Graph-cut methods for joint reconstruction and segmentation

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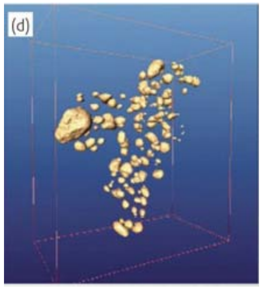
Outline

- Discrete amplitude inverse problems
- Use of graph cuts for discrete problems
- Graph-cuts and inverse problems
  - The inverse problem messes things up
- New ideas:
  - Idea 1: Separable approximation approach
  - Idea 2: Understand structure
    - Graph representable class of inverse problems
  - Idea 3: Exploit structure
    - Graph-cut based dual decomposition method for inverse problems
- Examples

Overview

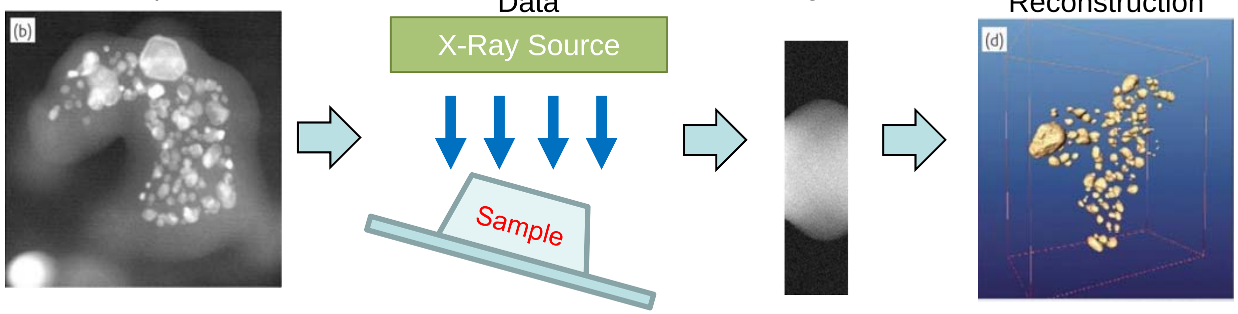


- Inverse problems:
  - Medical tomography – images from projections
  - Scanning electron tomography of materials – grains
  - Security scanning – imaging inside luggage
  - Deblurring, Super-resolution – fusion and deblurring

- Discrete-amplitude inverse problems
  - Medical tomography – bone, tissue, interventional instrument
  - Material science – grain volume fractions
  - Security – threat classification
  - Deblurring – segmentation

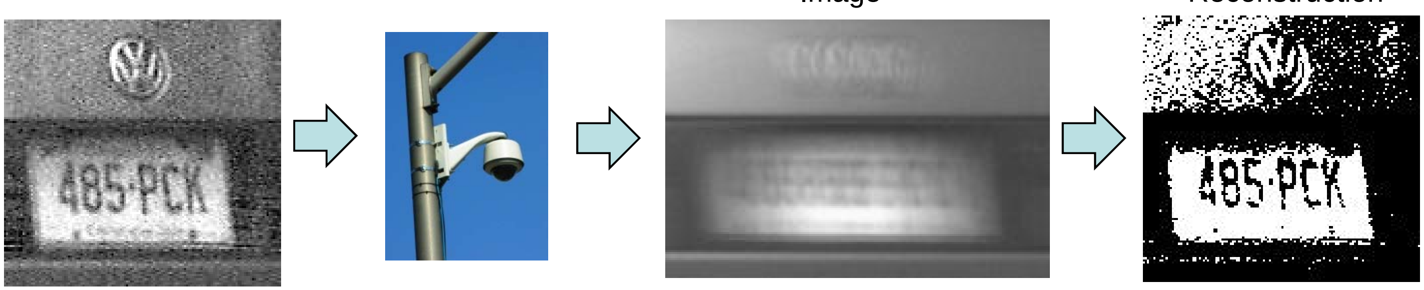
Examples

- Electron Tomography for Material Science:



Images from M. DeGraef, Tomography in Materials Science Workshop, Dayton, OH, Dec. 13-15, 2010

- Deblurring (coded aperture “flutter shutter”)



Images from Coded Exposure Photography, R. Raskar, A. Agrawal, J. Tumblin, ACM Siggraph 2006

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Common problem characteristics

- Indirect or “coded” observation y of latent variables x
- Few amplitudes are of interest: x discrete
- Scene structure is important
  - Focus on locations with specific values – where stuff is
- Data is limited, “information poor”
  - Ad hoc methods perform poorly

Problem Formulation

- Observations:

$$y = Cx + n$$

Data Vector | Imaging Operator | Discrete Amplitude Image | Noise

- Estimate discrete level image x by cost minimization:

$$\hat{x} = \underset{x \in \{\text{Disc Amp}\}}{\operatorname{argmin}} \; J(x) = \underset{x \in \{\text{Disc Amp}\}}{\operatorname{argmin}} \; J_{data}(x) + \lambda J_{prior}(x)$$

NP Hard!

