity as propositional logic
   - based on: M. C. Cooper, AH, F. Maffre, F. Maris, E. Perrotin, P. Régnier "A lightweight epistemic logic and its application to planning", Artificial Intelligence, 2021
3. analysis of of epistemic-doxastic situations
   - three independent dimensions
   - based on: AH & E. Perrotin, "True belief and mere belief about a proposition and the classification of epistemic-doxastic situations", Filosofiska Notiser 8:1, 2021

Part 1 Relating individual and common knowledge

Part 2 Lightweight fragments

Part 3 The three dimensions of epistemic-doxastic situations

Language of 'knowledge that' and 'common knowledge that'

- grammar:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}_i\varphi \mid \mathbf{EK}\,\varphi \mid \mathbf{CK}\,\varphi$$

where p ranges over a countable set of propositional variables
and i over a finite set of agents

- reading:

$$\begin{array}{rcl} \mathbf{K}_i\varphi & = & \text{"agent } i \text{ knows that } \varphi\text{"} \\ \mathbf{EK}\,\varphi & = & \text{"it is shared knowledge that } \varphi\text{"} \ = \ \bigwedge_{i \in Agt} \mathbf{K}_i\varphi \\ \mathbf{CK}\,\varphi & = & \text{"it is common knowledge that } \varphi\text{"} = \bigwedge_{k \geq 0} \mathbf{EK}^k\varphi \end{array}$$

Individual knowledge: S5

S5($\mathbf{K}$) = modal logic S5 for the modal operators $\mathbf{K}_i$

- truth axiom:

$$\mathbf{K}_i\varphi \to \varphi$$

- positive introspection axiom:

$$\mathbf{K}_i\varphi \to \mathbf{K}_i\mathbf{K}_i\varphi$$

- negative introspection axiom:

$$\neg\mathbf{K}_i\varphi \to \mathbf{K}_i\neg\mathbf{K}_i\varphi$$

Shared knowledge: contains KTB

- axiom $\mathtt{Def}(\mathbf{EK})$: $\mathbf{EK}\,\varphi \leftrightarrow \bigwedge_{i \in Agt} \mathbf{K}_i \varphi$
- normal modal operator:
  - axiom $\mathtt{K}(\mathbf{EK})$ provable
  - rule of necessitation $\mathtt{RN}(\mathbf{EK})$ derivable
- truth axiom provable:

$$\mathbf{EK}\,\varphi \rightarrow \varphi$$

- axiom $\mathtt{B}(\mathbf{EK})$ provable:

$$\varphi \rightarrow \mathbf{EK}\,\neg\mathbf{EK}\,\neg\varphi$$

- neither positive nor negative introspection provable
  - when knowledge is shared then this is not necessarily known

Common knowledge: should contain S5

- truth axiom:

$$\mathbf{CK}\,\varphi \to \varphi \text{ should be valid}$$

- positive introspection axioms:

$$\mathbf{CK}\,\varphi \to \mathbf{EK}\,\mathbf{CK}\,\varphi \text{ should be valid}$$
$$\mathbf{CK}\,\varphi \to \mathbf{CK}\,\mathbf{CK}\,\varphi \text{ should be valid}$$

$\Rightarrow$ fixed-point axiom follows:

$$\mathrm{FP} \quad \mathbf{CK}\,\varphi \to \mathbf{EK}\,(\varphi \wedge \mathbf{CK}\,\varphi)$$

Minimal axiom system with induction rule

S5(**K**) and Def(**EK**), plus:

FP	$\mathbf{CK}\,\varphi \to \mathbf{EK}\,(\varphi \wedge \mathbf{CK}\,\varphi)$
RGFP	from $\varphi \to \mathbf{EK}\,(\varphi \wedge \psi)$, infer $\varphi \to \mathbf{CK}\,\psi$

[Halpern and Moses, 1992, Fagin et al., 1995]

