Proof Theory: From the Foundations of Mathematics to Applications in Core Mathematics

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In particular: Show the consistency of $\mathcal{I}$ by finitistic means.

Impossibility of the program (in the narrow sense)

Theorem [K. Gödel 1931]

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In ordinary mathematics: the **"Gödel Phenomenon"** is extremely rare. Usually, "ideal" principles can be replaced by suitable more elementary ones. However: this can be very difficult to accomplish.

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'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true? (G. Kreisel, 50's)

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  - existence of bound that is independent from certain parameters,
  - generalizations of proofs: weakening of premises.

Proof Interpretations

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be theories with languages $\mathcal{L}(\mathcal{T}_1)$ and $\mathcal{L}(\mathcal{T}_2)$.

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Central Method: Modern extensions of **Gödel**'s 1958 (developed as part of modified Hilbert program) **Functional ('Dialectica') Interpretation!** Uses embedding into systems based in **intuitionistic logic** (**Brouwer**).

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- General **logical metatheorems** explain these applications as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).
- Some of the logical tools used have recently been rediscovered in special cases by Terence Tao prompted by concrete mathematical needs **"finitary analysis"!**

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Examples of such spaces X : metric, normed, Hilbert, uniformly convex uniformly smooth, hyperbolic, CAT(0) spaces (but e.g. not separable, strictly convex or uniformly Gâteaux differentiable spaces).

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Often a third approach to the foundations of mathematics

- **Formalism** (Hilbert)

is discussed as yet another option.

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- Mental constructions 'by abandoning Kant's a-priority of space but adhering more resolutely to the a-priority of time' (Brouwer 1912). 'Intuituitionistic mathematics is an esentially language-less mental structure which comes into being by the self-unfolding of the two-ity as the Primordial Intuition' (Brouwer 1947)

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- Onotological solipsism: 'the only thing that is real to me is my own self at this moment, surrounded by a wealth of images in which the self believes and which make the self live. ... A second reality, independent of my self and corresponding to these images, is out of the question.' (Brouwer 1898).

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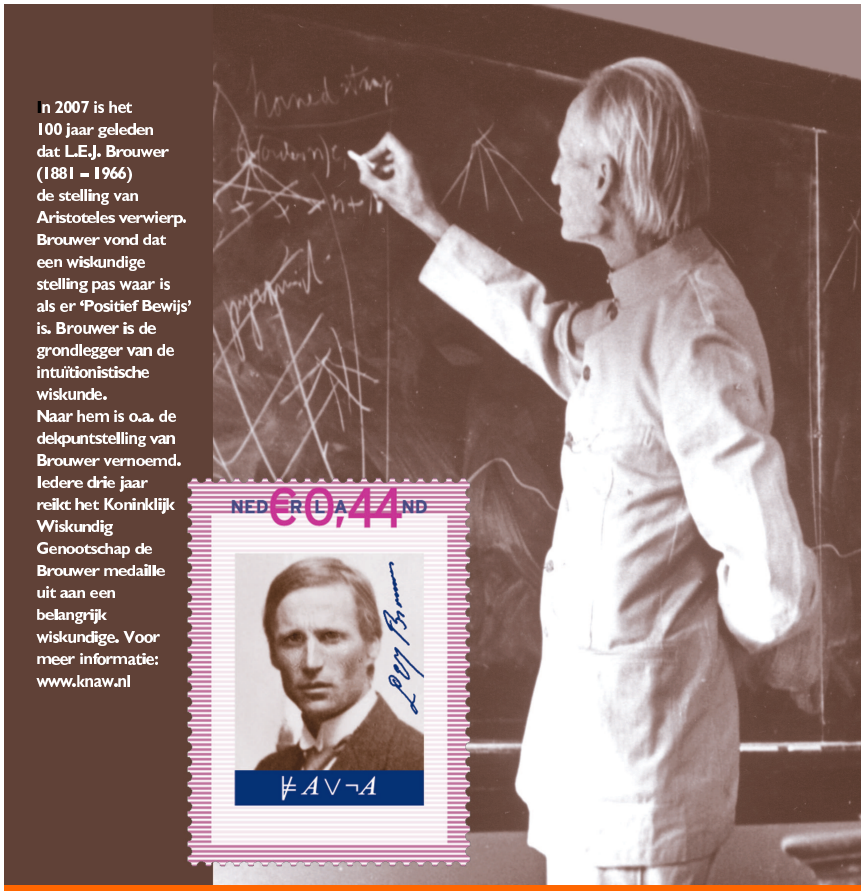
- Logic is secondary to mathematics.
- Law-of-excluded middle on infinite domains in general meaningless



Er is positief bewijs!

100 jaar na dato wordt de wiskundige L.E.J. Brouwer (1881-1966) geëerd met een eigen postzegel.

TNT post

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- Formalism is a way of viewing the **ideal elements** (Hilbert) of a metaphysical reading of platonism/intuitionism as mere **regulative ideas** (Kant) that are useful in proving facts about **real statements** (Hilbert) of mathematics rather than as real objects themselves.

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We will discuss

- **Mathematical uses** of a formalistic reading of platonism/intuitionism.

