Quadrature Rules, Discrepacies and their Relation to Halftoning

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History of Work

S. Electrostatic Halftoning,

Computer Graphics Forum, 2010,

C. Schmaltz, P. Gwosdek, A. Bruhn and J. Weickert

1. Dithering by Differences of Convex Functions,

SIAM J. Imaging Science, 2011

with T. Teuber (TU Kaiserslautern), P. Gwosdek, Ch. Schmaltz and J. Weickert (Saarland U)

2. Quadrature Rules, Discrepanies and their Relation to Dithering on the Torus and the Sphere,

SIAM J. Scientific Computing, 2012

with M. Gräf and D. Potts (U Chemnitz)

- Efficient Algorithms for the Computation of Optimal Quadrature Points on Riemannian Manifolds,

  PhD Thesis of M. Gräf, 2013.

3. Consistency of Variational Dithering via Kinetic Theory,

Applicable Analysis, 2013

with M. Fornasier (U München) and J. Haskovec (U Linz)

Outline

1. Introduction
2. Dithering by Differences of Convex Functions (DC)
3. Quadrature Errors in RKHSs
4. Discrepancies
5. Least Squares Functionals for Bandlimited Functions - Efficient Minimization
6. Continuous-Domain Quantization via Kinetic Theory

1. Introduction

Aim: creating the illusion of a continuous tone image having only a limited number of tones (black/white) available

Applications: printing, sampling problems occurring in rendering, re-lighting, image compression, artistic non-photorealistic image visualization

Setting: $u : G \to [0,1] \quad \Rightarrow \quad w := 1 - u$

How to place M black dots at appropriate positions $\mathbf{p_k}$?

Existing Methods

Dithering: grid points $\mathcal{G} := \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$

- add Gaussian noise prior to quantization (Godall 1951)
- error diffusion (Floyd-Steinberg 1976)
- iterative processes (Ostromoukhov 2001, Chang/Alain/Ostromoukhov 2009, Pang et al. 1997)
- ordered dithering (Purgathofer et al. 1994)

Stippling: arbitrary point positions

- weighted centroidal Voronoi tessellation

$$L((p_k, V_k)_{k=1}^M) := \sum_{k=1}^{M} \int_{V_k} w(x) \|x - p_k\|^2 \, dx$$

- Lloyd's algorithm (Lloyd 1982, Secord 2002, Du/Faber/Gunzberger 1999)
  - convergence in 1D for differentiable strictly logarithmical concave weights
  - arithmetic complexity: $\mathcal{O}(M \log M)$ per iteration
- capacity-constrained variant of Lloyd's algorithm (Balzer/Schlömer/Deussen 2009)
- stochastical inspired method (Fattal 2011)

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2. Dithering by Differences of Convex Functions (DC)

Find the point positions $p_k \in \mathbb{R}^2$ by minimizing the attraction–repulsion functional

$$E(p) = \underbrace{\sum_{i=1}^{M} \sum_{x \in \mathcal{G}} w(x) \|p_i - x\|_2}_{\substack{F(p) \\ \textbf{Attraction}}} - \underbrace{\frac{\lambda}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \|p_i - p_j\|_2}_{\substack{G(p) \\ \textbf{Repulsion}}}$$

with $p := ((p_{x,k}, p_{y,k})^\top)_{k=1}^{M} \in \mathbb{R}^{2M}$ and the equilibration parameter

$$\lambda := \frac{1}{M} \sum_{x \in \mathcal{G}} w(x)$$

General continuous functionals:

$$E_\varphi(p) = \frac{\lambda}{2} \sum_{i,j=1}^{M} \varphi(\|p_i - p_j\|_2) - \sum_{i=1}^{M} \int_{[0,1]^2} w(x) \varphi(\|p_i - x\|_2) \, \mathrm{d}x$$

- $\varphi(r) = -r$ (above setting!)
- $\varphi(r) = -\log(r)$ (Schmaltz et al. 2010)

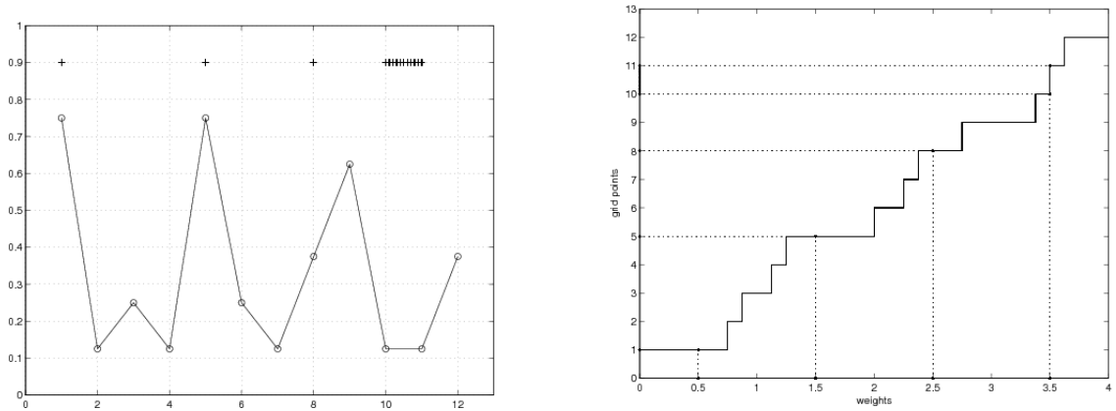
One-dimensional Problem

Given $w : \{1, \ldots, n\} \to [0, 1]$. Minimize

$$E(p) = \sum_{k=1}^{M} \sum_{j=1}^{n} w(j)\,|p_k - j| \;-\; \frac{\lambda}{2} \sum_{k,l=1}^{m} |p_k - p_l|.$$

Ordering of points leads to convex functional

$$\operatorname*{argmin}_{p_1 \le \ldots \le p_M} E(p) = \operatorname*{argmin}_{p} \sum_{k=1}^{M} \left(\sum_{j=1}^{n} w(j)\,|p_k - j| \;+\; (M - (2k-1))\,p_k \right).$$



Graphical illustration of the solution. Left: Signal w consisting of $n = 12$ points linearly connected. Right: Solution with $m = 4$ is $\hat{p}_1 = 1$, $\hat{p}_2 = 5$, $\hat{p}_3 = 8$, $\hat{p}_4 \in [10, 11]$.

Two-dimensional Problem

Minimize

$$
\begin{aligned}
E(p) &= \sum_{i=1}^{M} \sum_{x)\in\mathcal{G}} w(x)\|p_i - x\|_2 \; - \; \frac{\lambda}{2} \sum_{i,j=1}^{M} \|p_i - p_j\|_2 \\
&= F(p) - G(p)
\end{aligned}
$$

Necessary condition for $\hat{\boldsymbol{p}}$ to be a local minimizer

$$\partial G(\hat{p}) \subset \partial F(\hat{p})$$

where $\quad \partial F(p) := \{p^* \in \mathbb{R}^{2M} : F(q) - F(p) \geq \langle p^*, q - p\rangle \; \forall q \in \mathbb{R}^{2M}\}$

Relaxed condition

$$\partial F(\hat{p}) \cap \partial G(\hat{p}) \neq \emptyset \quad \Leftrightarrow \quad 0 \in \partial F(\hat{p}) - \partial G(\hat{p}) = \partial_C E(\hat{p})$$

$\hat{p}$ are critical points (Toland 1979, Hirriat-Urruty 1989)

Solution by proximal point (type) algorithm or Moreau–Yoshida regularization (Sun et al. 2003, Hamdi 2005, Maingé/Moudafi 2008), DC-algorithm (Pham Dinh et al. 1986...)

Algorithm (FBS) *Initialization*: $p^{(0)}$
For $r = 0, 1, \ldots$ repeat until a convergence criterion is reached

$$p^{(r+1)} = (I + \mu\partial F)^{-1}(p^{(r)} + \mu\, v^{(r)}), \quad v^{(r)} \in \partial G(p^{(r)})$$

Computation via

1. $v^{(r)} \in \partial G(p^{(r)})$: $\quad v_k^{(r)} = \sum_{\substack{l=1 \\ p_k^{(r)} \neq p_l^{(r)}}}^{M} \frac{p_k^{(r)} - p_l^{(r)}}{\|p_k^{(r)} - p_l^{(r)}\|_2}, \qquad k = 1, \ldots, M$
2. Set $y^{(r)} := p^{(r)} + \mu v^{(r)}$
3. Solve the proximation problem componentwise for each p_k, $k = 1, \ldots, M$

$$p_k^{(r+1)} = \operatorname*{argmin}_{q \in \mathbb{R}^2} \left\{ \frac{\mu}{2} \|q - y_k^{(r)}\|_2^2 + \sum_{x \in \mathcal{G}} w(x) \|q - x\|_2 \right\}$$

by splitting algorithms (PPXA by Combettes/Pesquet 2008) or Weiszfeld algorithm (Weiszfeld 1937, Ostresh 1978)

Fast Summation Algorithms

The above algorithms involve the computation of special sums of the form

$$s_1(p_k) := \sum_{(i,j)\in G(i,j)^{\mathsf{T}} \neq p_k} w(i,j) \frac{p_k - \binom{i}{j}}{|p_k - \binom{i}{j}|} \quad \text{and} \quad s_2(p_k) := \sum_{l=1, p_l \neq p_k}^{M} \frac{p_k - p_l}{|p_k - p_l|}$$

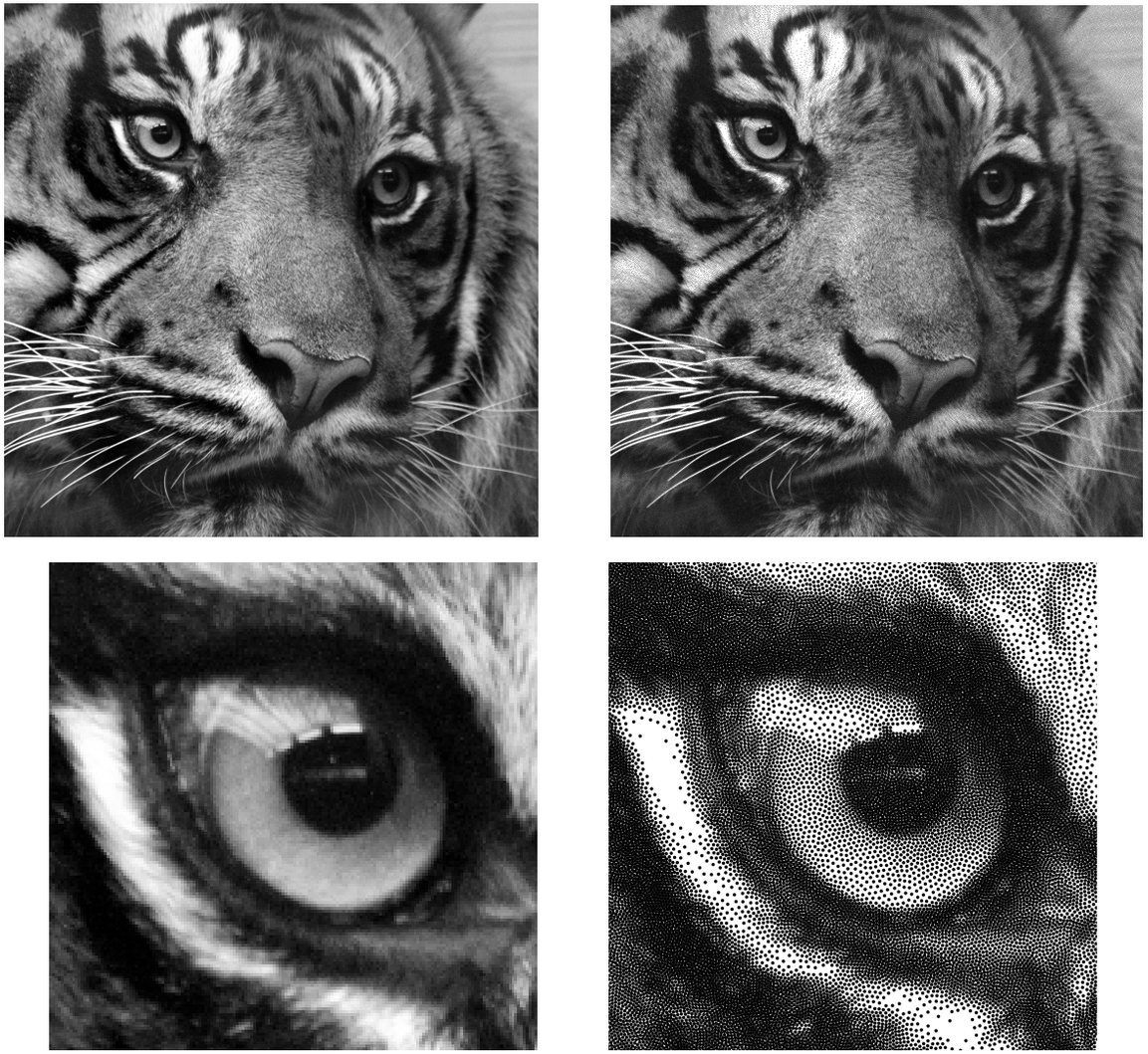
for $k = 1, \ldots, M$ in each iteration.

- FAST SUMMATION based on NFFT (Potts/Steidl 2002)

  NFFT/USFFT (Dutt/Rokhlin 1993, Beylkin 1995, Fessler et al. 2003, Kunis et al. ...)
- Alternative methods:
  - fast multipole method (Greengard/Rokhlin 1987, Beatson/Newsam 1992)
  - fast mosaic-skeleton matrix multiplication (Tyrtyshnikov 1996)
  - fast $\mathcal{H}$-matrix multiplication (Hackbusch 1999)
  - ...
- Arithmetic complexity $\mathcal{O}(M \log M)$

Numerical Examples

GPU implementation of NFFT based algorithm by P. Gwosdek, T. Teuber et al.
Fast Electrostatic Halftoning, J. of Real-Time Image Process, to appear.



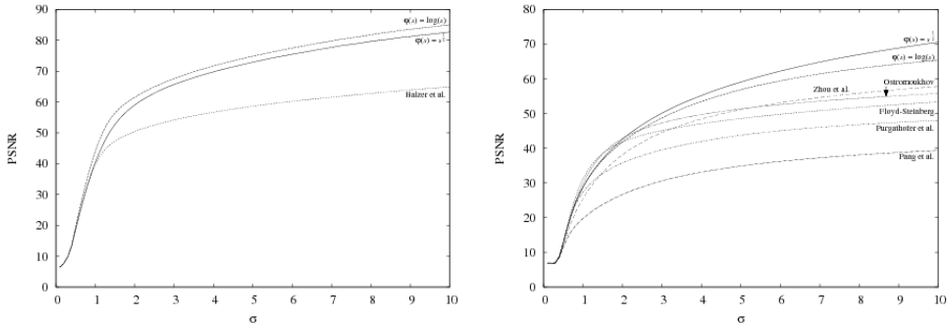
Dithering of a 1024×1024 image with $M = 647\,770$ points, $820ms$ per iteration.