- sound and complete for S5 models
  - rule of necessitation RN(**CK**) derivable
  - axioms K(**CK**), T(**CK**), 4(**CK**), 5(**CK**) provable
  - induction axiom schema GFP provable

Minimal axiom system with induction axiom

$\mathtt{S5}(\mathbf{K})$ and $\mathtt{Def}(\mathbf{EK})$, plus:

$\mathtt{K}(\mathbf{CK})$	modal logic $\mathtt{K}$ for $\mathbf{CK}$
$\mathtt{FP}$	$\mathbf{CK}\,\varphi \to \mathbf{EK}\,(\varphi \wedge \mathbf{CK}\,\varphi)$
$\mathtt{GFP}$	$\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to (\varphi \to \mathbf{CK}\,\varphi)$

[Lehmann, 1984, Halpern and Moses, 1985]

- sound and complete for $\mathtt{S5}$ models
  - induction rule $\mathtt{RGFP}$ provable
  - original presentation has moreover axioms $\mathtt{T}(\mathbf{CK})$, $\mathtt{4}(\mathbf{CK})$, $\mathtt{5}(\mathbf{CK})$ $\Rightarrow$ redundant!

Common knowledge: status of GFP/RGFP?

- induction axiom schema intuitive in temporal logics (well-founded orderings)
- epistemic logics:
  - difficult to justify
  - difficult to paraphrase

$$\texttt{RGFP} \quad \text{from } \varphi \to \mathbf{EK}\,(\varphi \wedge \psi)\text{, infer } \varphi \to \mathbf{CK}\,\psi$$

"If it is the case that φ is 'self-evident', in the sense that if it is true, then everyone knows it, and, in addition, if φ is true, then everyone knows ψ, we can show by induction that if φ is true, then so is $\mathbf{EK}^k(\psi \wedge \varphi)$ for all k." [van Ditmarsch et al., 2015]

A more intuitive axiomatisation of S5 common knowledge

$\mathtt{S5}(\mathbf{K})$ and $\mathtt{Def}(\mathbf{EK})$, plus:

$\mathtt{S4}(\mathbf{CK})$	modal logic $\mathtt{S4}$ for $\mathbf{CK}$
FP_0	$\mathbf{CK}\,\varphi \to \mathbf{EK}\,\varphi$
GFP_0	$\mathbf{CK}\,\mathbf{EKA}\,\varphi \to \mathbf{CKA}\,\varphi$

"If it is common knowledge that there is shared knowledge about φ then there is common knowledge about φ."

where:

$\mathbf{CKA}\,\varphi = \mathbf{CK}\,\varphi \vee \mathbf{CK}\,\neg\varphi$ "there is common knowledge about φ"

$\mathbf{EKA}\,\varphi = (\bigwedge_{i \in Agt} \mathbf{K}_i\varphi) \vee (\bigwedge_{i \in Agt} \mathbf{K}_i\neg\varphi)$ "there is shared knowledge about φ"

$\leftrightarrow \bigwedge_{i \in Agt} (\mathbf{K}_i\varphi \vee \mathbf{K}_i\neg\varphi)$

A more intuitive axiomatisation of S5 common knowledge

S5(**K**) and Def(**EK**), plus:

S4(**CK**)	modal logic S4 for **CK**
FP_0	**CK** $\varphi \to$ **EK** φ
GFP_0	**CK EKA** $\varphi \to$ **CKA** φ

- sound for S5 models
  - GFP_0 provable in the axiom system with induction axiom GFP
- complete for S5 models
  - induction axiom GFP provable
  - proof uses S4 axioms for **CK**

Soundness: proof of $\mathtt{GFP}_0$

Proposition

$\mathtt{GFP}_0$ is provable from $\mathtt{GFP}$.

Proof.