  - Data Fidelity: $J_{data}(x) = \|y - Cx\|_2^2$
  - Prior (MRFs, TV, sparse, etc): $J_{prior}(x) = \|Dx\|_1^1$

Approach #1

- Two-step decoupled process
- Continuous relaxation
  1. Relax discrete assumption on x
  2. Solve resulting continuous inverse problem (e.g. with TV)
  3. Segment continuous reconstruction result
- Ad hoc, wastes information

$$\underset{x \in \{\text{Disc Amp}\}}{\operatorname{argmin}} \|y - Cx\|_2^2 + \lambda J_{prior}(x) \Rightarrow \underset{x \in \Re^n}{\operatorname{argmin}} \|y - Cx\|_2^2 + \lambda J_{prior}(x)$$

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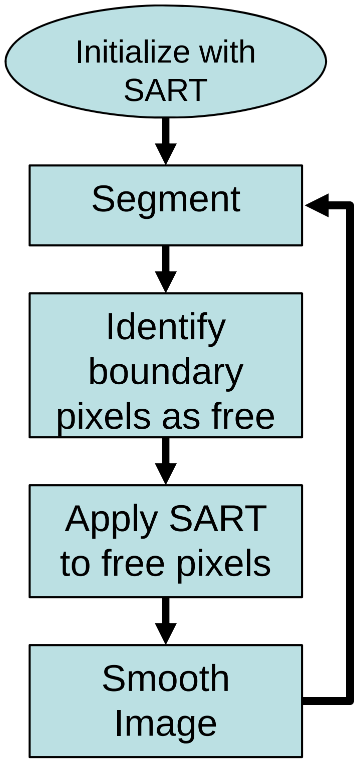
Approach #2: DART (Discrete ART)

- Algorithm for discrete tomography
- Idea: Interior regions are "reliable"
- DART Algorithm flow chart ➔
  - "Free pixels" are boundary + random pixels
- No explicit regularization
- Discretization decoupled from reconstruction
- *Very* popular in materials science
- Iterative variant of Approach #1



Approach #3: Graph-cut Methods

- Direct solution of underlying discrete problem
- All start by representing problem on a graph
- Powerful and efficient methods for these problems exist
- Some nice performance results ➔ **Global optimality**!
- Efficient algorithms
  - Min-cut/Max-flow
  - Polynominal time

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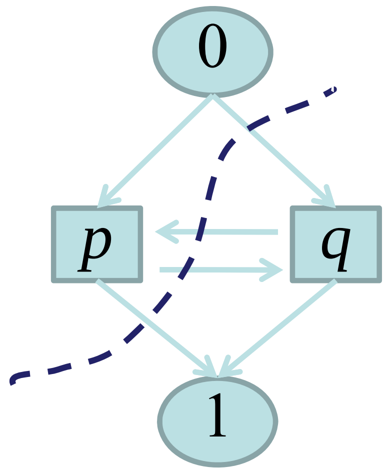
Graph Cuts when C=I, Binary Case

$$\operatorname*{argmin}_{x\in\{0,1\}} \|y-x\|_2^2 + \lambda J_{prior}(x) = \operatorname*{argmin}_{x\in\{0,1\}} \underbrace{\sum_p \theta_p(x_p)}_{\text{Unary Terms}} + \underbrace{\sum_{p,q} \theta_{p,q}(x_p, x_q)}_{\text{Pairwise Terms}}$$



- "Segmentation" case
- Vertices are pixel values
- Edges are costs
- Graph representation ←→ "**Submodularity**" of pairwise terms

$$\theta_{p,q}(0,0) + \theta_{p,q}(1,1) \le \theta_{p,q}(0,1) + \theta_{p,q}(1,0)$$

- Submodular when $C = I$ since pairwise terms arise from smoothness prior

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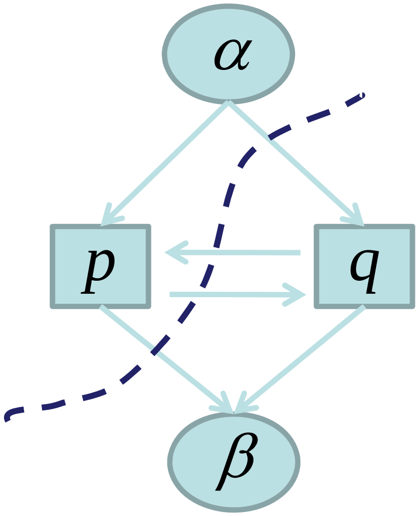
Graph Cuts when $C=I$, Multi-label case

$$\operatorname*{argmin}_{x \in L^n} \|y - x\|_2^2 + \lambda J_{prior}(x) = \operatorname*{argmin}_{x \in L^n} \underbrace{\sum_p \theta_p(x_p)}_{\text{Unary Terms}} + \underbrace{\sum_{p,q} \theta_{p,q}(x_p, x_q)}_{\text{Pairwise Terms}}$$



- Pixel values from set L^n
- Can't exploit binary graph-cut machinery
  - Multi-label extensions exist, but clumsy
- Typically break problem into sequence of binary problems
  - α-β swap – iteratively minimize wrt pairs of labels α,β
  - α expansion – iteratively minimize all labels wrt label α
- Only obtain approximate solutions
  - but good behavior in practice
- Sub-problems still need to be submodular!
  - Still true when $C = I$. Things work well in this case

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Graph Cuts when $C \neq I$ (Inversion)

$$\underset{x \in \{\text{Disc Amp}\}}{\operatorname{argmin}} \|y - Cx\|_2^2 + \lambda J_{prior}(x) = \underset{x \in \{\text{Disc Amp}\}}{\operatorname{argmin}} \underbrace{\sum_p \theta_p(x_p)}_{\text{Unary Terms}} + \underbrace{\sum_{p,q} \theta_{p,q}(x_p, x_q)}_{\text{Pairwise Terms}}$$

- Overall cost is **not** submodular (graph representable)
  ➔ Conventional graph-cut approaches do not work
- Problem is with data term:

$$J_{data}(x) = \|y - Cx\|_2^2 = y^T y - 2y^T Cx + x^T C^T Cx$$

Contains general cross terms involving $x_p x_q$

- [Raj `05] Coupling of pixels due to non-diagonal C makes J_{data} non-graph-representable