Numerical Examples

Stippling (left) and dithering (middle) results with our method and the Floyd-Steinberg (right)

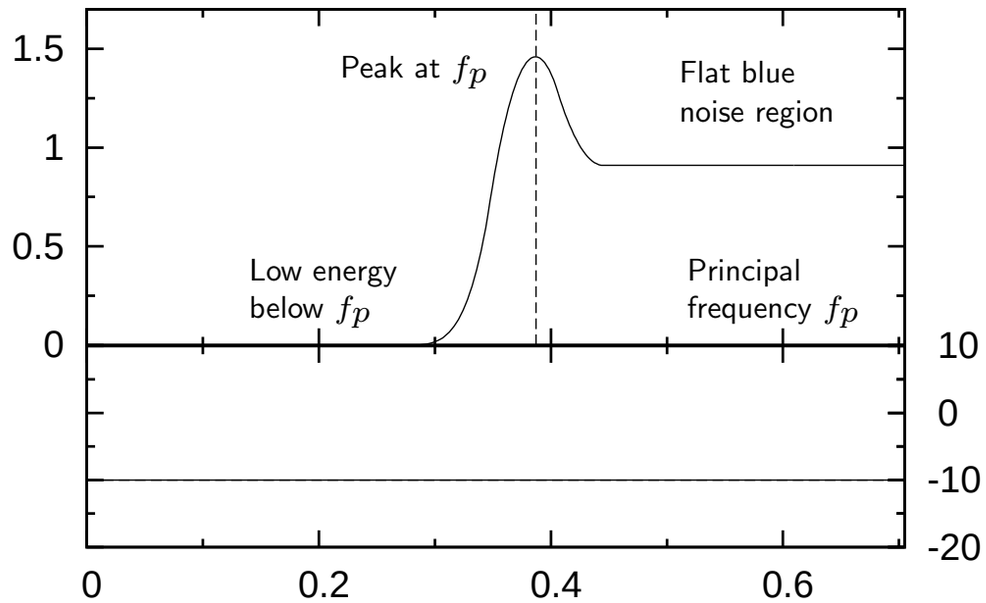
Comparison of Methods

1. Gaussian scale space properties: at a certain distance an observer should not recognize the difference between the halftoned and the original image

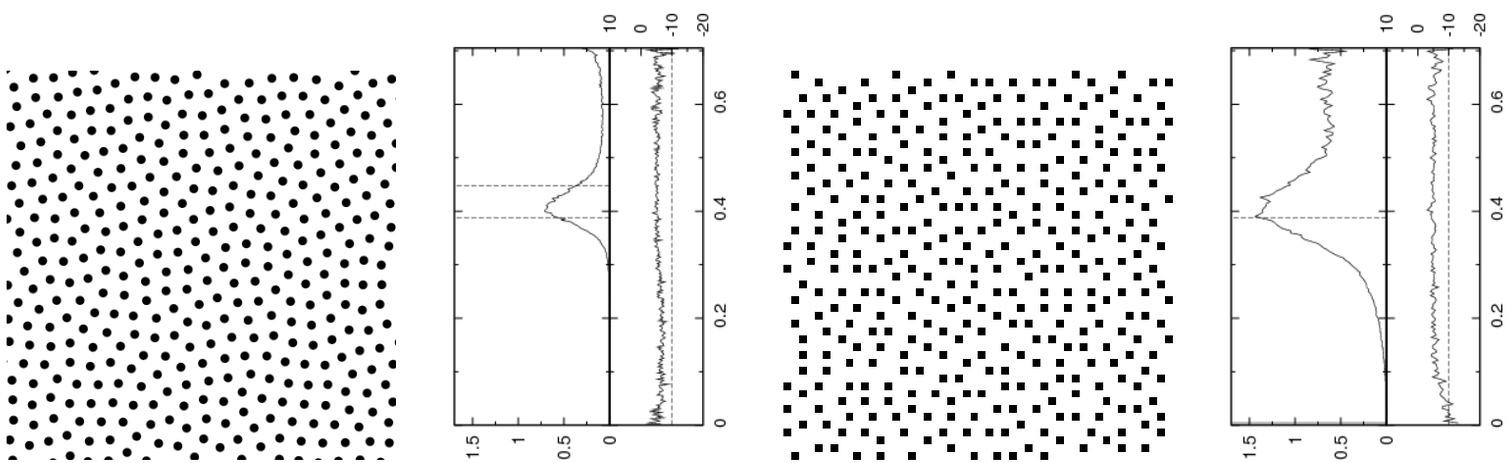


PSNR for various dithering results.

2. Blue noise spectrum: Fourier spectrum (periodogram) radially averaged



Blue noise behavior of an ideal halftone image by R. Ulichney 1987.



Results obtained by the stippling (left) and dithering method (right) for ℓ_1.

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General setting: $\mathcal{X} \in \{\mathbb{R}^d, [0,1]^d, \mathbb{T}^d, \mathbb{S}^{d-1}\}$, H_K reproducing kernel Hilbert space (RKHS) with positive semi-definite reproducing kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$:

$$K_x := K(\cdot, x) \in H_K \quad \text{and} \quad f(x) = \langle f, K(\cdot, x)\rangle_{H_K}, \quad \forall\, x \in \mathcal{X}, \forall f \in H_K.$$

Aim: Approximation of

$$I_w(f) := \int_{\mathcal{X}} f(x) w(x)\,\mathrm{d}x \quad \text{for } f \in H_K$$

by a **quadrature rule**

$$Q(f,p) := \lambda \sum_{i=1}^{M} f(p_i), \qquad \lambda := \frac{1}{M}\int_{\mathcal{X}} w(x)\,\mathrm{d}x$$

Worst case quadrature error:

$$\mathrm{err}_K(p) := \sup_{\substack{f \in H_K \\ \|f\|_{H_K} \le 1}} |I_w(f) - Q(f,p)| = \|I_w - Q(\cdot,p)\|_{H_K^*}.$$

Theorem (Quadrature Error and Attraction-Repulsion Functional)
The minimizers of err_K and E_K coincide, where

$$\mathrm{err}_K(p)^2 = 2\lambda E_K(p) + \|h_w\|_{H_K}^2,$$

$$E_K(p) := \frac{\lambda}{2}\sum_{i,j=1}^{M} K(p_i,p_j) - \sum_{i=1}^{M}\int_{\mathcal{X}} w(x)K(p_i,x)\,\mathrm{d}x.$$

Remember: Attraction-Repulsion Functional

$$E_\varphi(p) = \frac{\lambda}{2}\sum_{i,j=1}^{M} \varphi(\|p_i-p_j\|_2) - \sum_{i=1}^{M}\int_{[0,1]^2} w(x)\varphi(\|p_i-x\|_2)\,\mathrm{d}x$$

- Bad news: The kernels $K(x,y) = \Phi(x-y) = \varphi(\|x-y\|_2), \quad \varphi(r) = -r^\tau$ are not positive semi-definite.
- Good news: These kernels are conditionally positive definite, radial functions of order 1 for $0 < \tau < 2 \longrightarrow$ proof that functional of related positive definite kernel has the same minimizer

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Setting:

- $\mathcal{X} \in \{\mathbb{T}^d, \mathbb{S}^{d-1}\}$, $D := \mathcal{X} \times [0, R]$
- $\mathcal{B}(c, r) := \{x \in \mathcal{X} : d_{\mathcal{X}}(c, x) \leq r\}$ ball centered at $c \in \mathcal{X}$ with radius $0 \leq r \leq R$

Positive semi-definite discrepancy kernels:

$$K_{\mathcal{B}}(x, y) := \int_0^R \int_{\mathcal{X}} 1_{\mathcal{B}(c,r)}(x) 1_{\mathcal{B}(c,r)}(y) \, \mathrm{d}\mu_{\mathcal{X}}(c) \, \mathrm{d}r = \int_0^R \mu_{\mathcal{X}}(\mathcal{B}(x, r) \cap \mathcal{B}(y, r)) \, \mathrm{d}r$$

L_2-discrepancy: (see books of Novak/Woźniakowski on tractability I - III)

$$\mathrm{disc}_2^{\mathcal{B}}(p) := \left(\int_D \left(\int_{\mathcal{X}} w(x) 1_{\mathcal{B}(c,r)}(x) \, dx - \lambda \sum_{i=1}^{M} 1_{\mathcal{B}(c,r)}(p_i) \right)^2 \mathrm{d}\mu_{\mathcal{X}}(c) \, \mathrm{d}r \right)^{\frac{1}{2}}.$$

Theorem (Quadrature Error and L_2-Discrepancy)

$$\mathrm{err}_{K_{\mathcal{B}}}(p) = \mathrm{disc}_2^{\mathcal{B}}(p).$$

Kernels on $\mathbb{S}^1$

Geodesic distance:

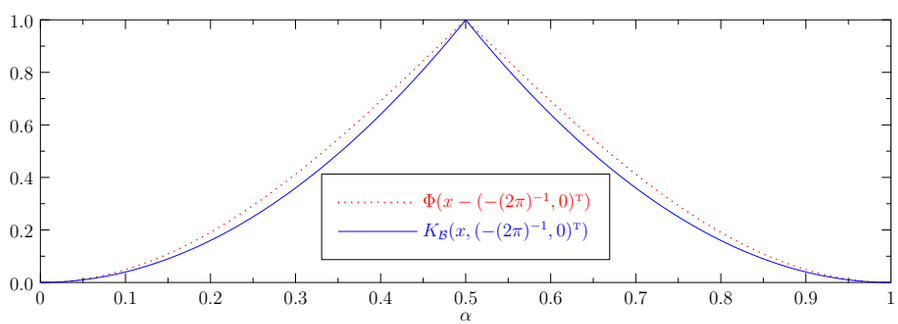
$$\mathrm{d}_{\mathbb{S}^1}(x,y) = (2\pi)^{-1}\arccos\left(\cos 2\pi(\alpha-\beta)\right), \quad x := (2\pi)^{-1}(\cos 2\pi\alpha, \sin 2\pi\alpha)^\top \in \mathbb{R}^2$$

Discrepancy kernel:

$$\begin{aligned} K_{\mathcal{B}}(x,y) &= \int_0^{\frac{1}{2}} \mu_{\mathbb{S}^1}(\mathcal{B}(x,r)\cap\mathcal{B}(y,r))\,\mathrm{d}r \\ &= \int_{\frac{d}{2}}^{\frac{1}{2}} 2\left(r-\frac{d}{2}\right)\mathrm{d}r + \int_{\frac{1}{2}-\frac{d}{2}}^{\frac{1}{2}} 2\left(r-\frac{1}{2}+\frac{d}{2}\right)\mathrm{d}r \\ &= \frac{1}{4}+\frac{1}{2}d(d-1) \;=\; \frac{1}{2}B_2(d)+\frac{1}{6} \end{aligned}$$

with Bernoulli polynomial B_2 (Wahba's smoothing spline)

There are relations between distance kernels and discrepancy kernels.



Kernels on $\mathbb{S}^2$

Geodesic distance:

$$\mathrm{d}_{\mathbb{S}^2}(x,y) = \arccos(x \cdot y), \qquad x, y \in \mathbb{S}^2.$$

with

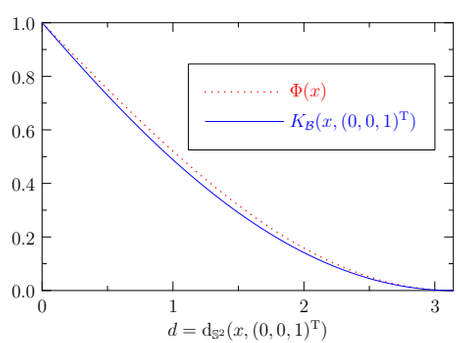
$$x = x(\theta,\varphi) := (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)^\mathsf{T} \in \mathbb{R}^3, \qquad (\varphi,\theta) \in [0,2\pi) \times [0,\pi].$$

Discrepancy kernel:

$$\tilde{K}_{\mathcal{B}}(d) = \int_0^\pi a(r,d)\,\mathrm{d}r.$$

with

$$a(r,d) = \begin{cases} 0, & 0 \le r \le d/2, \\ 4\left[\arccos\left(\sin(d/2)/\sin r\right) - \cos r \arccos\left(\tan(d/2)\cot r\right)\right], & d/2 < r < \pi/2, \\ 4r - 2d, & r = \pi/2, \\ 4\left[\arccos\left(\sin(d/2)/\sin r\right) - \cos r \arccos\left(\tan(d/2)\cot r\right)\right], & \pi/2 < r < \pi - d/2, \\ -4\pi\cos r, & \pi - d/2 \le r < \pi. \end{cases}$$



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5. Least Squares Functionals for Bandlimited Functions - Efficient Minimization

Setting:

- $\mathcal{X} \in \{\mathbb{S}^1, \mathbb{T}^2, \mathbb{S}^2\}$
- $\{\psi_l : l \in \mathbb{N}\}$ be an orthonormal basis of $L_2(\mathcal{X})$
- $\Pi_N(\mathcal{X}) := \text{span}\{\psi_l : l = 1, \ldots, d_N\}$ spaces of bandlimited functions

  $\Pi_N(\mathbb{S}^1) := \text{span}\{\mathrm{e}^{-2\pi \mathrm{i} n(\cdot)} : n = -N/2, \ldots, N/2\}$,

  $\Pi_N(\mathbb{T}^2) := \text{span}\{\mathrm{e}^{-2\pi \mathrm{i} n(\cdot)} : n = (n_1, n_2), n_j = -N/2, \ldots, N/2,\ j = 1, 2\}$

  $\Pi_N(\mathbb{S}^2) := \text{span}\{Y_n^k : n = 0, \ldots, N; k = -n, \ldots, n\}$

  Y_n^k denote the spherical harmonics
- $K_N(x, y) := \sum_{l=1}^{d_N} \lambda_l \psi_l(x) \overline{\psi_l(y)}, \quad \lambda_l > 0$
- RKHS: $H_{K_N} := \Pi_N(\mathcal{X})$ with $\langle f, g \rangle_{H_{K_N}} = \sum_{l=1}^{d_N} \hat{f}_l \overline{\hat{g}_l} / \lambda_l$

Theorem (Quadrature Error and Least Squares Functional)

$$\operatorname{err}_{K_N}(p)^2 = \mathcal{E}_N(p)$$

$$\mathcal{E}_N(p) := \sum_{l=1}^{d_N} \lambda_l \Big| \underbrace{\lambda \sum_{i=1}^{M} \overline{\psi_l(p_i)} - \hat{w}_l}_{F_l(p)} \Big|^2 = \|\mathbf{\Lambda}^{\frac{1}{2}} F(p)\|_2^2$$

- Spherical N-Designs $M = (N+1)^2$ (Delsarte/Goethals/Seidel 1977, Chen/Frommer/Lang 2010, $N = 100$)

$$\int_{\mathbb{S}^2} f(x)\,\mathrm{d}x = \frac{4\pi}{M} \sum_{i=1}^{M} f(p_i), \qquad \text{for all } f \in \Pi_N(\mathbb{S}^2)$$

Equivalent to (Sloan/Womersley 2009) for $\mathcal{X} := \mathbb{S}^2$ and $w \equiv 1$

$$\mathcal{E}_N(p) = \lambda^2 \sum_{n=1}^{N} \sum_{k=-n}^{n} \lambda_n \left| \sum_{i=1}^{M} Y_n^k(p_i) \right|^2 = 0$$

Efficient Minimization Algorithms

Task: Minimization of

$$\mathcal{E}_N(p) := \sum_{l=1}^{d_N} \lambda_l \left| \lambda \sum_{i=1}^{M} \overline{\psi_l(p_i)} - \hat{w}_l \right|^2 = \|\boldsymbol{\Lambda}^{\frac{1}{2}} F(p)\|_2^2$$

Computation of a local minimizers by nonlinear CG on Riemannian manifolds (here $\mathcal{M} := (\mathbb{S}^2)^M$), (see Daniel 1967, Smith 1994)