1. $\mathbf{EKA}\,\varphi \to ((\varphi \to \mathbf{EK}\,\varphi) \land (\neg\varphi \to \mathbf{EK}\,\neg\varphi))$ — $\mathtt{T}(\mathbf{EK})$
2. $\mathbf{CK}\,\mathbf{EKA}\,\varphi \to \big(\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \land \mathbf{CK}\,(\neg\varphi \to \mathbf{EK}\,\neg\varphi)\big)$ — from 1
3. $\mathbf{CK}\,\mathbf{EKA}\,\varphi \to \big((\varphi \to \mathbf{CK}\,\varphi) \land (\neg\varphi \to \mathbf{CK}\,\neg\varphi)\big)$ — from 2 by $\mathtt{GFP}$
4. $\mathbf{CK}\,\mathbf{EKA}\,\varphi \to (\mathbf{CK}\,\varphi \lor \mathbf{CK}\,\neg\varphi)$ — from 3

Completeness: a key lemma

Lemma

The schema $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{CK}\,(\neg\varphi \to \mathbf{EK}\,\neg\varphi)$ *is provable from the schemas* $\mathtt{K}(\mathbf{CK})$, $\mathtt{4}(\mathbf{CK})$, $\mathtt{RN}(\mathbf{CK})$, $\mathtt{T}(\mathbf{CK})$, *and* $\mathtt{FP}$.

Proof.

1. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{EK}\,(\varphi \to \mathbf{EK}\,\varphi)$ — by $\mathtt{FP}$
2. $\mathbf{EK}\,(\varphi \to \mathbf{EK}\,\varphi) \to (\mathbf{EK}\,\neg\mathbf{EK}\,\varphi \to \mathbf{EK}\,\neg\varphi)$
3. $\neg\varphi \to \mathbf{EK}\,\neg\mathbf{EK}\,\varphi$ — $\mathtt{B}(\mathbf{EK})$
4. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to (\neg\varphi \to \mathbf{EK}\,\neg\varphi)$ — from 1, 2, 3
5. $\mathbf{CK}\,\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{CK}\,(\neg\varphi \to \mathbf{EK}\,\neg\varphi)$ — from 4 by $\mathtt{RN}(\mathbf{CK})$
6. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{CK}\,\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi)$ — $\mathtt{4}(\mathbf{CK})$
7. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{CK}\,(\neg\varphi \to \mathbf{EK}\,\neg\varphi)$ — from 5 and 6

Completeness: proof of GFP

Proposition

GFP *is provable from* $\mathtt{GFP}_0$*.*

Proof.

1. $(\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \land \mathbf{CK}\,(\neg\varphi \to \mathbf{EK}\,\neg\varphi)) \to \mathbf{CK}\,\mathbf{EKA}\,\varphi$
2. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{CK}\,\mathbf{EKA}\,\varphi$ — from 1 by key lemma
3. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to \mathbf{CKA}\,\varphi$ — from 2 by $\mathtt{GFP}_0$
4. $\mathbf{CK}\,(\varphi \to \mathbf{EK}\,\varphi) \to (\varphi \to \mathbf{CK}\,\varphi)$ — from 3 by $\mathtt{T}(\mathbf{CK})$

Conclusion of Part 1

- more intelligible axiomatisation of the relation between individual and common knowledge
  - more intuitive than the standard induction principles
  - intuitive axiomatisation of the pure logic of knowlege about
    - fragment with only $\mathbf{KA}_i$, $\mathbf{CKA}$ (no $\mathbf{K}_i$, $\mathbf{CK}$)
- hypothesis: logic of individual knowledge is $\mathtt{S5}$
  - $\mathtt{GFP}_0$ is sound for knowledge (logics with $\mathtt{T}(\mathbf{K})$ axiom)
    - conjecture: incomplete
  - $\mathtt{GFP}_0$ is unsound for logics without $\mathtt{T}(\mathbf{K})$!
    - suppose $\mathbf{B}_1\,\mathbf{CB}\,p \wedge \mathbf{B}_2\,\mathbf{CB}\,\neg p$ $\Rightarrow$ no common belief about p
    - consequence: $\mathbf{B}_1\,\mathbf{CB}\,\mathbf{EB}\,p \wedge \mathbf{B}_2\,\mathbf{CB}\,\mathbf{EB}\,\neg p$
    - consequence: $\mathbf{B}_1\,\mathbf{CB}\,\mathbf{EBA}\,p \wedge \mathbf{B}_2\,\mathbf{CB}\,\mathbf{EBA}\,p$
      (where $\mathbf{EBA}\,p = \mathbf{EB}\,p \vee \mathbf{EB}\,\neg p$)
    - consequence: $\mathbf{CB}\,\mathbf{EBA}\,p$
    - $\mathtt{GFP}_0$ would allow to infer common belief about p!