Related work on graph-cuts, inversion

- Graph Cuts
  - [Raj `05] : Deconvolution
    - Approximate non-submodular cross-terms in J_{data}
    - Replace non-submodular cross-terms $C_{pq}x_p x_q$ with linear terms $C_{pq}\,[x_p^{(0)}x_q + x_p x_q^{(0)}]$
    - Method is for α-β swap algorithm
    - Assumes non-negative C
  - [Raj `06, Rother `07] : MRI Reconstruction
    - Use of “roof-duality” to remap some non-submodular terms (QPBO)
    - Cannot always be done
    - Can leave pixels unlabeled
  - [Cremers 06] : Learned submodular models
- Inverse Problems
  - [Wright , Nowak , Figueiredo `09] : Sparse reconstruction
    - Continuous amplitude problems (not focused on graph cuts)
    - Separable approximation of data term
- Discrete Tomography
  - [Batenburg, `11]: DART
    - Ad hoc iterative thresholding method

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Idea #1: Use separable approximation of J_{data}

- Relax discrete assumption on x
- Perform Taylor expansion of $J_{data}(x)$ around x_0 (Wright `09)
- Approximate Hessian by scaled diagonal
  - **Separable approximation → eliminates cross-pixel coupling**

$$J_{data}(x) = \|y - Cx\|_2^2 = y^T y - 2y^T Cx + x^T C^T Cx$$

$$\hat{J}_{data}(x \mid x_0, \gamma) = J_{data}(x_0) + \nabla J_{data}(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T [\gamma D](x - x_0)$$

$$\nabla J_{data}(x_0) = -2C^T y + 2C^T C x_0$$
$$D = \mathrm{diag}(2C^T C) = \mathrm{diag}(\nabla^2 J_{data})$$
$$\gamma = \text{Approximation Parameter}$$

Diagonal of Hessian
diag[C^TC]

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Overall new approach

- Replace original cost with surrogate cost

$$\boxed{J(x) = J_{data}(x) + \lambda J_{prior}(x)}$$

Not Graph Representable

$$\Downarrow$$

$$\boxed{\hat{J}(x) = \hat{J}_{data}(x \mid x_0, \gamma) + \lambda J_{prior}(x)}$$

Graph Representable

- **<u>Surrogate cost is graph representable</u>** (submodular)
- Any multi-label graph-cut method can now minimize
  - ➔ α-expansion in this work
- Iteratively minimize surrogate cost

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Graph Cuts w/Separable Approximation (GCWSA)

GCWSA Algorithm:

1. Initialize: $x^{(0)}$
2. Use graph-cut algorithm to minimize surrogate cost

$$\boxed{x^{(k+1)} = \underset{x \in \{\text{Disc Amp}\}}{\operatorname{argmin}} \; \hat{J}_{data}(x \mid x^{(k)}, \gamma_k) + \lambda J_{prior}(x)}$$

3. γ_k Update (Acceptance test):
   a. If $J(x^{(k+1)}) > J(x^{(k)})$ increase γ_k and redo 2. (forces smaller step size)
   b. If $J(x^{(k+1)}) < J(x^{(k)})$ accept updated estimate, set $k \rightarrow k+1$ and goto 2.

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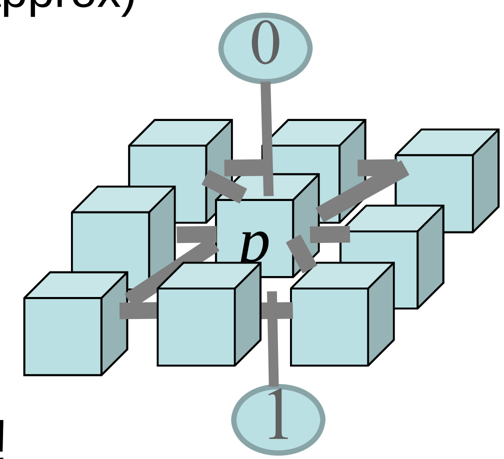
Efficient computation for relaxation parameter γ_k

$$\hat{J}_{data}(x \mid x_0, \gamma) = J_{data}(x_0) + \nabla J_{data}(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T [\gamma D](x - x_0)$$

Diagonal of $C^T C$ scaled by γ

- Good choice of γ improves solution
  - Repeated computation for different γ
- Observation:
  - Changing γ_k only affects unary terms (via diagonal approx)
  - Pairwise terms remain unchanged (many of these)
- Flow Recycling [Kohli, 2005]
  - Reuse min-flow results
  - Modify previous residual graph
  - Most terms remain unchanged
- Order of magnitude faster than naïve way!