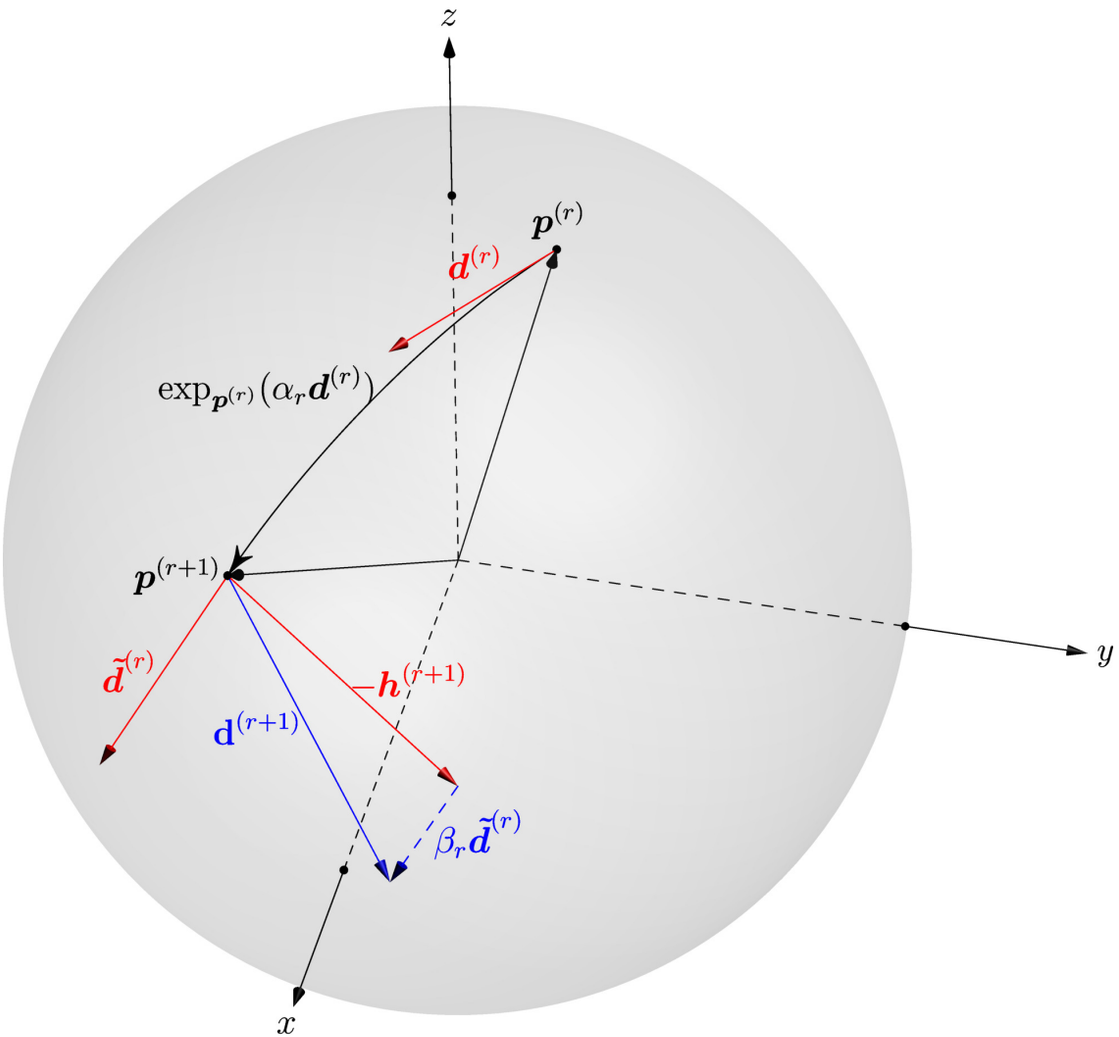
Algorithm: (CG algorithm on Riemannian manifolds)

Initialization: $\boldsymbol{p}^{(0)}$, $\boldsymbol{h}^{(0)} := \nabla \mathcal{E}_N(\boldsymbol{p}^{(0)})$, $\boldsymbol{d}^{(0)} = -\boldsymbol{h}^{(0)}$

For $r = 0, 1, \ldots$ repeat until a convergence criterion is reached

1. $\alpha_r := -\langle \boldsymbol{d}^{(r)}, \boldsymbol{h}^{(r)} \rangle / \langle \boldsymbol{d}^{(r)}, \mathrm{H}\mathcal{E}_N(\boldsymbol{p}^{(r)}) \boldsymbol{d}^{(r)} \rangle$
2. $\boldsymbol{p}^{(r+1)} := \exp_{\boldsymbol{p}^{(r)}} \left(\alpha_r \boldsymbol{d}^{(r)} \right)$
3. $\boldsymbol{h}^{(r+1)} := \nabla \mathcal{E}_N(\boldsymbol{p}^{(r+1)})$
4. Compute β_r by $\quad \beta_r := \frac{\langle \boldsymbol{h}^{(r+1)}, \mathrm{H}\mathcal{E}_N(\boldsymbol{p}^{(r+1)}) \tilde{\boldsymbol{d}}^{(r)} \rangle}{\langle \tilde{\boldsymbol{d}}^{(r)}, \mathrm{H}\mathcal{E}_N(\boldsymbol{p}^{(r+1)}) \tilde{\boldsymbol{d}}^{(r)} \rangle}, \quad \tilde{\boldsymbol{d}}^{(r)} := \boldsymbol{P}_{\boldsymbol{g}(\alpha_r)}(\boldsymbol{d}^{(r)}).$
5. $\boldsymbol{d}^{(r+1)} := -\boldsymbol{h}^{(r+1)} + \beta_r \tilde{\boldsymbol{d}}^{(r)}$

- $\exp_p : \mathrm{T}_{\boldsymbol{p}}\mathcal{M} \to \mathcal{M}$ denotes the *exponential map* from the tangent space $\mathrm{T}_{\boldsymbol{p}}\mathcal{M}$ to the manifold
- $\boldsymbol{P}_{\boldsymbol{g}(\alpha_r)}(\boldsymbol{d}^{(r)})$ the *parallel transport* of $\boldsymbol{d}^{(r)} \in \mathrm{T}_{\boldsymbol{p}^{(r)}}\mathcal{M}$ along the geodesics $\boldsymbol{g}$



An iteration step of the nonlinear CG method on the sphere $\mathbb{S}^2$.

Efficient Minimization Algorithms

Basically we need the evaluation of bandlimited functions

$$f(p_i) = \sum_{l=1}^{d_N} \hat{f}_l \psi_l(p_i); i = 1, \ldots, M \quad \Leftrightarrow \quad \boldsymbol{f} = \boldsymbol{A}_N \hat{\boldsymbol{f}}$$

where

$$\boldsymbol{A}_N := \begin{cases} \boldsymbol{F}_N = \left(\mathrm{e}^{-2\pi \mathrm{i} n p_i} \right)_{i=1,\ldots,M; n=-N/2,\ldots,N/2} & \text{for } \mathbb{S}^1, \\ \boldsymbol{F}_{2,N} = \left(\mathrm{e}^{-2\pi \mathrm{i} (n_1,n_2)^\mathsf{T} \cdot p_i} \right)_{i=1,\ldots,M; n_i=-N/2,\ldots,N/2, i=1,2} & \text{for } \mathbb{T}^2, \\ \boldsymbol{Y}_N = \left(Y_k^n(p_i) \right)_{i=1,\ldots,M; n=0,\ldots,N, |k| \le n} & \text{for } \mathbb{S}^2. \end{cases}$$

Multiplication of a vector with the matrix $\boldsymbol{A}_N$, resp. $\overline{\boldsymbol{A}}_N^\mathsf{T}$ (NFFT 3.0 software library (Kunis, Potts et al.)):

* $\mathcal{O}(N \log N + M \log(1/\epsilon))$ for $\mathbb{S}^1$ by NFFT,
* $\mathcal{O}(N^2 \log N + M \log^2(1/\epsilon))$ for $\mathbb{T}^2$ by NFFT
* $\mathcal{O}(N^2 \log^2 N + M \log^2(1/\epsilon))$ for $\mathbb{S}^2$ by NFSFT

Theorem (Fast Evaluation of $\nabla\mathcal{E}_N$ and Multiplication with $\mathrm{H}\mathcal{E}_N$)
The gradient $\nabla\mathcal{E}_N(p)$ and the multiplication of a vector with the Hessian $\mathrm{H}\mathcal{E}_N(\boldsymbol{p})$ can be computed with the arithmetic complexity (*).

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Setting: $\Omega = \mathbb{R}^d$, $w \geq 0$ is compactly supported, $\phi, \psi \in C^1(\Omega)$ and

$$\int_\Omega w(x)\,dx = 1, \quad N\lambda = 1$$

Passage to the Mean-Field Limit

Consider evolution of an N-particle system according to the gradient flow of

$$E(p) := -\sum_{k=1}^{N} \int_\Omega w(x)\phi(p_k - x)\,dx + \frac{1}{2N}\sum_{k,\ell=1}^{N} \psi(p_k - p_\ell)$$

which is given by

$$\begin{aligned} \frac{d}{dt}p_i(t) &= -\nabla_{p_i} E(p(t)) \\ &= \int_\Omega w(x)\nabla\phi(p_i(t) - x)\,dx - \frac{1}{N}\sum_{\ell=1}^{N} \nabla\psi(p_i(t) - p_\ell(t))\,, \qquad i = 1, \ldots, N \end{aligned}$$

subject to the initial condition

$$p_i(0) = p_i^0\,, \qquad i = 1, \ldots, N.$$

Passage to the Mean Field Limit

- mean field limit is obtained as the number of particles N tends to infinity
- the vector of time-dependent particle positions $p(t) \in \Omega^N$ is replaced by the time-dependent probability measure $f(x,t)$

  (Roughly speaking, $f(x,t)\,dx$ the probability that a particle is located in the space element dx around the position $x \in \Omega$ at time $t \geq 0$)
- empirical measure

$$f^N(\cdot,t) := \frac{1}{N}\sum_{i=1}^{N} \delta(\cdot - p_i(t))$$

  corresponds to the above system

Notation:
$\mathcal{M}(\Omega)$: Radon measures on Ω
$\mathcal{M}^1(\Omega)$: probability measures on Ω
$L^\infty(\mathbb{R}_+, \mathcal{M}(\Omega))$: essentially bounded functions from $\mathbb{R}_+$ to $\mathcal{M}(\Omega)$

Theorem (Mean Field Limit)

Let $\varphi, \psi \in C^1(\Omega)$, where $\nabla\psi$ is bounded.
Assume that there exists a probability measure $f_0 \in \mathcal{M}^1(\Omega)$ such that $f^N(\cdot, 0) \to f_0(\cdot)$ weakly-* in $\mathcal{M}(\Omega)$ as $N \to \infty$.

Then there exists a subsequence $\left(f^{N_k}\right)_{k\in\mathbb{N}}$ which converges weakly-* in $L^\infty(\mathbb{R}_+, \mathcal{M}(\Omega))$ to a time-dependent probability measure $f \in L^\infty(\mathbb{R}_+, \mathcal{M}^1(\Omega))$ which solves, in the sense of distributions, the mean-field equation

$$\begin{aligned} \partial_t f &= -\nabla_y \cdot \left(\int_\Omega (w(x)\nabla\phi(y-x) - f(x,t)\nabla\psi(y-x)) f(y,t)\, dx \right) \\ &= -\nabla \cdot (\nabla \mathcal{K}[f] f) \,, \end{aligned}$$

where

$$\mathcal{K}[f](y,t) := \int_\Omega (w(x)\phi(y-x) - f(x,t)\psi(y-x))\, dx \,,$$

subject to

$$f(\cdot, 0) = f_0 \,.$$

Lemma (Classical Solution)

Let $\nabla\phi$ and $\nabla\psi$ be globally Lipschitz continuous on $\mathbb{R}^d$. Let $f_0 \in C_c^1(\Omega)$ be nonnegative, compactly supported and fulfill $\int_\Omega f_0(x)\,dx = 1$.

Then the corresponding distributional solution f of the mean field equation is in fact the unique classical solution with $f \in C^1(\Omega \times \mathbb{R}_+)$ and $f(\cdot, t) \geq 0$ for all $t \geq 0$.

Moreover, $f(\cdot, t)$ is compactly supported on Ω for any $t \in \mathbb{R}_+$.

Energy Dissipation

Consider formal limit of

$$E(p) = E_{\varphi,\psi}(p) := -\sum_{k=1}^{N} \int_\Omega w(x)\phi(p_k - x)\,dx + \frac{1}{2N}\sum_{k,\ell=1}^{N} \psi(p_k - p_\ell)$$

given by

$$\mathcal{E}[f] = -\int_\Omega \int_\Omega w(x)\phi(p-x)f(p)\,dx\,dp + \frac{1}{2}\int_\Omega \int_\Omega f(x)\psi(p-x)f(p)\,dp\,dx\,,$$

Lemma

Let φ, ψ be radial functions fulfilling the assumptions of the previous lemma.

Let $f \in C^1(\Omega \times \mathbb{R}_+)$ be a classical solution of the mean field equation. Then

$$\frac{d\mathcal{E}[f(x,t)]}{dt} \leq 0\,.$$

holds true.

Energy Dissipation

Note that

$$\frac{d\mathcal{E}[f(x,t)]}{dt} = -\int_\Omega |\nabla\mathcal{K}[f](p,t)f(p,t)|^2 f(p,t)\,dp \le 0$$

Lemma

Let φ, ψ be of the form of the previous lemma and let $f \in C^1(\Omega \times \mathbb{R}_+)$ be a classical solution of the mean field equation. Then $f(\cdot,t) \to f^*$ weakly-* in $\mathcal{M}^1(\Omega)$ as $t \to \infty$, where f^* satisfies

$$|\nabla\mathcal{K}[f^*](p)f^*(p)|^2 f^*(p) = 0 \quad \text{a.e. on } \Omega.$$

Lemma

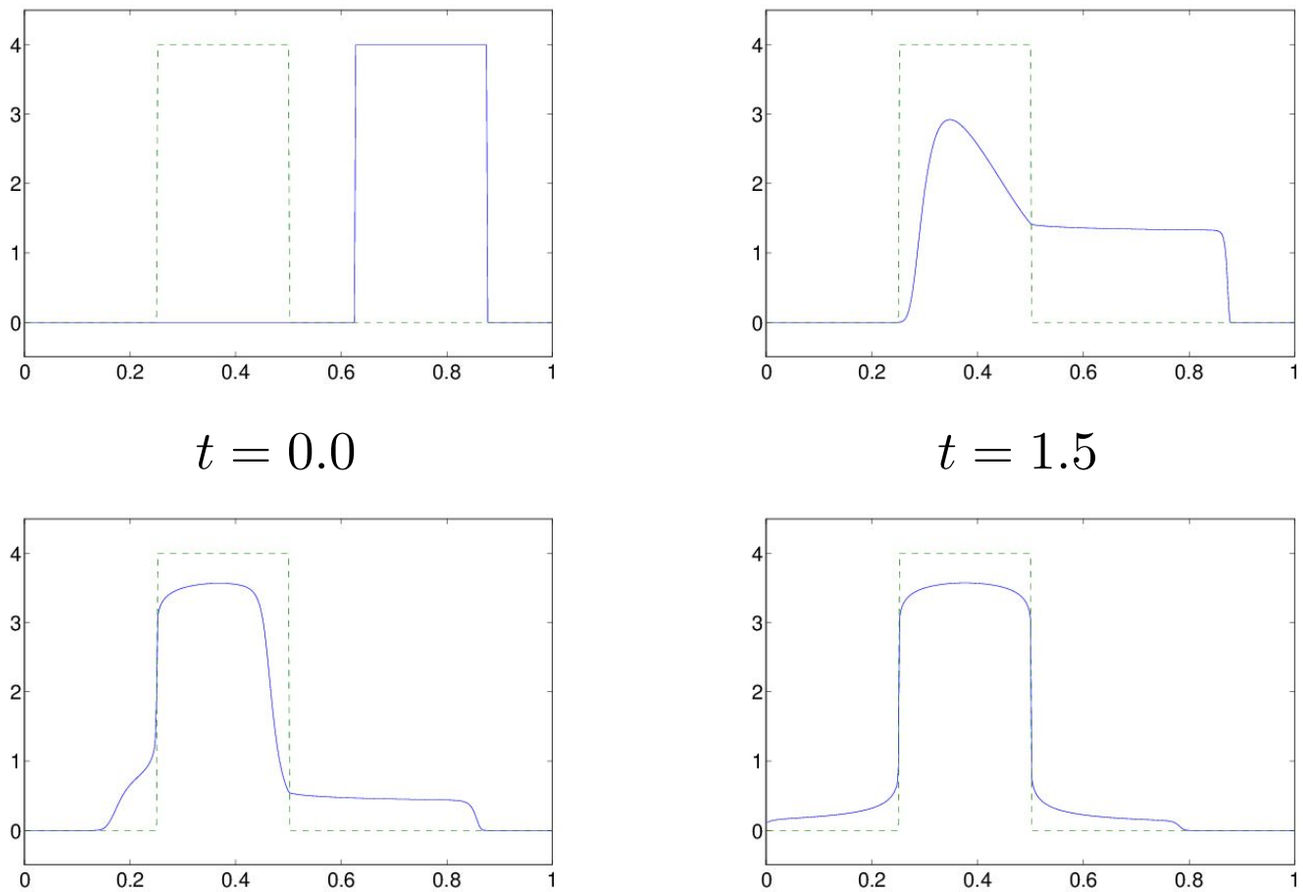
Let $\Omega = \mathbb{R}$ and $\phi(\cdot) = \psi(\cdot) = -|\cdot|$. Let $w \ge 0$ be compactly supported in Ω and such that $\int_\Omega w(x)\,dx = 1$.