Part 1 Relating individual and common knowledge

Part 2 Lightweight fragments

Part 3 The three dimensions of epistemic-doxastic situations

Lightweight fragments: motivation

- epistemic reasoning is difficult:
  - satisfiability is PSPACE hard if there are multiple agents; EXPTIME complete if formulas may contain **CK** [Halpern and Moses, 1992, Fagin et al., 1995]
  - planning is undecidable with DEL event models [Bolander and Andersen, 2011, Aucher and Bolander, 2013]
    - even for heavily restricted event models [Bolander et al., 2015, Bolander et al., 2020]
- quest for lightweight fragments of the epistemic language
  - cf. description logics

'Knowledge that' literals [Lakemeyer and Lespérance, 2012, Muise et al., 2015]

$$\lambda ::= p \mid \neg\lambda \mid \mathbf{K}_i\lambda$$

- formula = boolan combination of epistemic literals
  - no conjunction or disjunction in scope of epistemic operators
- complexity: same as propositional logic
  - view epistemic atoms as propositional variables
  - plus theory: $\neg(\mathbf{K}_i\lambda \wedge \mathbf{K}_i\neg\lambda)$, $\mathbf{K}_i\mathbf{K}_i\lambda \leftrightarrow \mathbf{K}_i\lambda$, etc.
- cannot express "I know you know more than me"

$$\neg\mathbf{K}_i p \wedge \neg\mathbf{K}_i\neg p \wedge \mathbf{K}_i(\mathbf{K}_j p \vee \mathbf{K}_j\neg p)$$

however: is fundamental in dialogues (and more generally in interaction between agents)

‘Knowledge about’ atoms [Herzig et al., 2015, Cooper et al., 2021]

- grammar:

$$\alpha ::= p \mid \mathbf{KA}_i\alpha \mid \mathbf{CKA}\,\alpha$$

- formula = boolan combination of epistemic atoms
- can express some disjunctions in scope of epistemic operator:

$$\neg\mathbf{K}_i p \land \neg\mathbf{K}_i\neg p \land \mathbf{K}_i(\mathbf{K}_j p \lor \mathbf{K}_j\neg p)$$

expressed as

$$\neg\mathbf{KA}_i p \land \mathbf{K}_i\,\mathbf{KA}_j p$$
$$= \neg\mathbf{KA}_i p \land \mathbf{KA}_j p \land \mathbf{KA}_i\mathbf{KA}_j p$$

‘Knowledge about’ atoms: axiomatisation

$\mathbf{KA}_i\mathbf{KA}_i\alpha$
$\mathbf{CKA}\,\mathbf{CKA}\,\alpha$
$\mathbf{CKA}\,\mathbf{KA}_i\mathbf{KA}_i\alpha$
$\mathbf{CKA}\,\alpha \to \mathbf{KA}_i\alpha$
$\mathbf{CKA}\,\alpha \to \mathbf{CKA}\,\mathbf{KA}_i\alpha$
$\bigwedge_{i\in Agt}(\mathbf{KA}_i\alpha \wedge \mathbf{CKA}\,\mathbf{KA}_i\alpha) \to \mathbf{CKA}\,\alpha \qquad (\mathtt{GFP}_0)$