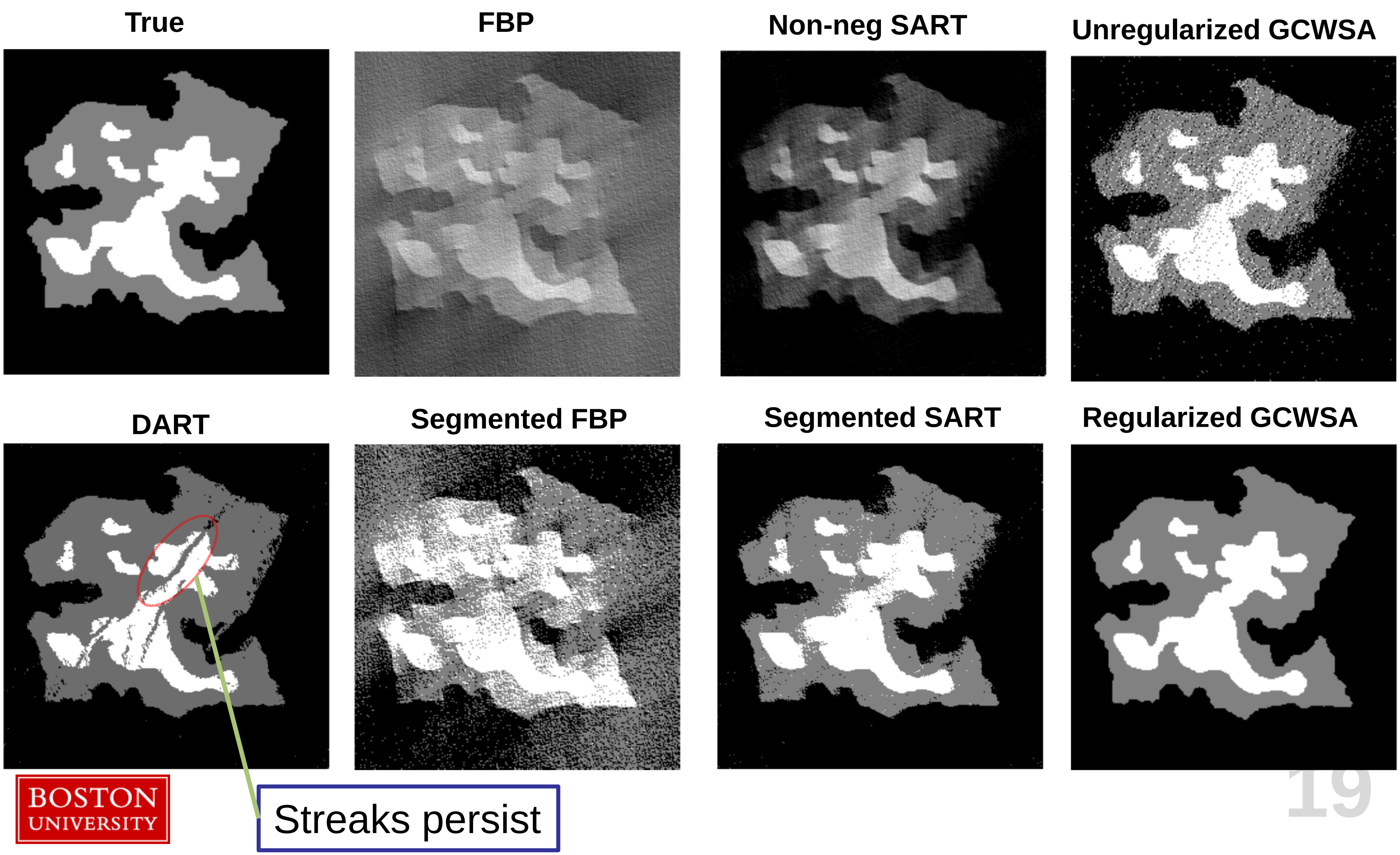


Experiment: Noisy Multi-level

Ground truth image

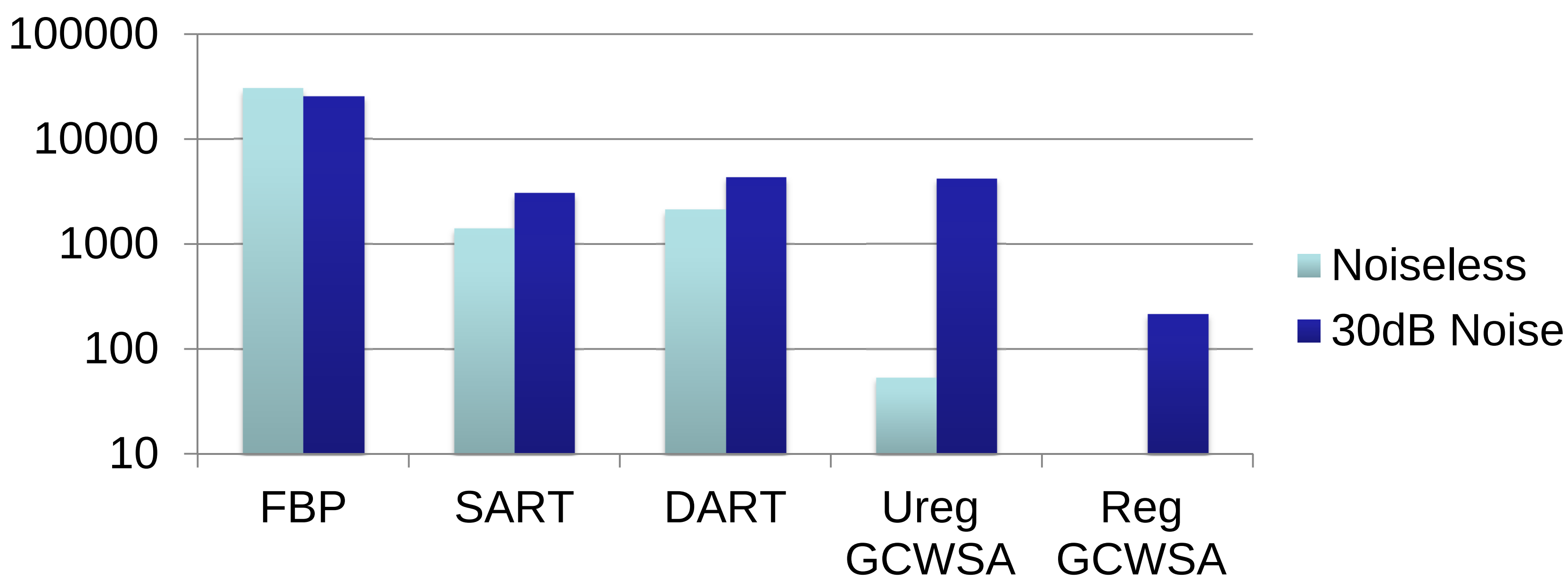
- 256 x 256 Image
- Known Intensity Levels {0, 5, 10}.
- Projections over 120 degree angular span
- 1 degree separation between projections
- 362 detector values per angle
- 30 dB AWGN