Then the solution $f^* \equiv w$ of the mean field equation is unique in the class $\mathcal{X} := \left\{f \in L^1(\Omega) : f \ge 0 \text{ a.e. on } \Omega, \int_\Omega f(x)\,dx = 1\right\}$.

Numerical Examples 1D - Proof of the Concept

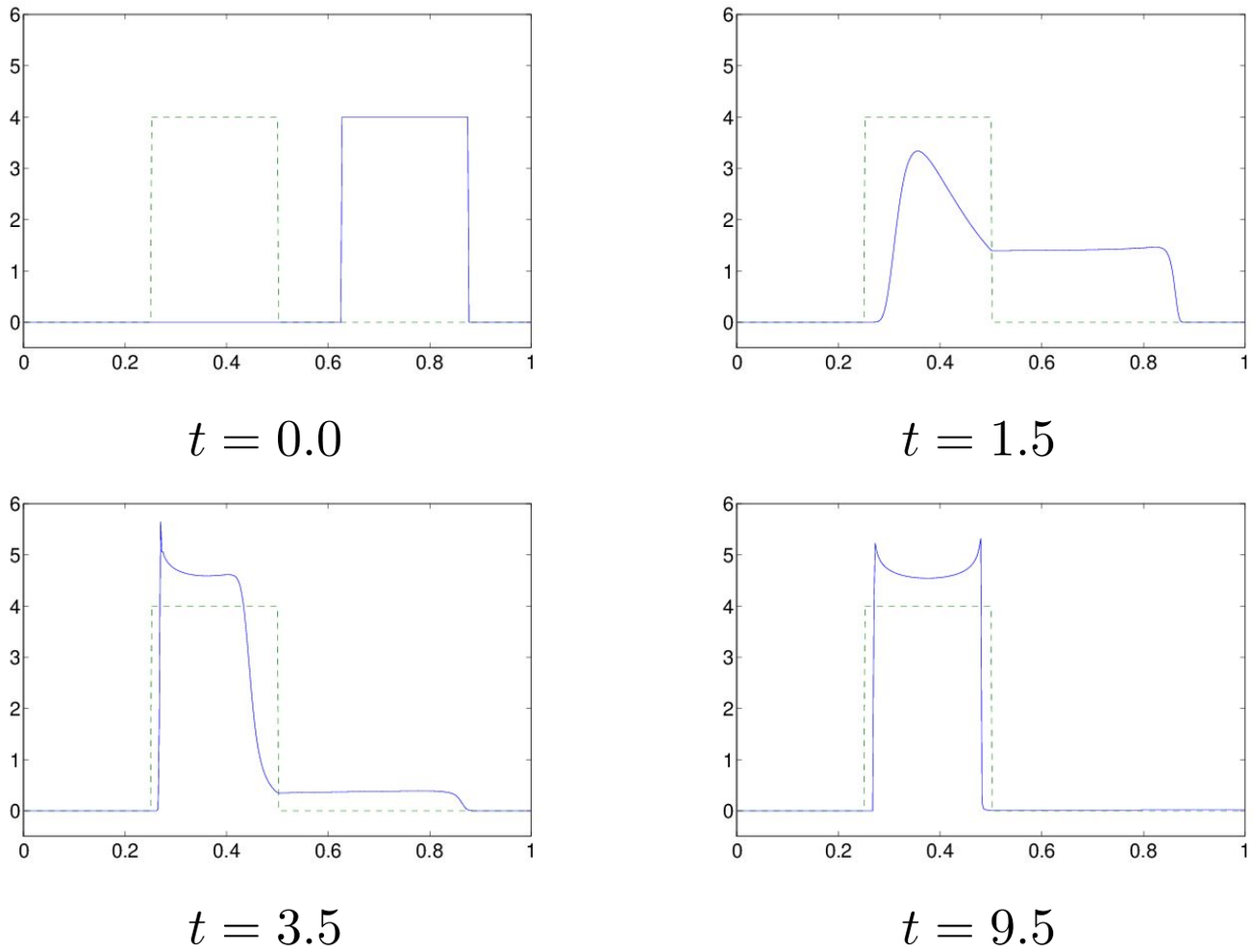
We consider two cases:

- Smoothing case: $\phi(s) = -|s|^{1.1}$ and $\psi(s) = -|s|$
- Sharpening case: $\phi(s) = -|s|$ and $\psi(s) = -|s|^{1.1}$



Smoothing case of the mean-field limit. The solid line represents the solution f, the dashed line the data w. The upper left panel shows the initial condition, the lower right panel is the steady state.

Numerical Examples 1D - Proof of the Concept



Sharpening case of the mean-field limit. The solid line represents the solution f, the dashed line the data w. The upper left panel shows the initial condition, the lower right panel is the steady state.

Conclusions

- Halftoning as minimization problem of non-convex functionals
- Relations between minimizers of

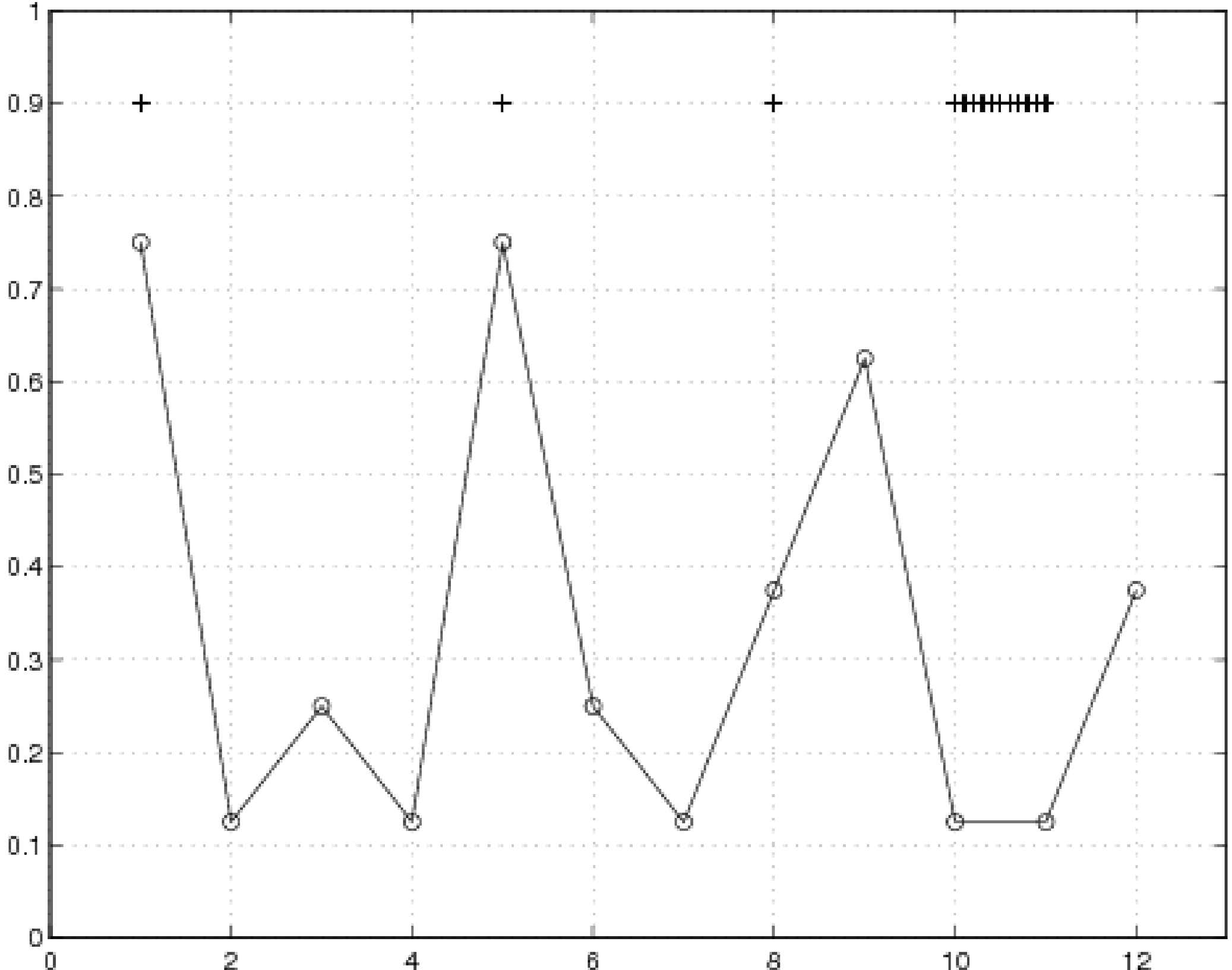
$$\boxed{\text{Quadrature Errors in RKHSs}}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\boxed{\text{Attraction-Repulsion}} \leftrightarrow \boxed{\text{Discrepancies}} \leftrightarrow \boxed{\text{LSQ}}$$

- Efficient numerical algorithms
  - DC-algorithms in conjunction with fast summation
  - nonlinear CG algorithm on on $\mathbb{T}^2$, $\mathbb{S}^2$ in conjunction with NFFT, NFSFT
- Mean Field Kinetic Equation
  - kinetic mean field limit of the discrete system of interacting particles $\longrightarrow$ kinetic mean-field equation
  - in the long time limit the solution tends to an equilibrium given by a local minimum of the system energy
  - in a very special case the equilibrium is unique and identical to the prescribed image profile