- sound and complete axiomatisation of the validities of the fragment
- N.B: axiom $\mathtt{GFP}_0$ is in the fragment (while $\mathtt{GFP}$ is not)

‘Knowledge about’ atoms: complexity

- basically: epistemic atoms can be viewed as propositional logic variables
  - take care of introspection: simulated by truth conditions
  - take care of inductive closure: inductively closed valuations of ‘knowledge about’ atoms
- complexity of model checking, satisfiability, planning: same as propositional logic
  - 1. prove fmp
    2. guess valuation and model check

Conclusion of Part 2

- interesting fragment of epistemic logic
  - based on ‘knowledge about’ atoms
  - satisfiability NP-complete
  - planning PSPACE-complete
- enough for many applications
  - gossip problem (including higher-order knowledge)
  - ...

Part 1 Relating individual and common knowledge

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Part 3 The three dimensions of epistemic-doxastic situations

Which possible relations between state of affairs and agent?

- cf. act positions [Demolombe and Jones, 2002]:

$\varphi \wedge \mathbf{E}_i\varphi$	$\neg\varphi \wedge \mathbf{E}_i\neg\varphi$
$\varphi \wedge \neg\mathbf{E}_i\varphi \wedge \neg\mathbf{E}_i\neg\varphi$	$\neg\varphi \wedge \neg\mathbf{E}_i\varphi \wedge \neg\mathbf{E}_i\neg\varphi$

where $\mathrm{E}_i\varphi$ = "i brings it about that φ"

- cf. Kanger-Lindahl theory of normative positions:

  "method for mapping out in a systematic and exhaustive fashion the complete space of all logically possible normative relations" [Sergot and Richards, 2001, Sergot, 2001]

- here:
  - which epistemic situations?
  - which doxastic situations?
  - which epistemic-doxastic situations?

$\Rightarrow$ 'knowledge/belief about' modalities provide interesting insights

Which epistemic situations?

- 4 possible relations between state of affairs and knowledge state:

$\varphi \wedge \mathbf{K}_i\varphi$	$\neg\varphi \wedge \mathbf{K}_i\neg\varphi$
$\varphi \wedge \neg\mathbf{K}_i\varphi \wedge \neg\mathbf{K}_i\neg\varphi$	$\neg\varphi \wedge \neg\mathbf{K}_i\varphi \wedge \neg\mathbf{K}_i\neg\varphi$

- with 'knowledge about':
  - 2^2 independent combinations of φ and $\mathbf{KA}_i\varphi$

$\varphi \wedge \mathbf{KA}_i\varphi$	$\neg\varphi \wedge \mathbf{KA}_i\varphi$
$\varphi \wedge \neg\mathbf{KA}_i\varphi$	$\neg\varphi \wedge \neg\mathbf{KA}_i\varphi$

Which doxastic situations?

- 6 possible relations between state of affairs and belief state

$\varphi \wedge \mathbf{B}_i\, \varphi$	$\neg\varphi \wedge \mathbf{B}_i\, \neg\varphi$
$\varphi \wedge \neg\mathbf{B}_i\, \varphi \wedge \neg\mathbf{B}_i\, \neg\varphi$	$\neg\varphi \wedge \neg\mathbf{B}_i\, \varphi \wedge \neg\mathbf{B}_i\, \neg\varphi$
$\varphi \wedge \mathbf{B}_i\, \neg\varphi$	$\neg\varphi \wedge \mathbf{B}_i\, \varphi$

- requires 3 dimensions
  - cannot be independent

Which epistemic-doxastic situations?