Experiment: Noisy Multi-level



Reconstruction Errors of Different Methods

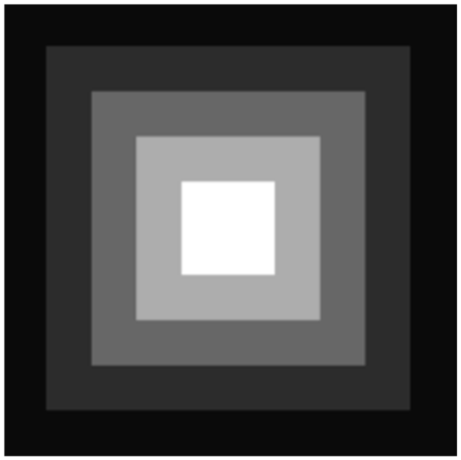
- Mislabeled Pixels: Multi-label Phantom



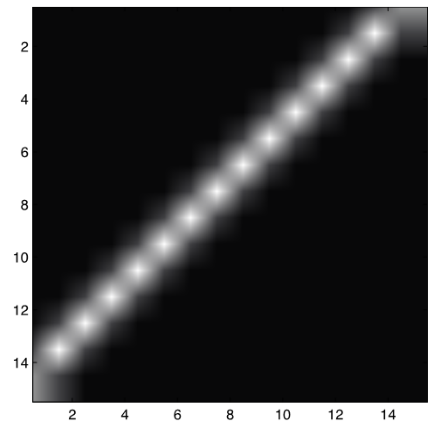
Log scale!

Experiment: Multi-level Deblurring

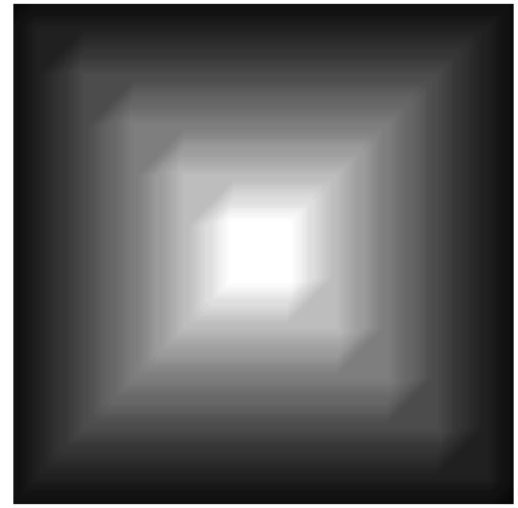
Ground truth image



20 pixel motion blur at 45 deg.

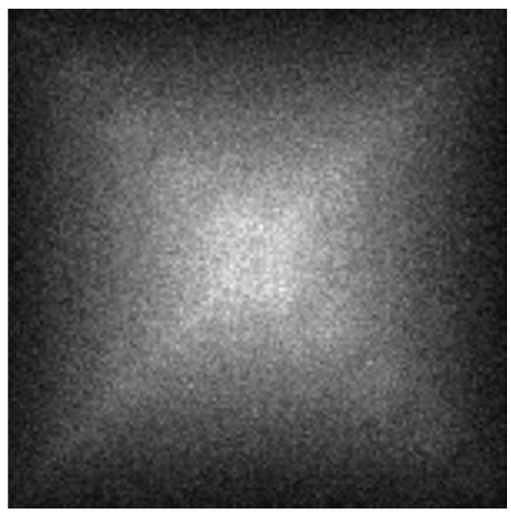


Blurred image



- 150X150 synthetic image
- Pixel values, $L = \{1, 2, 3, 4, 5\}$
- Linear motion blur, length 20 @ 45^{o}
- AWGN for 10 dB SNR.
- J_{prior} : Total-variation

Noisy blurred image



Joint Inversion and Labeling

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Summary so far…

- **GCWSA:**
  - New approximation-based method for graph-cut solution of discrete linear inverse problems
  - Based on separable approximation of data term
- **Question:**
  - Can structure be exploited?
  - Are there special classes of operators C which lead to submodular problems?
    - No approximation is necessary in this case!
- Answer: Yes!... next

Variable flipping/re-labeling

- Submodularity condition (binary case):

$$\theta_{p,q}(0,0)+\theta_{p,q}(1,1)\le\theta_{p,q}(0,1)+\theta_{p,q}(1,0)$$

- Idea of variable re-labeling/flipping (Hammer 1984, Kolmogorov 2007):

$p \backslash q$	0	1
0	10	0
1	0	10

Non-submodular term

re-label q $\longrightarrow$ $\bar{q}=1-q$

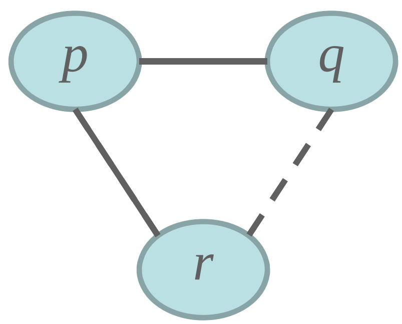
$p \backslash \bar{q}$	0	1
0	0	10
1	10	0

Submodular term

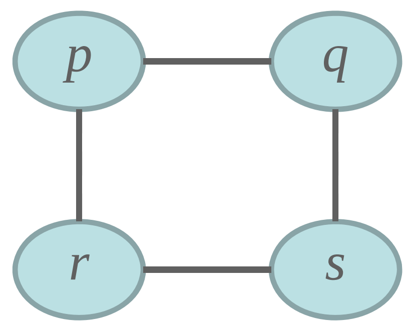
- In which graphs can you get all *submodular* interactions through relabelling?