THANK YOU FOR YOUR ATTENTION!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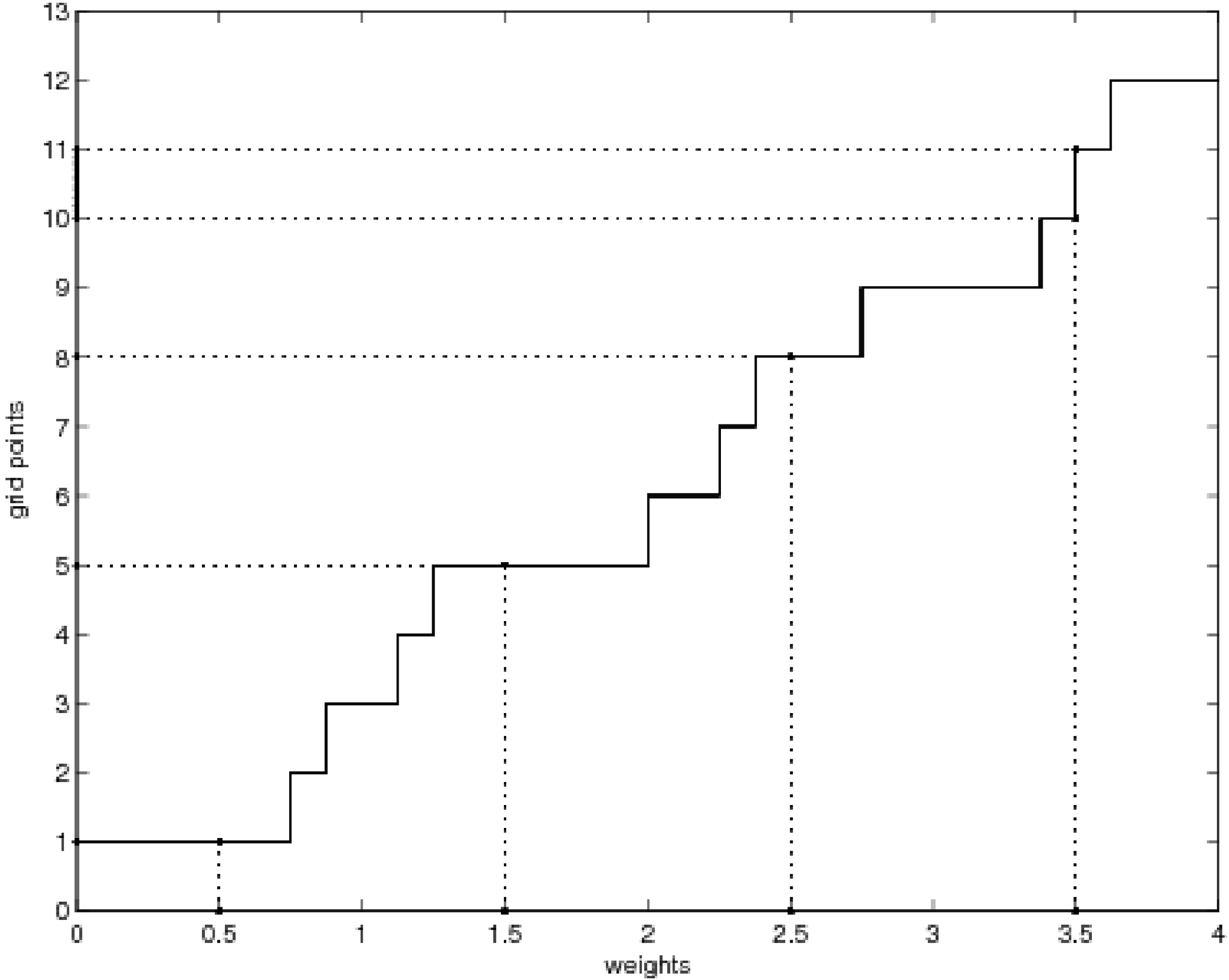
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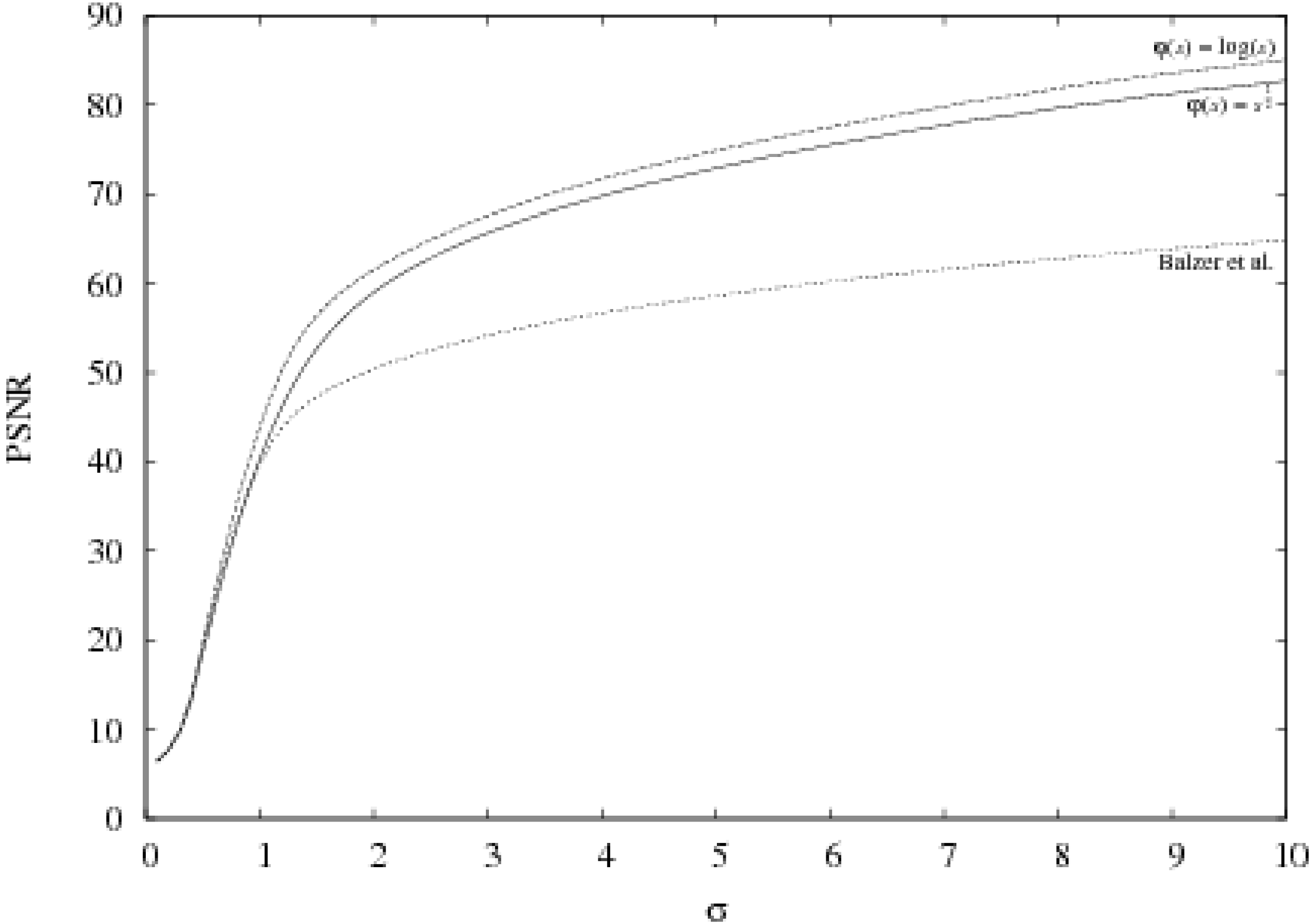
PSNR
σ
φ(x) = log(x)
φ(x) = x²
Balzer et al.
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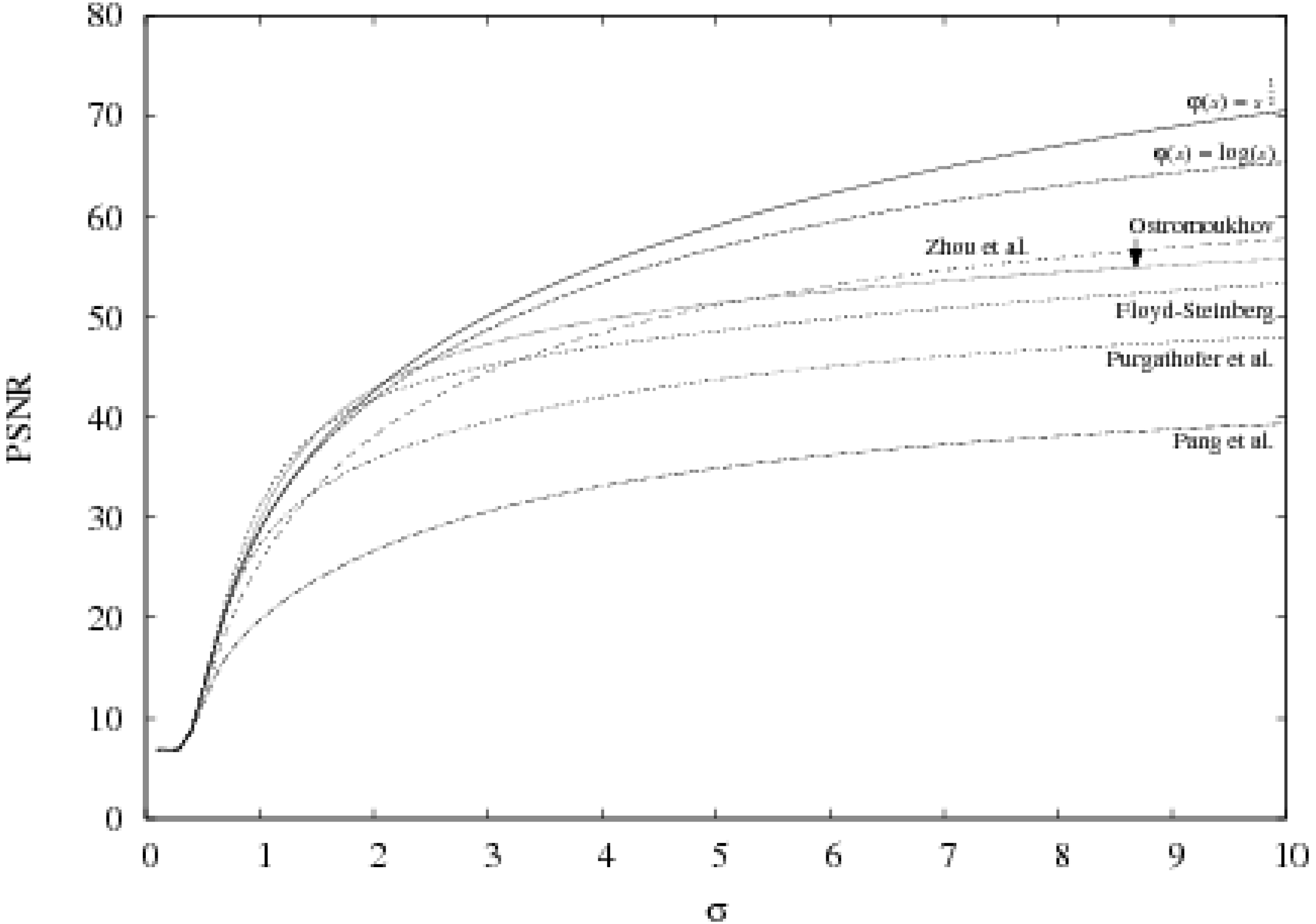
PSNR
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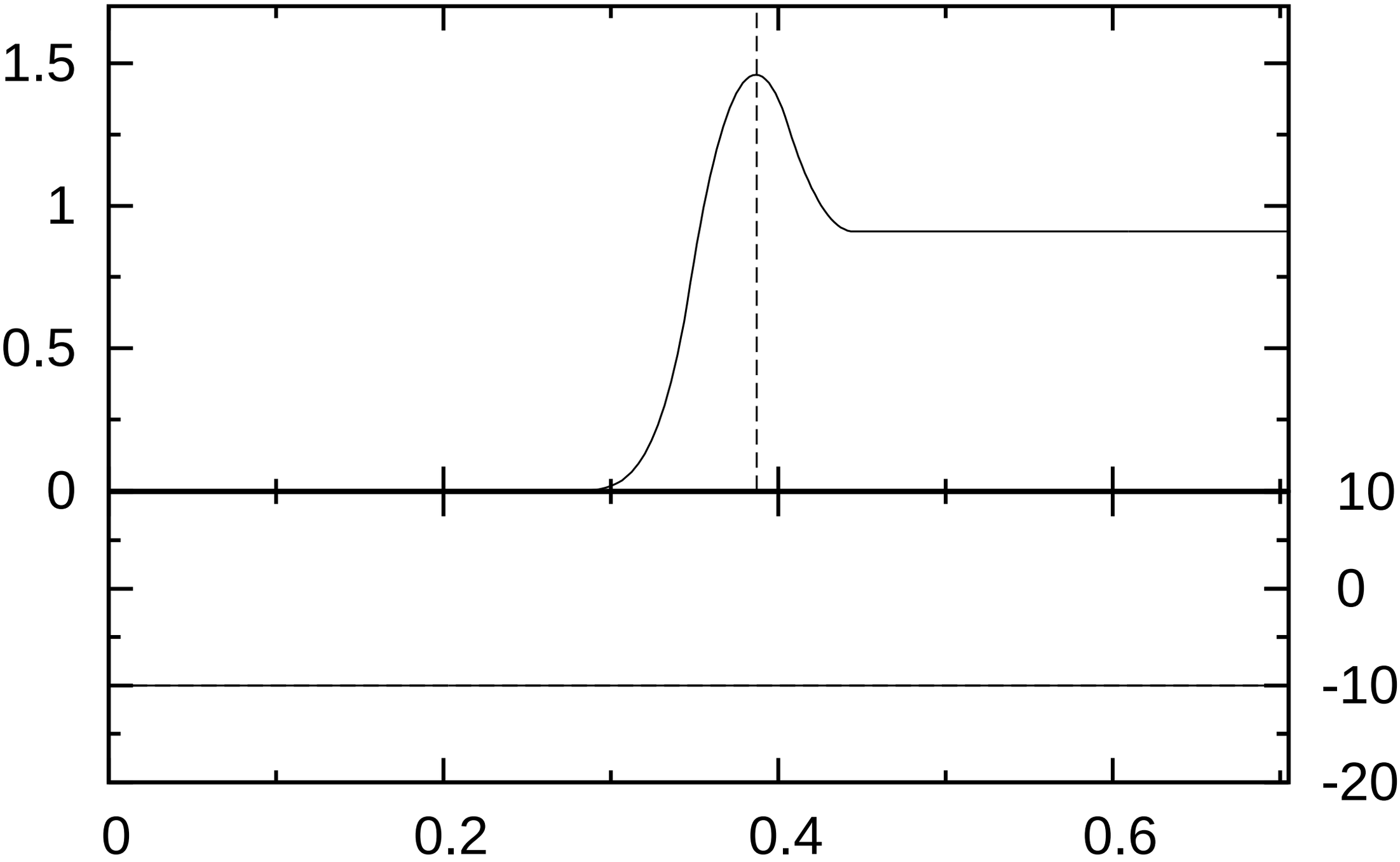
φ(x) = x
φ(x) = log(x)
Ostromoukhov
Zhou et al.
Floyd-Steinberg
Purgathofer et al.
Pang et al.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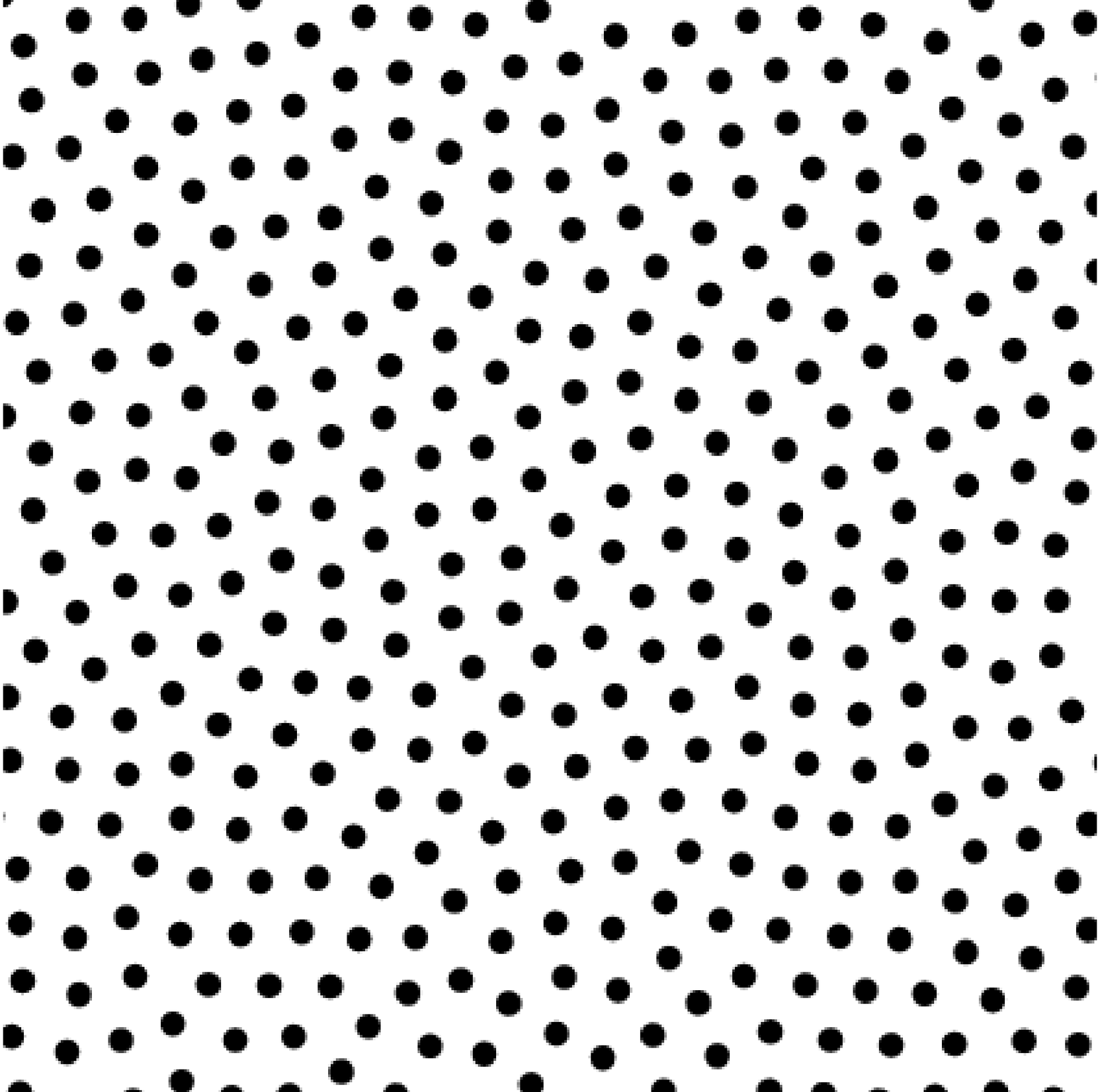

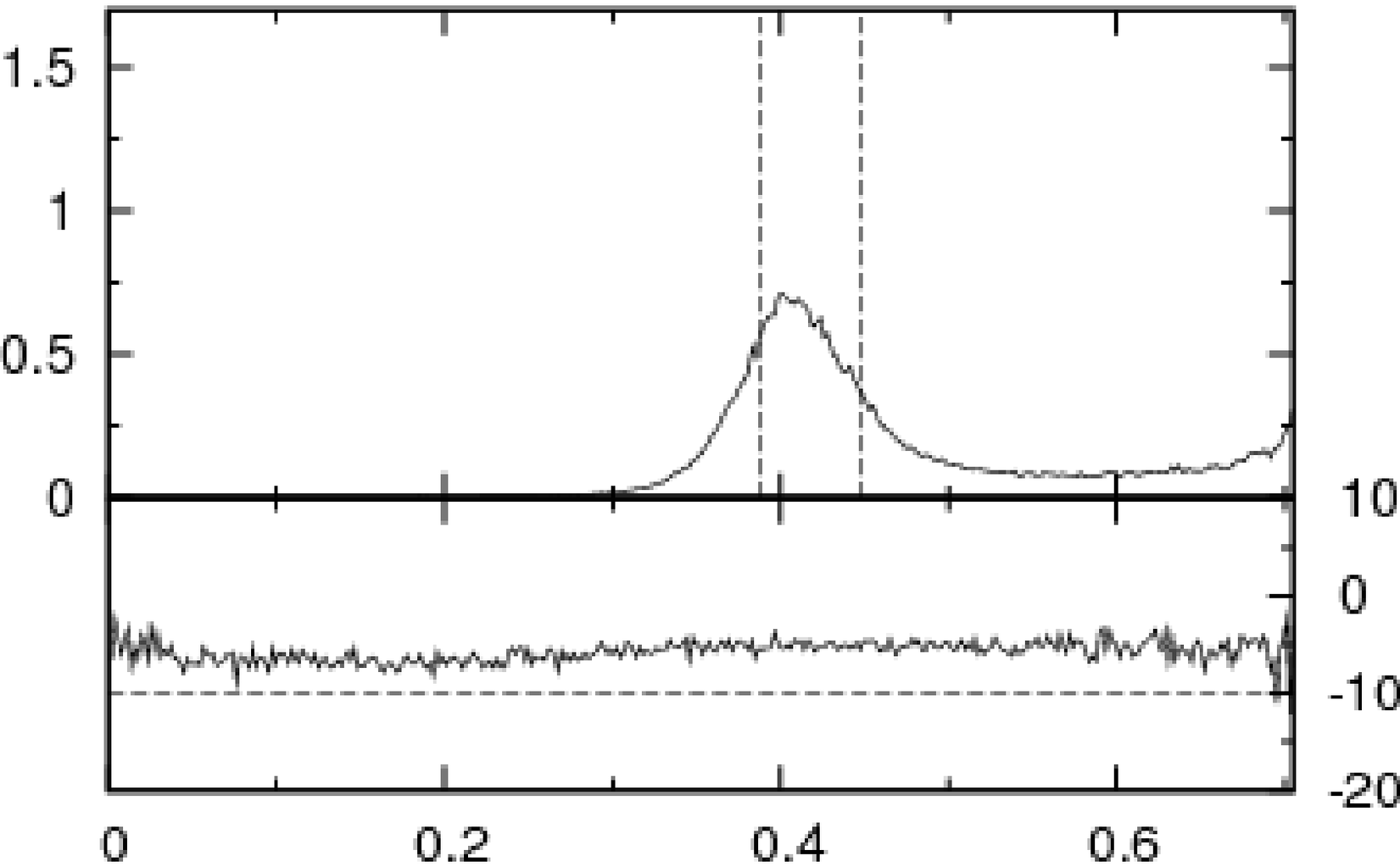
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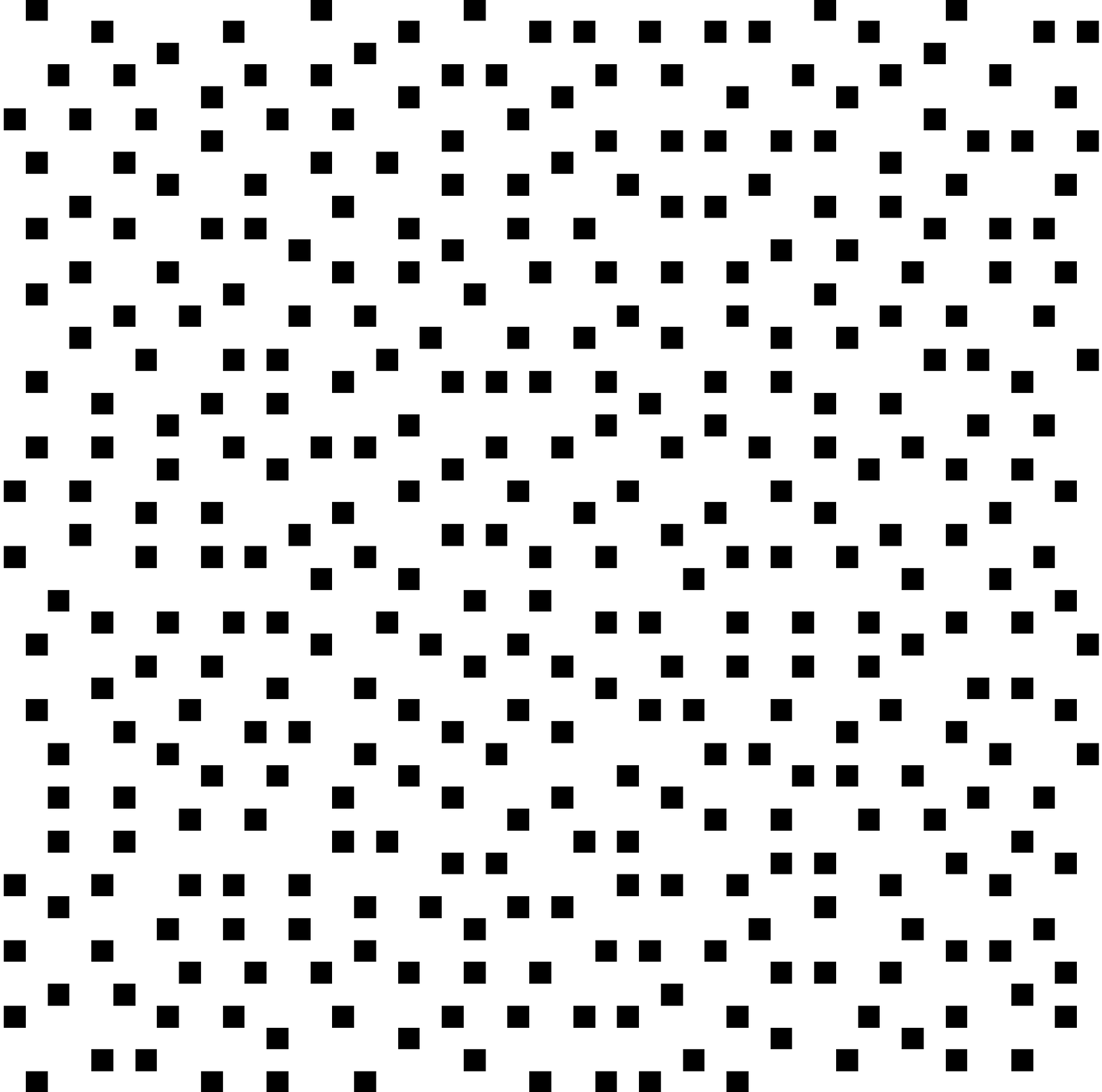