- 8 possible relations:

$\varphi \wedge \mathbf{K}_i\varphi$	$\neg\varphi \wedge \mathbf{K}_i\neg\varphi$
$\varphi \wedge \mathbf{B}_i\,\varphi \wedge \neg\mathbf{K}_i\varphi$	$\neg\varphi \wedge \mathbf{B}_i\,\neg\varphi \wedge \neg\mathbf{K}_i\neg\varphi$
$\varphi \wedge \neg\mathbf{B}_i\,\varphi \wedge \neg\mathbf{B}_i\,\neg\varphi$	$\neg\varphi \wedge \neg\mathbf{B}_i\,\varphi \wedge \neg\mathbf{B}_i\,\neg\varphi$
$\varphi \wedge \mathbf{B}_i\,\neg\varphi$	$\neg\varphi \wedge \mathbf{B}_i\,\varphi$

- $8 = 2^3 \;\Rightarrow\;$ which are the 3 dimensions?

Which epistemic-doxastic situations?

- two new modalities:

$$\begin{aligned}
\mathbf{TBA}_i\,\varphi &= (\varphi \wedge \mathbf{B}_i\,\varphi) \vee (\neg\varphi \wedge \mathbf{B}_i\,\neg\varphi) \\
&= \text{``}i \text{ has a } \textbf{true} \text{ belief about } \varphi\text{''} \\
\mathbf{MBA}_i\,\varphi &= (\mathbf{B}_i\,\varphi \wedge \neg\mathbf{K}_i\varphi) \vee (\mathbf{B}_i\,\neg\varphi \wedge \neg\mathbf{K}_i\neg\varphi) \\
&= \text{``}i \text{ has a } \textbf{mere} \text{ belief about } \varphi\text{''} \\
&= \text{``}i \text{ has a falsifiable belief about } \varphi\text{''} \\
&= \text{``}i \text{ has a belief about } \varphi \text{ but does not know whether } \varphi\text{''}
\end{aligned}$$

- just as 'belief about':

$$\mathbf{TBA}_i\,\neg\varphi \leftrightarrow \mathbf{TBA}_i\,\varphi$$
$$\mathbf{MBA}_i\,\neg\varphi \leftrightarrow \mathbf{MBA}_i\,\varphi$$

Epistemic-doxastic situations: 3 dimensions

- 2^3 epistemic-doxastic situations:

$\varphi \wedge \mathbf{TBA}_i\,\varphi \wedge \neg\mathbf{MBA}_i\,\varphi$	$\neg\varphi \wedge \mathbf{TBA}_i\,\varphi \wedge \neg\mathbf{MBA}_i\,\varphi$
$\varphi \wedge \mathbf{TBA}_i\,\varphi \wedge \mathbf{MBA}_i\,\varphi$	$\neg\varphi \wedge \mathbf{TBA}_i\,\varphi \wedge \mathbf{MBA}_i\,\varphi$
$\varphi \wedge \neg\mathbf{TBA}_i\,\varphi \wedge \neg\mathbf{MBA}_i\,\varphi$	$\neg\varphi \wedge \neg\mathbf{TBA}_i\,\varphi \wedge \neg\mathbf{MBA}_i\,\varphi$
$\varphi \wedge \neg\mathbf{TBA}_i\,\varphi \wedge \mathbf{MBA}_i\,\varphi$	$\neg\varphi \wedge \neg\mathbf{TBA}_i\,\varphi \wedge \mathbf{MBA}_i\,\varphi$

- needs getting used to, but is natural...

Example: the Sally-Ann Test

false belief task
[Wimmer and Perner, 1983, Baron-Cohen et al., 1985]

1. Sally puts the marble in the basket

$$\mathbf{TBA}_S\, \mathrm{b} \wedge \neg \mathbf{MBA}_S\, \mathrm{b}$$

2. Sally goes out for a walk

$$\mathbf{TBA}_S\, \mathrm{b} \wedge \mathbf{MBA}_S\, \mathrm{b}$$

3. Ann takes the marble out of the basket and puts it into the box

$$\neg \mathbf{TBA}_S\, \mathrm{b} \wedge \mathbf{MBA}_S\, \mathrm{b}$$

Full expressivity

- knowledge:

$$\mathbf{KA}_i\varphi \leftrightarrow \mathbf{TBA}_i\,\varphi \wedge \neg\mathbf{MBA}_i\,\varphi$$
$$\mathbf{K}_i\varphi \leftrightarrow \mathbf{TBA}_i\,\varphi \wedge \neg\mathbf{MBA}_i\,\varphi \wedge \varphi$$