Non-submodular - - - -

Submodular ———

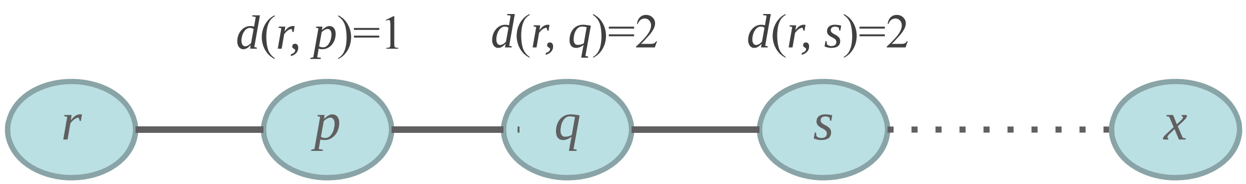


frustrated cycle



Flipping Schemes

Lemma 1. Flipping scheme for a set of nodes represented by an acyclic graph to make all interactions submodular:



$$w(i, j) = \begin{cases} 1 \text{ if the pair } i, j \text{ is nonsubmodular} \\ 0 \text{ if the pair } i, j \text{ is submodular} \end{cases}$$

➢ Flip nodes with an odd distance to a root node

Non-submodular - - - -

Submodular ——

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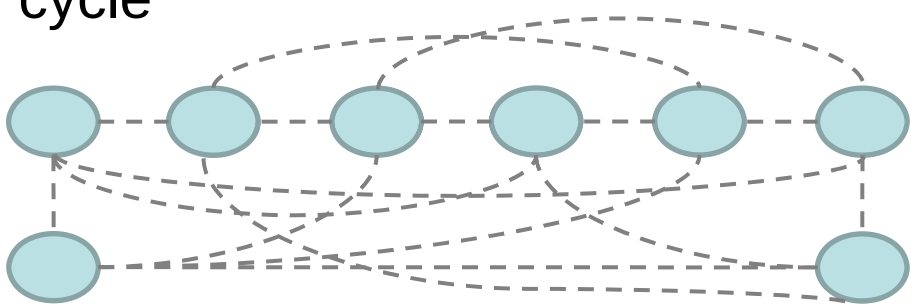
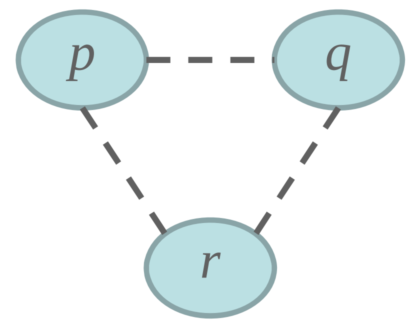
Flipping Schemes

Theorem. A flipping scheme for a set of arbitrarily connected nodes exists iff there is an *<u>even</u>* number of non-submodular interactions in each cycle



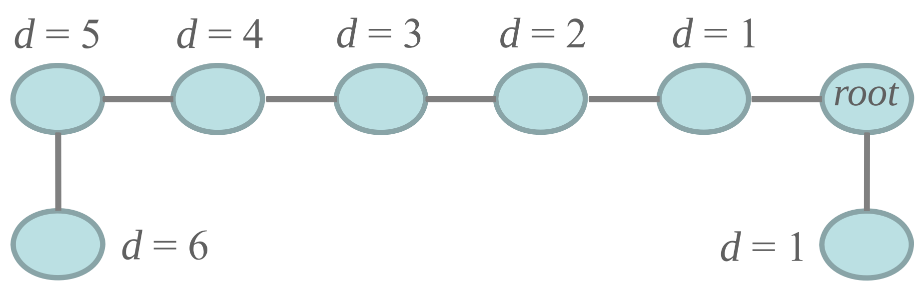
Yes



No

<u>Flipping Scheme:</u> Flip any spanning tree of the graph as given in *Lemma 1*.

Non-submodular
Submodular



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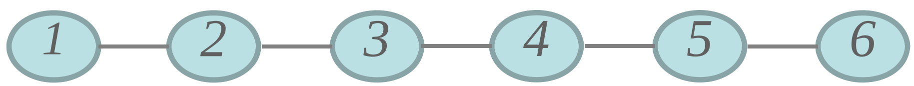
Implications for discrete inverse problems

$$J_{\text{data}}(x) = \|y - Cx\|_2^2 = y^T y - 2y^T Cx + x^T C^T Cx$$

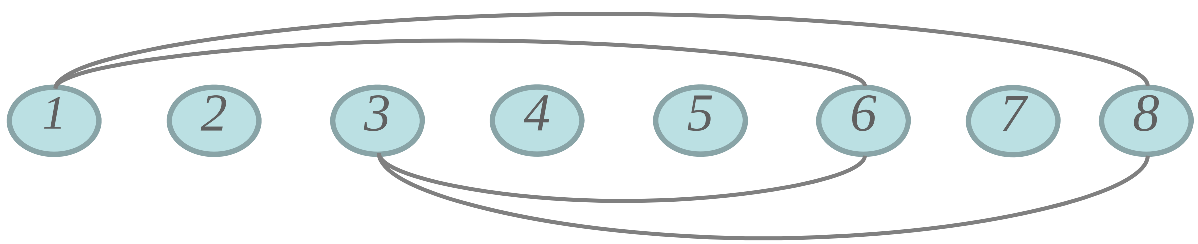
- Structure of $C^T C$:
  1. First off-diagonal ➔ acyclic interactions
  2. Odd off-diagonals ➔ even ("flipable") cycles
  3. Even off-diagonals ➔ odd (frustrated) cycles

$$\begin{bmatrix} x_1 x_1 & \boxed{x_1 x_2} & x_1 x_3 & \boxed{x_1 x_4} & \cdots & \boxed{x_1 x_N} \\ \boxed{x_2 x_1} & x_2 x_2 & \boxed{x_2 x_3} & x_2 x_4 & \cdots & x_2 x_N \\ x_3 x_1 & \boxed{x_3 x_2} & x_3 x_3 & \boxed{x_3 x_4} & \cdots & \boxed{x_3 x_N} \\ \boxed{x_4 x_1} & x_4 x_2 & \boxed{x_4 x_3} & x_4 x_4 & \cdots & x_4 x_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{x_N x_1} & x_N x_2 & \boxed{x_N x_3} & x_N x_4 & \cdots & x_N x_N \end{bmatrix}$$