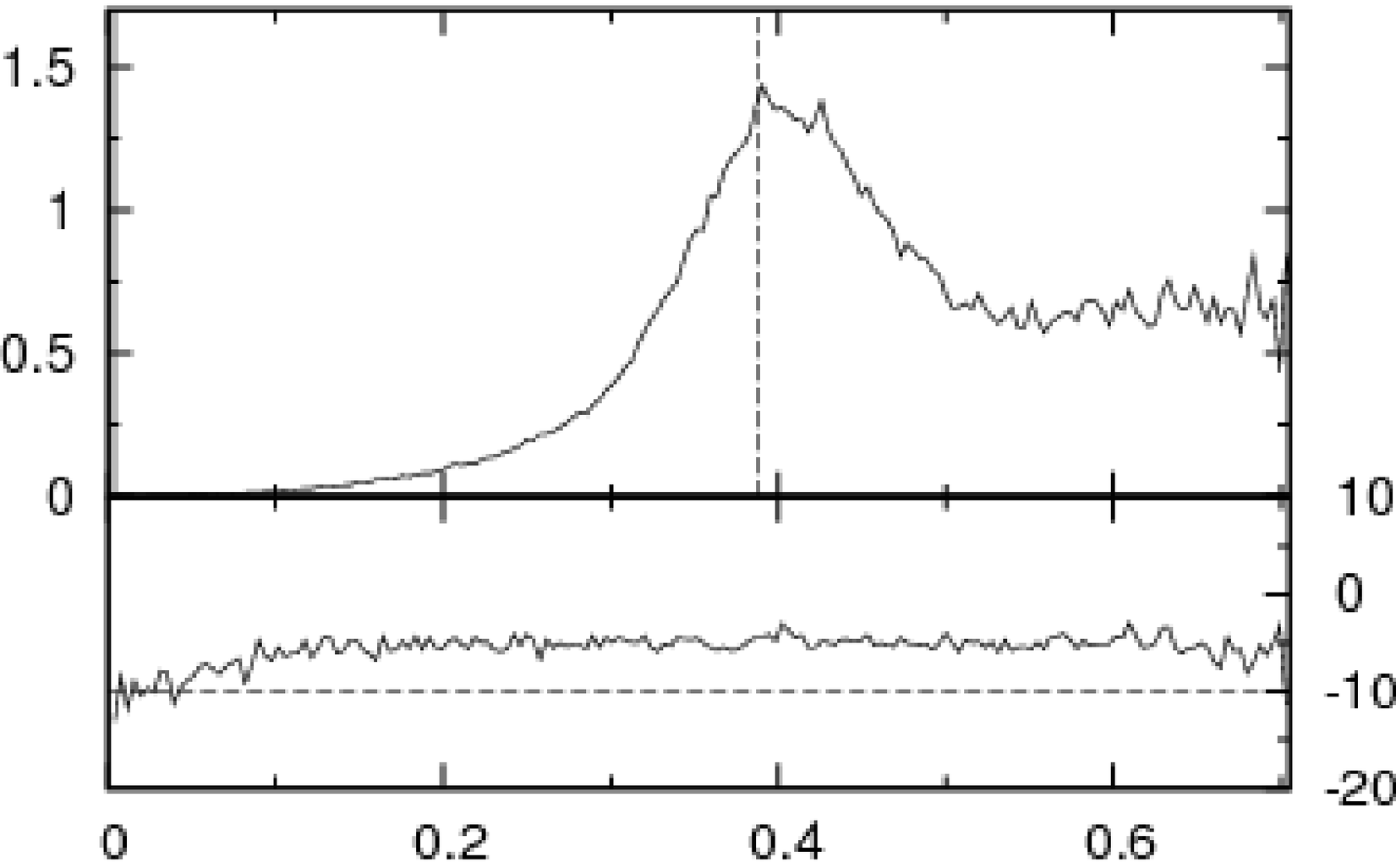
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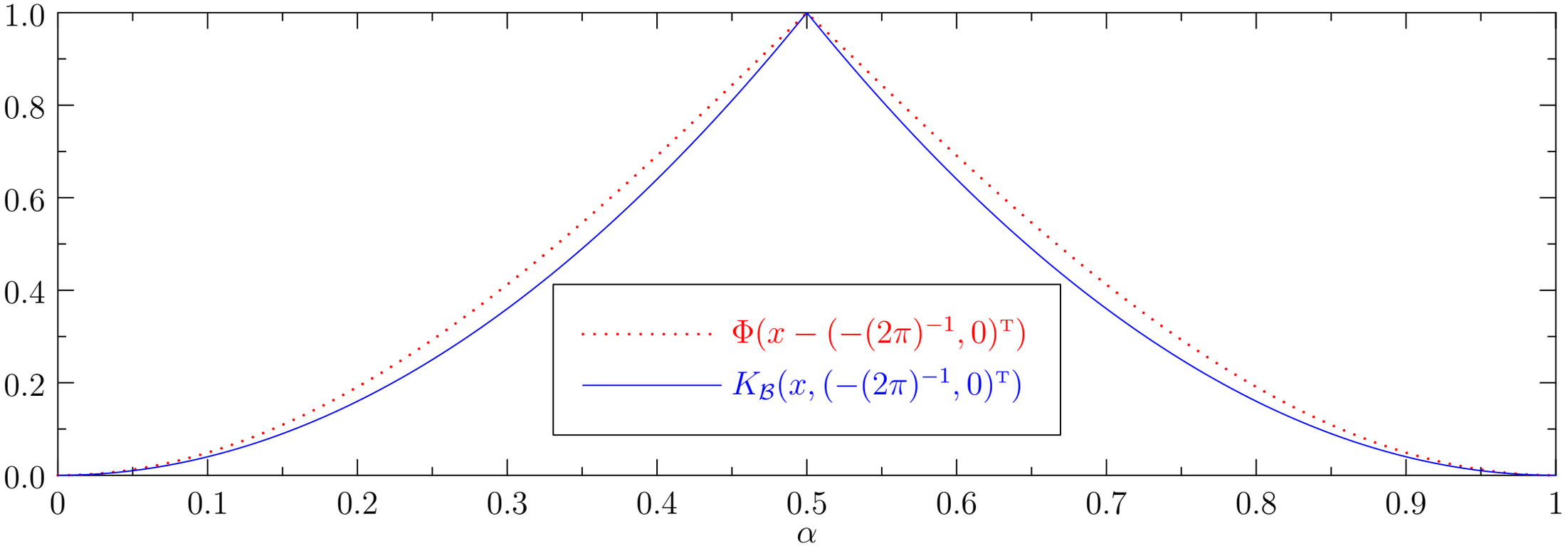
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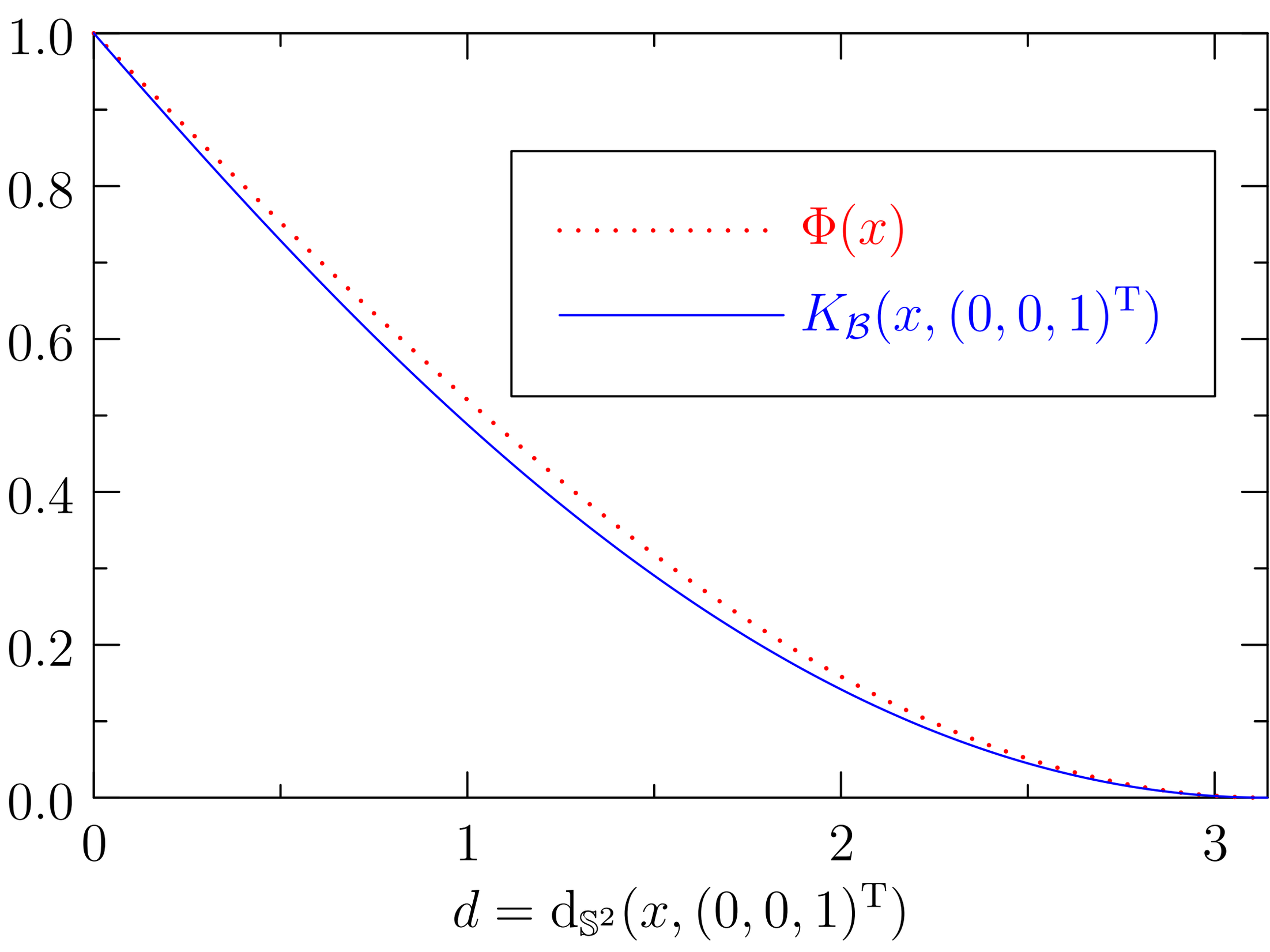
α
$\Phi(x - (-(2\pi)^{-1}, 0)^{\mathrm{T}})$
$K_{\mathcal{B}}(x, (-(2\pi)^{-1}, 0)^{\mathrm{T}})$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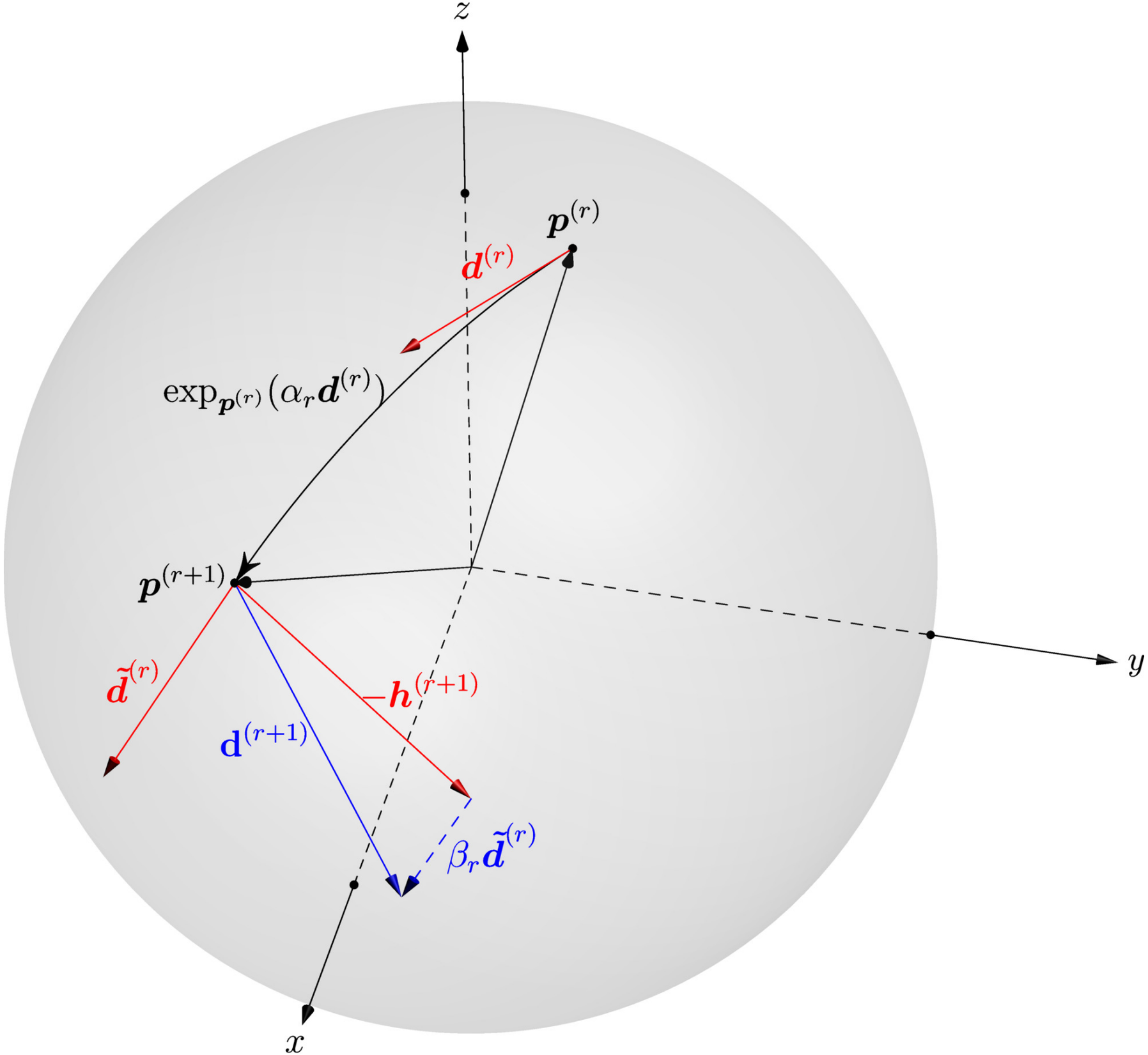
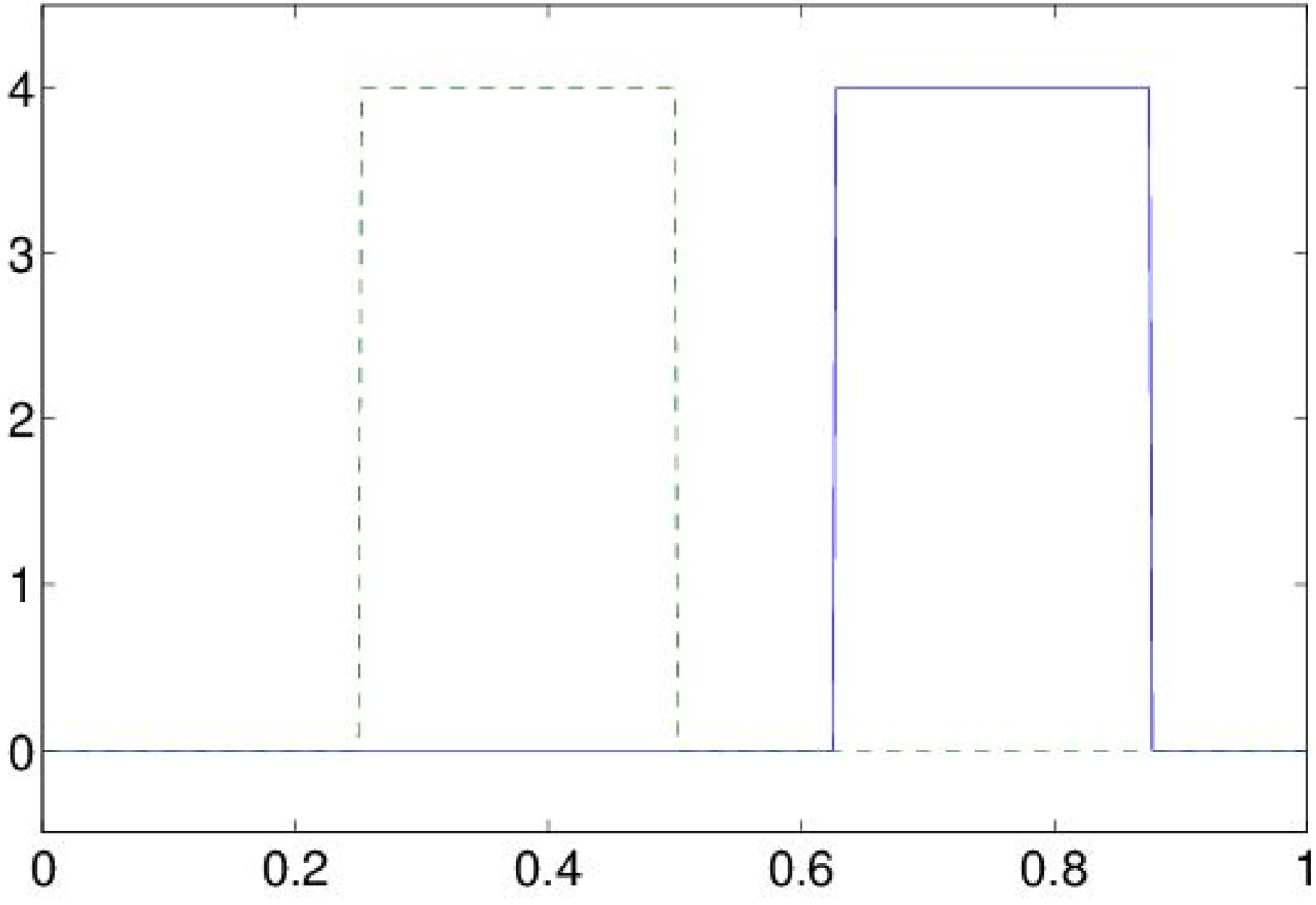
$\Phi(x)$
$K_{\mathcal{B}}(x, (0,0,1)^{\mathrm{T}})$
$d = \mathrm{d}_{\mathbb{S}^2}(x, (0,0,1)^{\mathrm{T}})$