- belief:

$$\mathbf{BA}_i\varphi \leftrightarrow \mathbf{TBA}_i\,\varphi \vee \mathbf{MBA}_i\,\varphi$$
$$\mathbf{B}_i\,\varphi \leftrightarrow (\varphi \wedge \mathbf{TBA}_i\,\varphi) \vee (\neg\varphi \wedge \neg\mathbf{TBA}_i\,\varphi \wedge \mathbf{MBA}_i\,\varphi)$$

... remember: $\mathbf{B}_i\,\varphi$ cannot be expressed with $\mathbf{BA}_i$ alone

An epistemic-doxastic logic

- logic:

KD5($\mathbf{B}$)	the principles of modal logic KD5 for $\mathbf{B}_i$
S4($\mathbf{K}$)	the principles of modal logic S4 for $\mathbf{K}_i$
KiB	$\mathbf{K}_i\varphi \to \mathbf{B}_i\,\varphi$
BiKB	$\mathbf{B}_i\,\varphi \to \mathbf{K}_i\mathbf{B}_i\,\varphi$
BiBK	$\mathbf{B}_i\,\varphi \to \mathbf{B}_i\,\mathbf{K}_i\varphi$

- belief definable from knowledge [Lenzen, 1978, Lenzen, 1995]:

$$\mathbf{B}_i\,\varphi \leftrightarrow \neg\mathbf{K}_i\neg\mathbf{K}_i\varphi$$

  - alternative axiomatisation: S4.2($\mathbf{K}$) plus $\mathbf{B}_i\,\varphi \leftrightarrow \neg\mathbf{K}_i\neg\mathbf{K}_i\varphi$
- complexity of satisfiability: PSPACE-complete [Shapirovsky, 2004, Chalki et al., 2021]

Reduction of ‘about’ modalities

- reduction of consecutive modal operators to length 1:

$$\begin{aligned}\mathbf{TBA}_i\,\mathbf{TBA}_i\,\varphi &\leftrightarrow \mathbf{TBA}_i\,\varphi \vee \neg\mathbf{MBA}_i\,\varphi\\ \mathbf{MBA}_i\,\mathbf{TBA}_i\,\varphi &\leftrightarrow \mathbf{MBA}_i\,\varphi\\ \mathbf{TBA}_i\,\mathbf{MBA}_i\,\varphi &\leftrightarrow \neg\mathbf{MBA}_i\,\varphi\\ \mathbf{MBA}_i\,\mathbf{MBA}_i\,\varphi &\leftrightarrow \mathbf{MBA}_i\,\varphi\end{aligned}$$

- cf. ‘know that’ modalities: length 2
  - ≥ 2 because $\neg\mathbf{K}_i\neg\mathbf{K}_i\varphi$ not reducible
  - ≤ 2 because all $\mathtt{S4}(\mathbf{K})$ axioms are valid

Conclusion of Part 3

- logic of ‘true/mere belief about’:
  - natural in knowledge representation
  - nice combinatorics:
    - boolean combinations of φ, $\mathbf{TBA}_i\,\varphi$, $\mathbf{MBA}_i\,\varphi$ are exclusive and exhaustive
  - paves the road towards lightweight fragment
    - formulas = boolean combinations of true/mere belief atoms
    - reduction to propositional logic

Conclusion: new perspectives provided by ‘knowledge/belief about’ modalities

1. alternative to greatest fixed-point axiom `GFP` that ‘talks’
   - sound for knowledge
   - complete if individual knowledge is `S5`
   - unsound for belief
2. interesting lightweight fragments of epistemic logic
   - same complexity as propositional logic
3. ‘true belief about’ and ‘mere belief about’ modalities
   - epistemic-doxastic situations

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