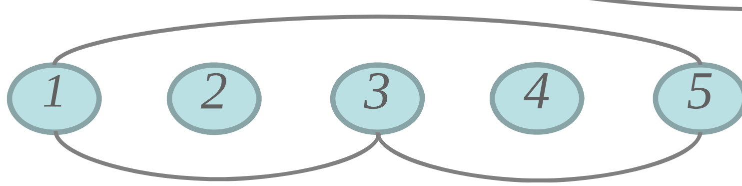
1. First off-diagonal



2. Odd off-diagonals



3. Even off-diagonals



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Idea #2: Special class of systems: No odd cycles

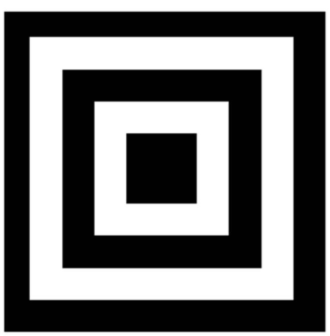
$$J_{\text{data}}(x) = \|y - Cx\|_2^2 = y^T y - 2y^T Cx + x^T C^T Cx$$

Odd-banded and non-negative $C^T C$ → Submodular

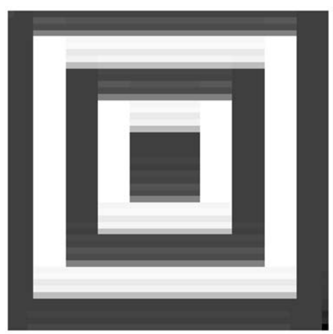
$$C^T C = \begin{bmatrix} * & * & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 \\ 0 & * & * & * & 0 & * \\ * & 0 & * & * & * & 0 \\ 0 & * & 0 & * & * & * \\ * & 0 & * & 0 & * & * \end{bmatrix}$$

- These systems are guaranteed graph representable!

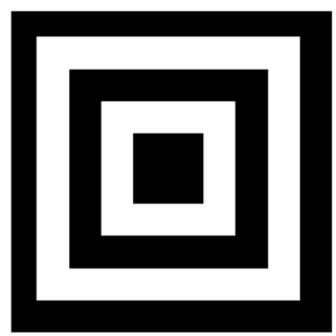
Ground truth image



Blurred image



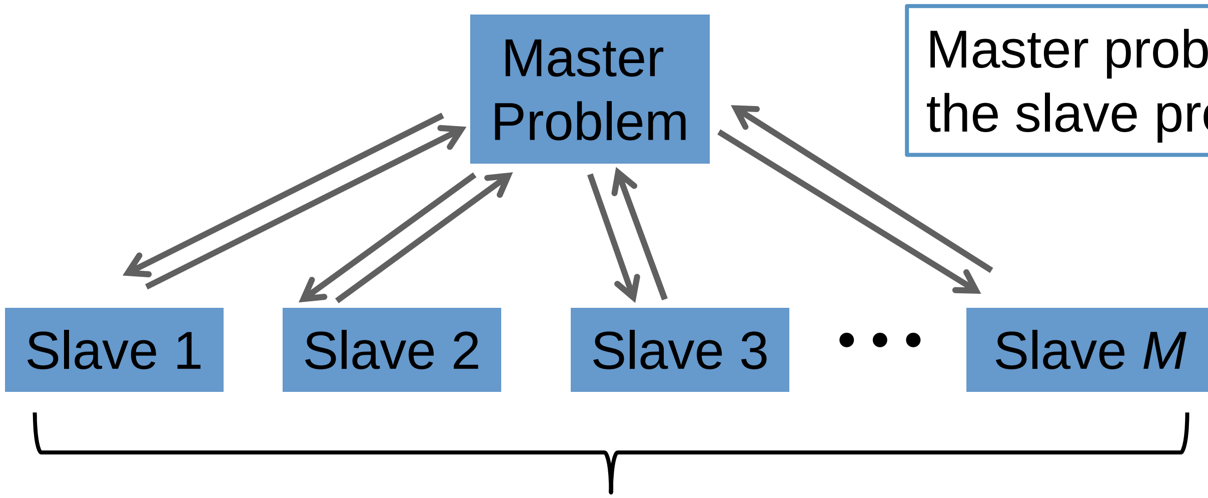
Graph Cut Reconstruction



e.g. Cholesky

Dual Decomposition Inversion Approach

- DD idea: Break single hard problem into multiple easy to solve problems
  - Slave problems are solved independently
  - Master coordinates, enforces consistency via Lagragian dual
  - E.g. Loopy graph solution via overlapping trees, message passing



Master problem coordinates the slave problems.