z
$\boldsymbol{p}^{(r)}$
$\boldsymbol{d}^{(r)}$
$\exp_{\boldsymbol{p}^{(r)}}(\alpha_r \boldsymbol{d}^{(r)})$
$\boldsymbol{p}^{(r+1)}$
y
$\tilde{\boldsymbol{d}}^{(r)}$
$-\boldsymbol{h}^{(r+1)}$
$\mathbf{d}^{(r+1)}$
$\beta_r \tilde{\boldsymbol{d}}^{(r)}$
x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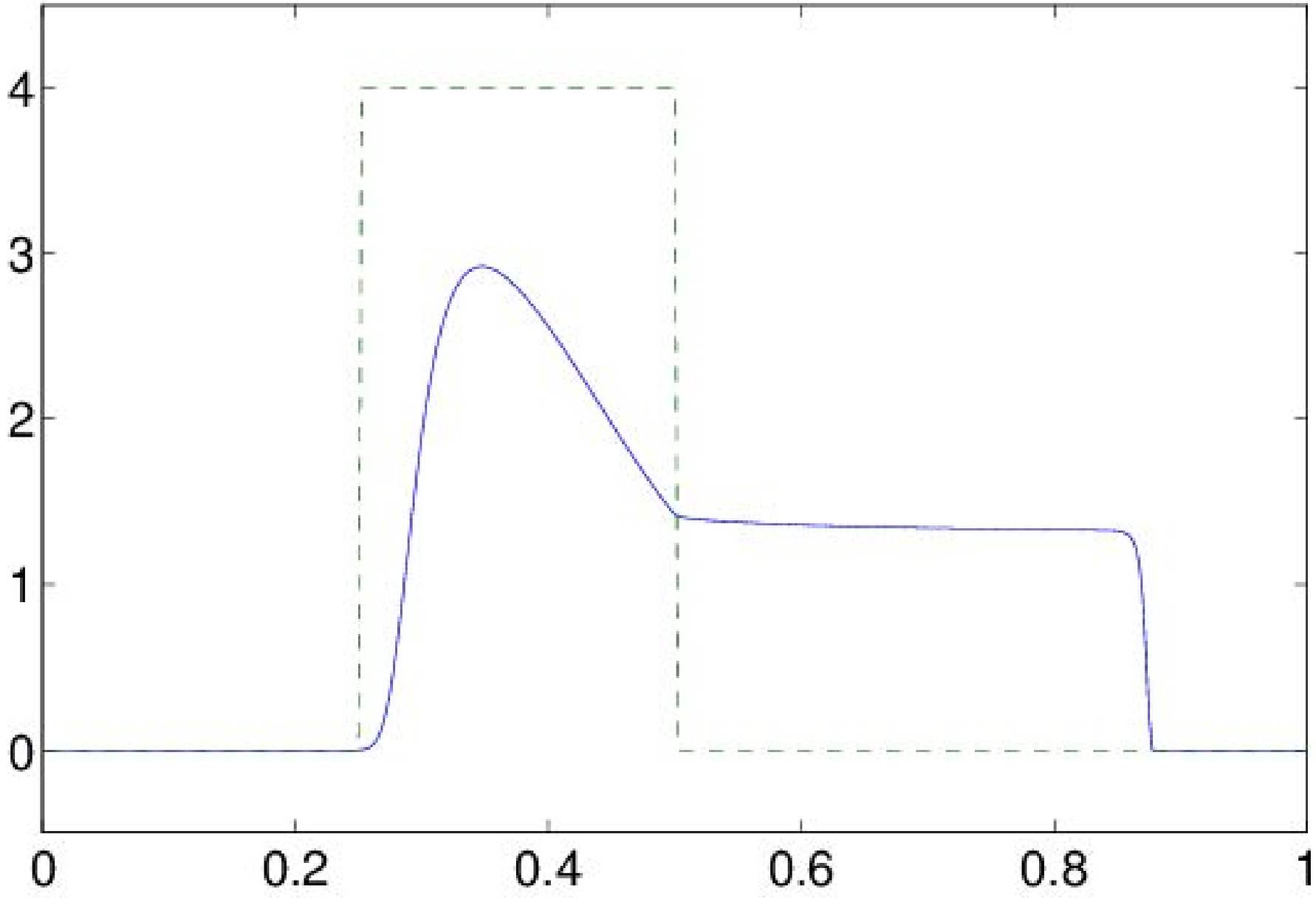
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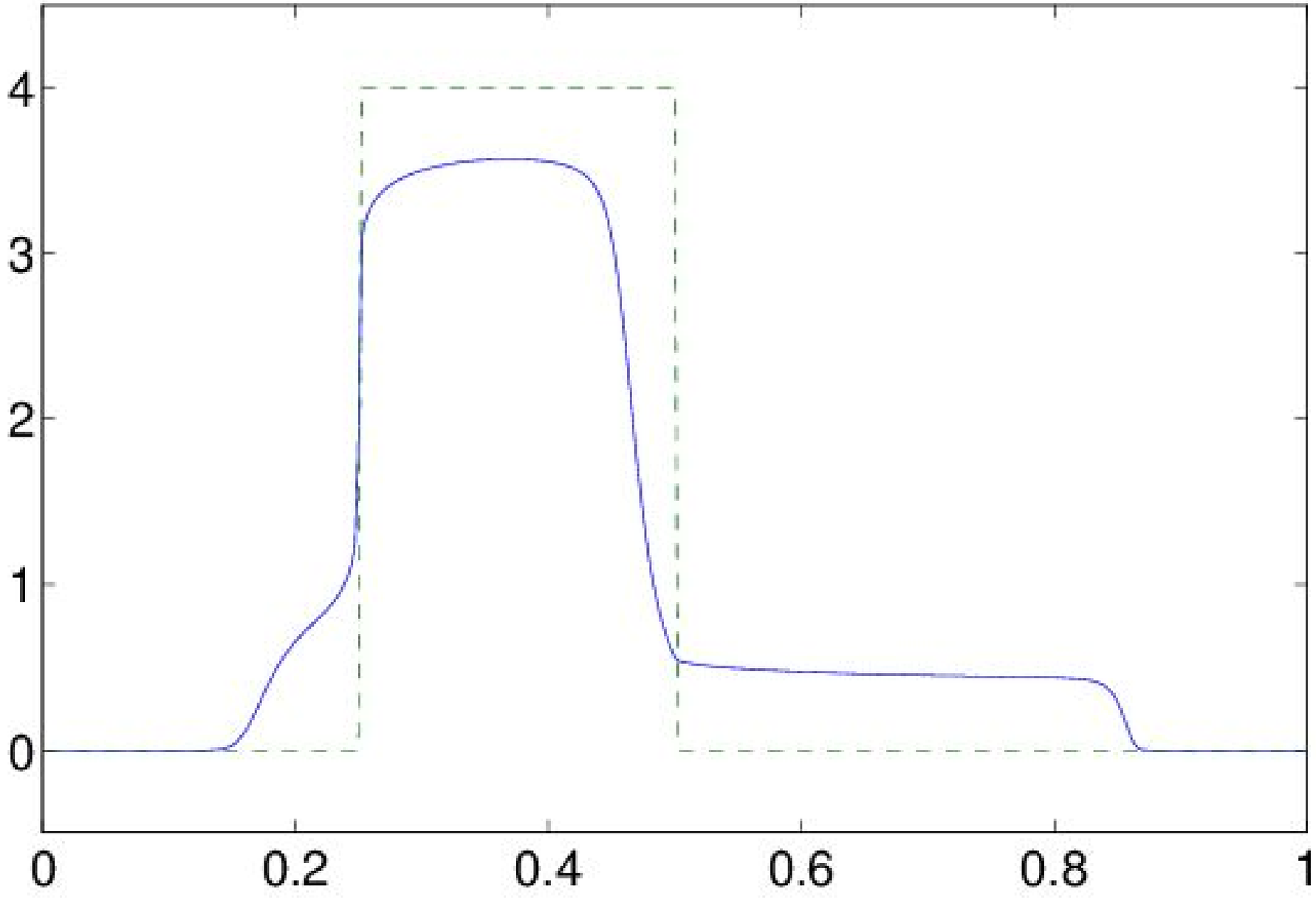