Easily optimizable slave problems

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Idea #3: Efficient DD slaves for inverse problems

- Decomposition exploiting structure
  - All odd off-diagonals (boxes) form a single, submodular subproblem
  - Each even off-diagonal forms a separate tree subproblem
- Number of subproblems ~ number of bands
- For FIR convolution problem, number of subproblems limited by size of kernel (not size of image)

$$\begin{bmatrix} \boxed{x_1 x_1} & \boxed{x_1 x_2} & x_1 x_3 & \boxed{x_1 x_4} & \cdots & \boxed{x_1 x_N} \\ \boxed{x_2 x_1} & \boxed{x_2 x_2} & \boxed{x_2 x_3} & x_2 x_4 & \cdots & x_2 x_N \\ x_3 x_1 & \boxed{x_3 x_2} & \boxed{x_3 x_3} & \boxed{x_3 x_4} & \cdots & \boxed{x_3 x_N} \\ \boxed{x_4 x_1} & x_4 x_2 & \boxed{x_4 x_3} & \boxed{x_4 x_4} & \cdots & x_4 x_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{x_N x_1} & x_N x_2 & \boxed{x_N x_3} & x_N x_4 & \cdots & \boxed{x_N x_N} \end{bmatrix}$$

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Efficient master huristics

- After many iterations some node labels obtain "obvious" consensus
- "Strong Nodes":
  - Nodes where slaves agree
- Fix strong nodes and remove them from problem
  - Greatly reduces frustrated cycles

After 1000 iterations:

Pixel	Label 1	Label 2
p	900	100
q	950	50
r	450	550

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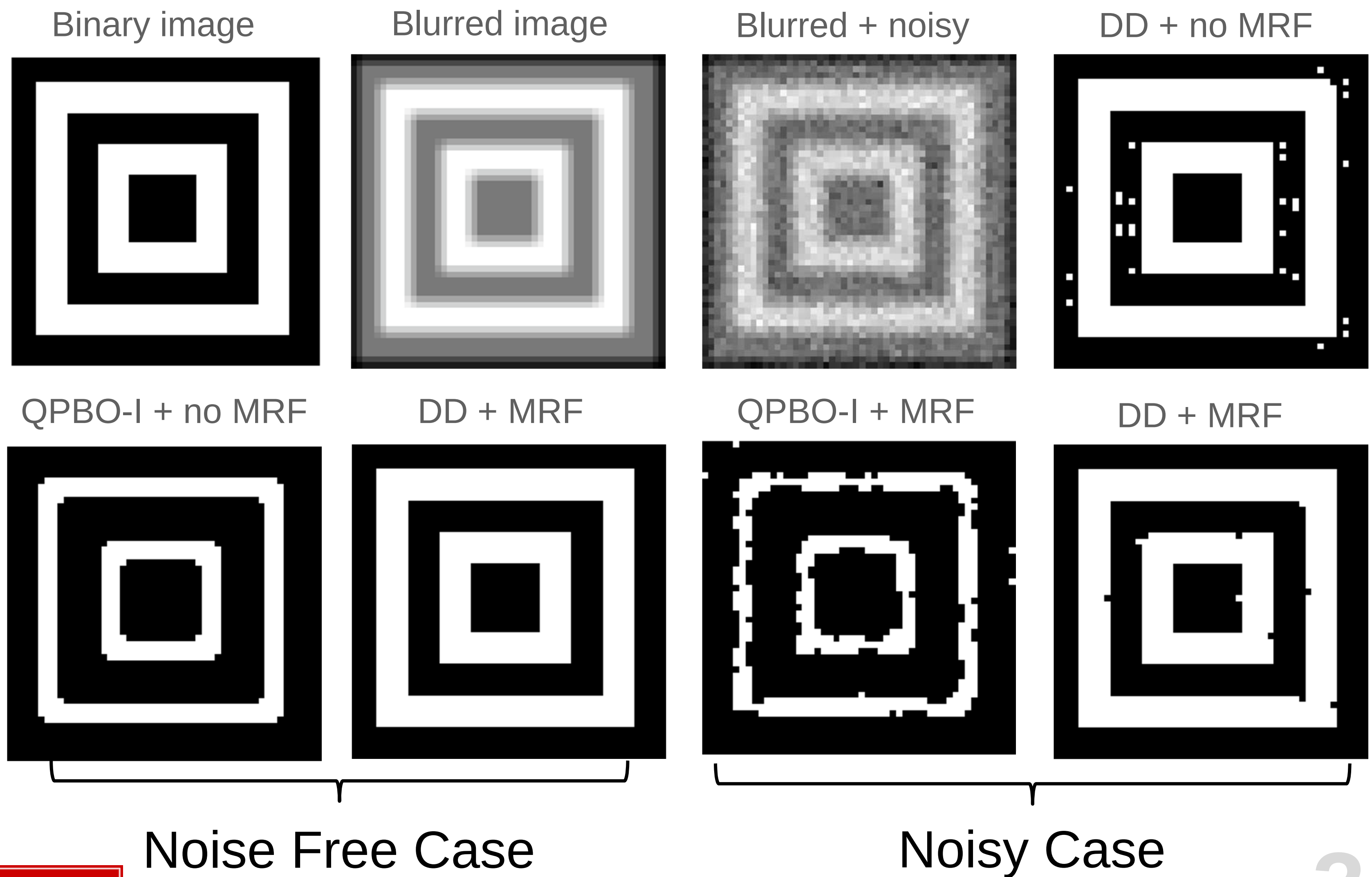
Experiment: Dual Decomposition



Experiment: Dual Decomposition Performance

3x3 Convolution kernel

Method	Missed Pixels	Unlabeled Pixels
QPBO	-	%65
QPBO-P	%21	%0
QPBO-I	%13	%0
GCDD	%0	%0

5x5 Convolution kernel

Method	Missed Pixels	Unlabeled Pixels
QPBO	-	%95
QPBO-P	%48	%0
QPBO-I	%19	%0
GCDD	%7	%0

Summary

- Idea #1: GCWSA graph-cut algorithm for general inverse problems.
  - Based on separable approximation of data term
- Idea #2: Specification of conditions and flipping scheme for obtaining submodularity for an inverse problem
- Idea #3: Identification of special banded inverse problem structure for graph representability
- Idea #4: Exploitation of special structure for efficient graph-cut based dual decomposition

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