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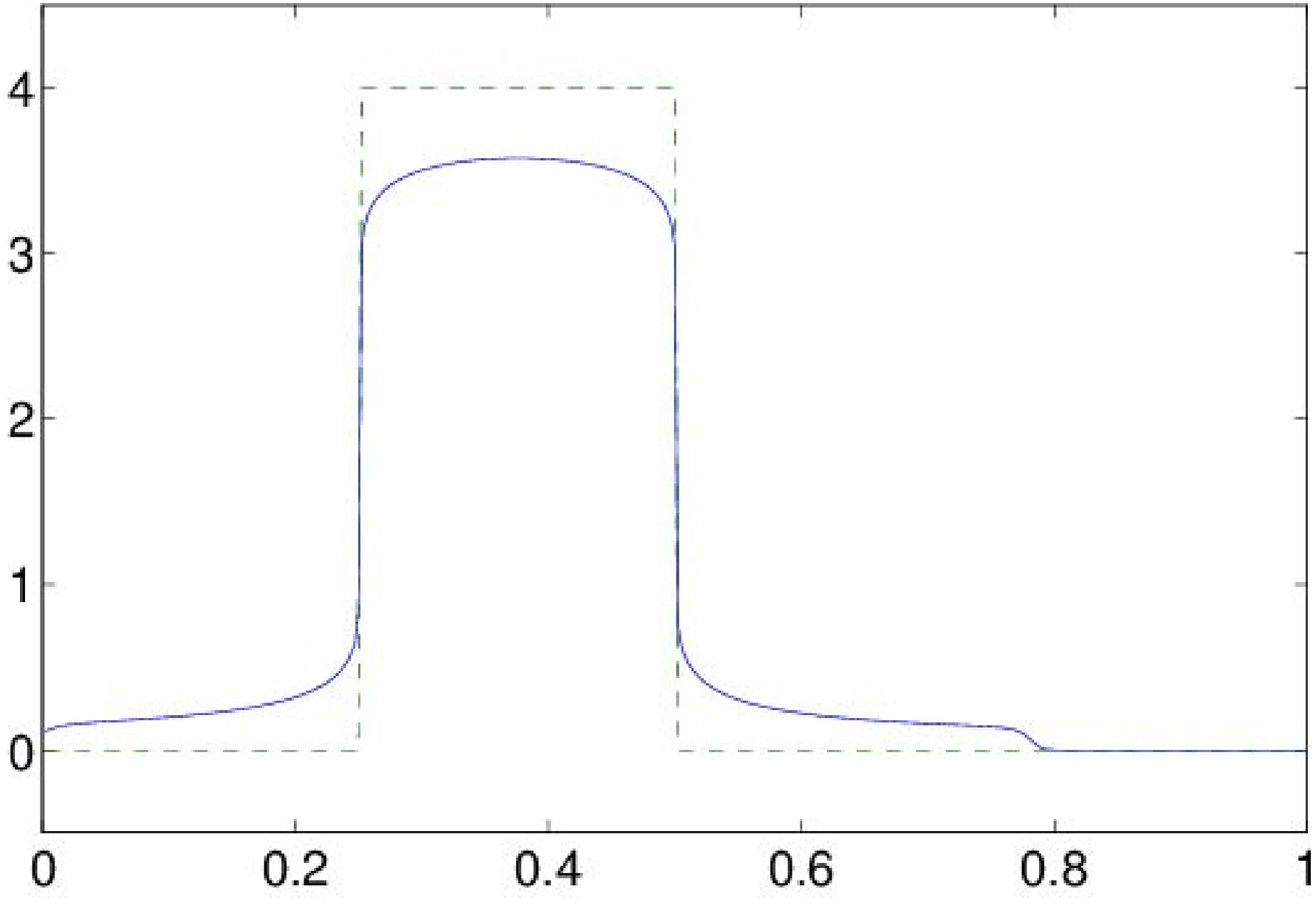
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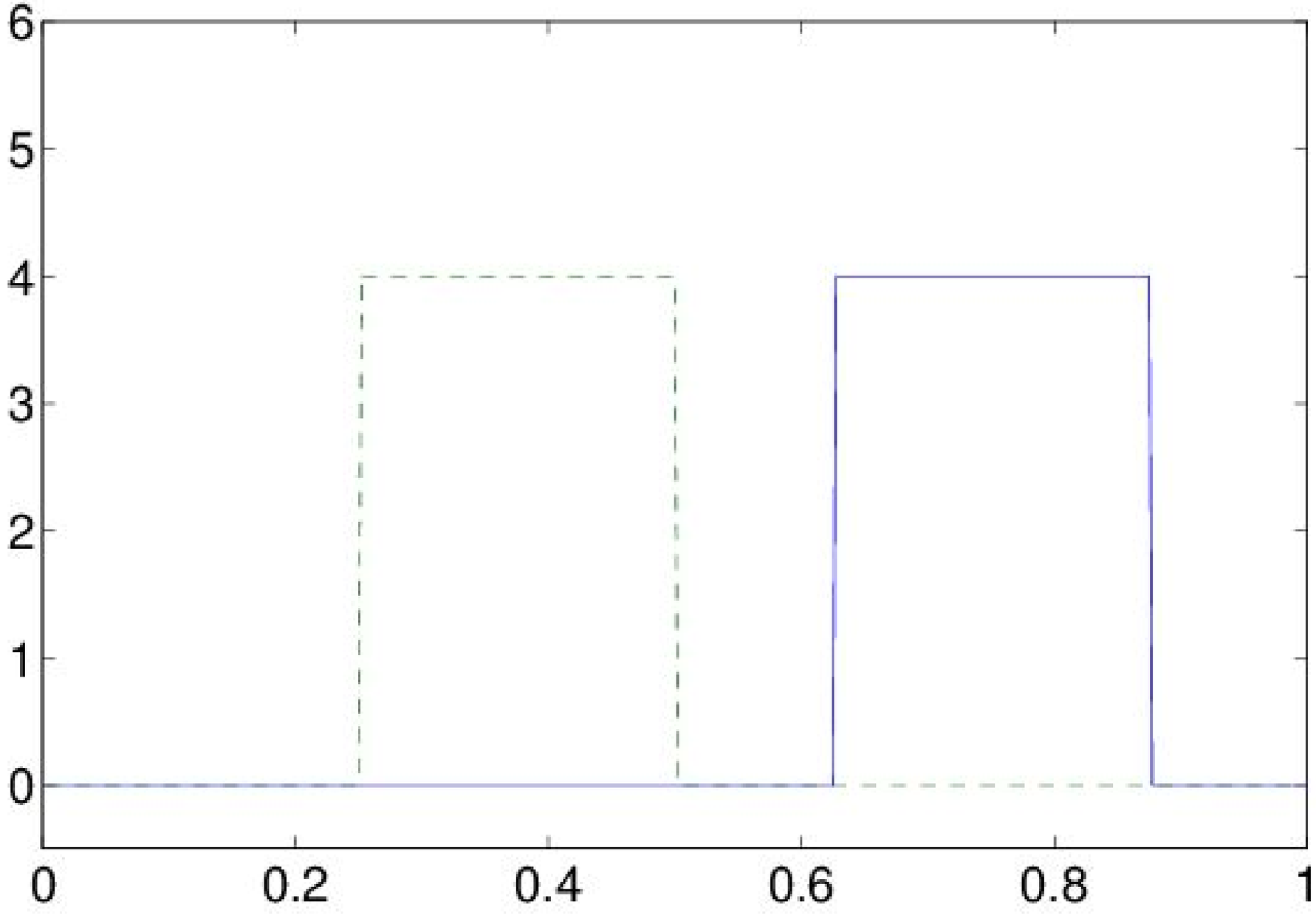
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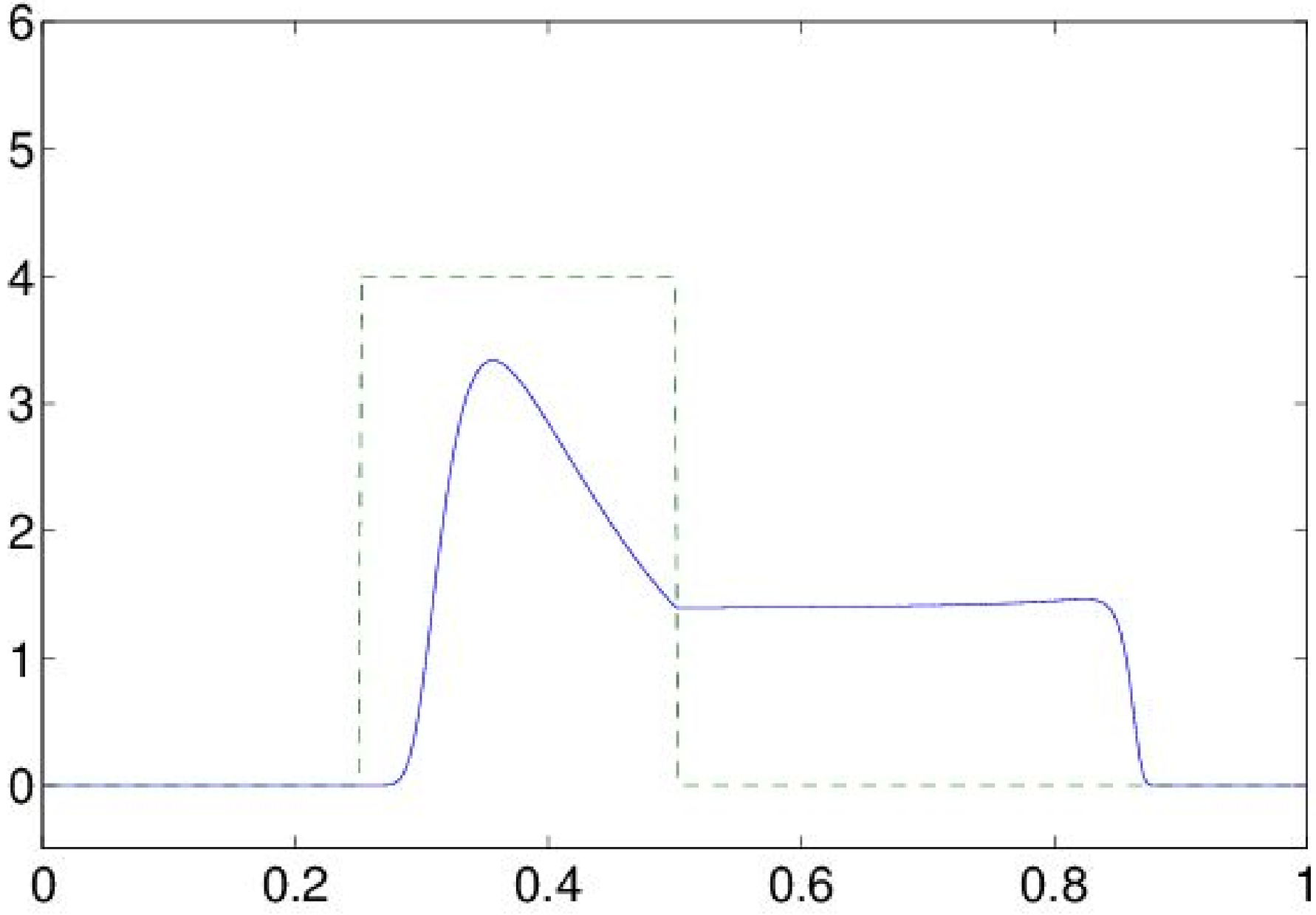
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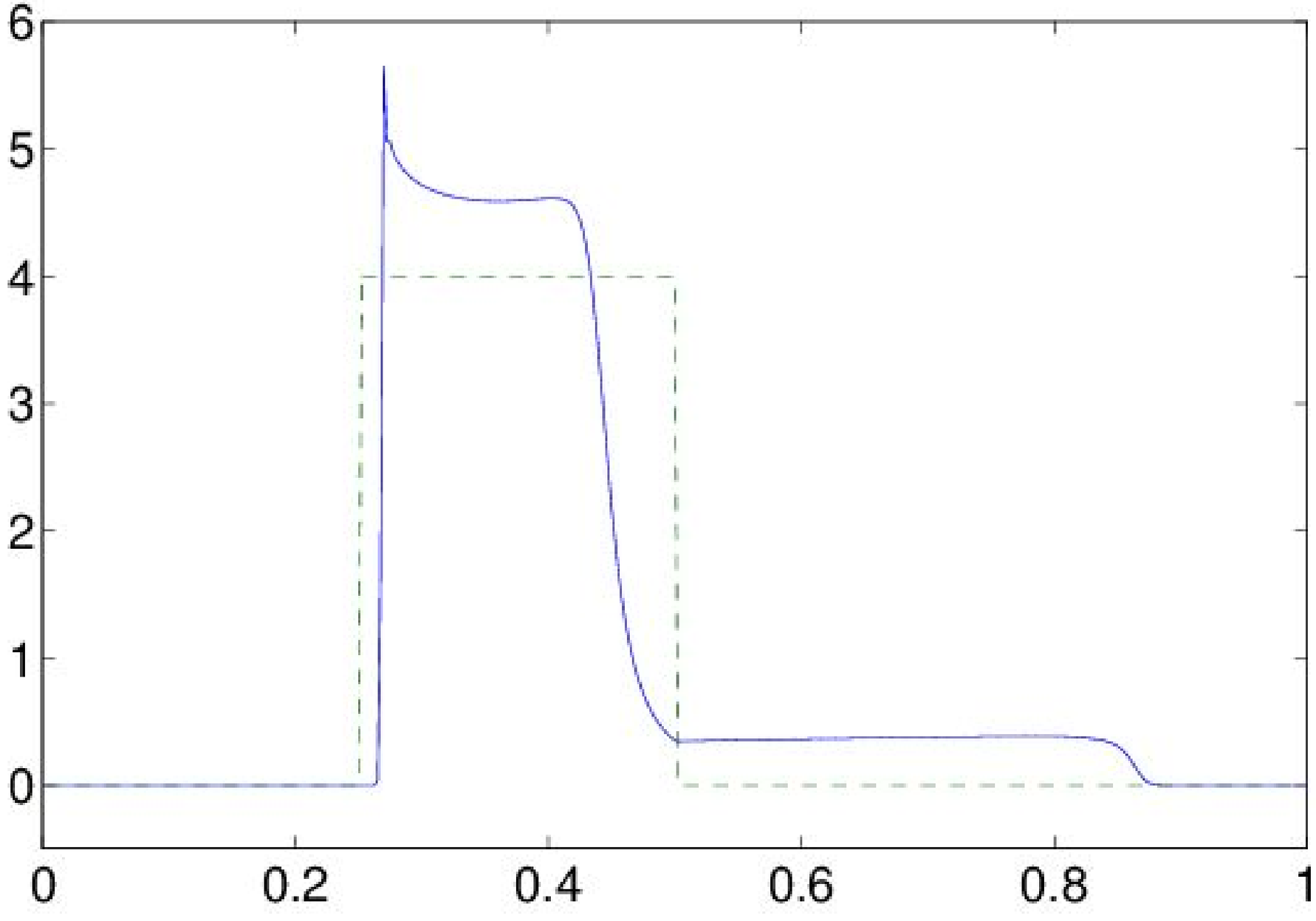
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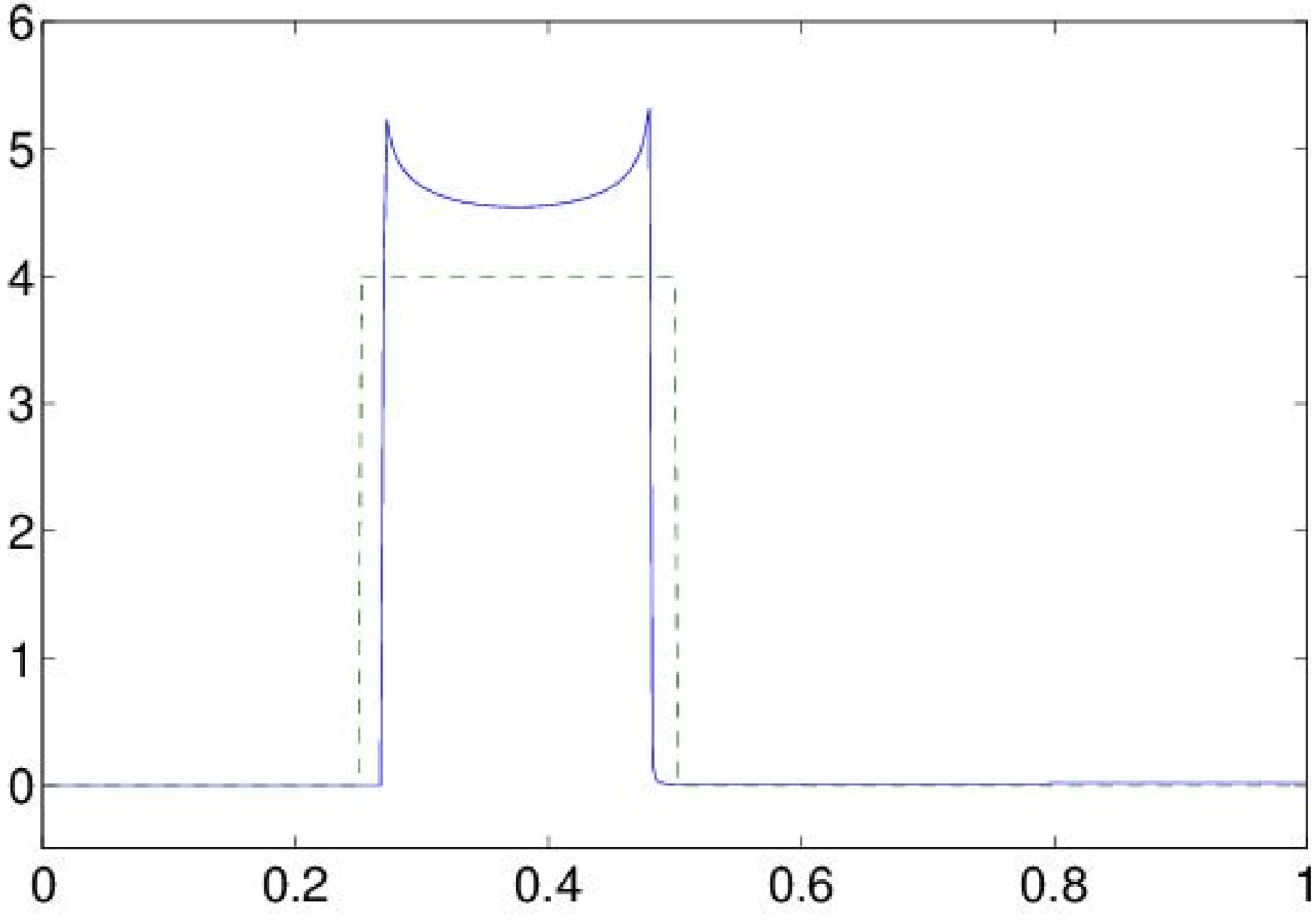
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