	Platonism	Intuitionism
Ontology (Semantics, Models)	Naive set theory Universe of all sets	Intuitionistic universe of mental constructions
Formalism (Syntax, Proofs)	Elimination of 'ideal' elements in proofs 'real' statements	Elimination of choice sequences in arithmetical statements

$\Downarrow$ $\Downarrow$

Finitism

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- This in the end gives some form of **finitistic realization of $\forall\exists$-theorems** poven together with an elementary proofs.
- For **semi-intuitionistic proofs** the restriction to $\forall\exists$-theorem can be overcome: **arbitary logical complexity allowed!**

The running theme: convergence statements in analysis

Let (x_n) be a Cauchy sequence in a metric space (X, ρ), i.e.

$$\forall \mathbf{k} \in \mathbb{N}\, \exists \mathbf{n} \in \mathbb{N}\, \forall \mathbf{i}, \mathbf{j} \geq \mathbf{n}\, (\rho(\mathbf{x_i}, \mathbf{x_j}) \leq \mathbf{2}^{-\mathbf{k}}) \in \forall\exists\forall$$

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A bound $\Phi(k, g)$ on '$\exists n$' in the latter formula is a **rate of metastability** (introduced by **Kreisel** in 1951 as **no-counterexample interpretation).**

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- Possible for (x_n) if sequence converges to **unique** fixed point/solution. Extraction of **modulus of uniqueness** $\Phi : \mathbb{R}^*_+ \to \mathbb{R}^*_+$

$$\forall \varepsilon > 0\, \forall x, y \in X\, \big(\rho(x, f(x)), \rho(y, f(y)) < \Phi(\varepsilon) \to \rho(x, y) < \varepsilon\big)$$

  gives rate of convergence (or – in the noncompact case – existence at all)! Numerous applications in analysis!

Rates of Asymptotic Regularity

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$\mathbf{A} := \mathbf{I} - \mathbf{f}$ is **accretive** (and $-\mathbf{A}$ **dissipative**) which in Hilbert space is equivalent to being **monotone**

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In 1974 Bruck considered the following iteration schema

$$\mathbf{x_n} + 1 := (1 - \lambda_n)\mathbf{x_n} + \lambda_n \mathbf{f}(\mathbf{x_n}) - \lambda_n \theta_n (\mathbf{x_n} - \mathbf{x_1}),$$

for suitable $(\lambda_n), (\theta_n)$ in $(0, 1]$ and showed asymptotic regurality and strong convergence (towards a fixed point) results in Hilbert space.

Asymptotic regularity for Lipschitzian pseudocontractions in arbitrary Banach spaces

Theorem (Chidume,Zegeye 2004): $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0$, where

$$\text{(i) } \lim \theta_n = 0, \ \text{(ii) } \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \ \text{(iii) } \lim \frac{\lambda_n}{\theta_n} = 0,$$

$$\text{(iv) } \lim \frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n \theta_n} = 0, \ \text{(v) } \lambda_n(1+\theta_n) \leq 1.$$

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Let $M \geq diam(C)$ and $(\lambda_n), (\theta_n) \subset (0,1]$ with rates of conv./div. $R_i : (0,\infty) \to \mathbb{N}$

1. $\forall \varepsilon > 0 \forall n \geq R_1(\varepsilon)(\theta_n \leq \varepsilon)$,
2. $\forall x \in (0,\infty) \left(\sum_{n=1}^{R_2(x)} \lambda_n \theta_n \geq x \right)$,
3. $\forall \varepsilon > 0 \forall n \geq R_3(\varepsilon)(\lambda_n \leq \theta_n \varepsilon)$,
4. $\forall \varepsilon > 0 \forall n \geq R_4(\varepsilon) \left(\frac{\left| \frac{\theta_{n-1}}{\theta_n} - 1 \right|}{\lambda_n \theta_n} \leq \varepsilon \right)$.

Rate of convergence extracted from Chidume/Zegeye (2004):

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where

$$\Psi\left(M, L, R_1, R_2, R_3, R_4, \varepsilon\right) = \max\left\{N_2\left(C\right) + 1, R_1\left(\frac{\varepsilon}{2M}\right) + 1\right\}$$

and

$$N_1\left(\varepsilon\right) := \max\left\{R_3\left(\frac{\varepsilon}{4M^2\left(2 + L\right)}\right), R_4\left(\sqrt{\frac{\varepsilon}{M^2} + 1} - 1\right)\right\},$$

$$N_2\left(x\right) := R_2\left(\frac{x}{2}\right) + 1,$$

$$C := \frac{8\left(1 + L\right)^2 M^2}{\varepsilon^2} + 2\left(N_1\left(\frac{\varepsilon^2}{8\left(1 + L\right)^2}\right) - 1\right).$$

Rates of Full Convergence in Case of Uniqueness

Example 1: Kirk's theorem for asymptotic contractions

Definition (Kirk JMAA03)

(X, d) metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if Φ, Φ_n are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$\forall \mathbf{n} \in \mathbb{N} \forall \mathbf{x}, \mathbf{y} \in \mathbf{X}(\mathbf{d}(\mathbf{f}^{\mathbf{n}}(\mathbf{x}), \mathbf{f}^{\mathbf{n}}(\mathbf{y})) \leq \mathbf{\Phi}_{\mathbf{n}}(\mathbf{d}(\mathbf{x}, \mathbf{y})),$$

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Theorem (Kirk JMAA03)

(X, d) complete metric space, $f : X \to X$ continuous asymptotic contraction with some orbit bounded. Then f has a unique fixed point $p \in X$ and $(f^n(x_0))$ converges to p for each $x_0 \in X$.

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(Proof uses ultrapower structures and hence the axiom of choice!)

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- As a consequence of his analysis E.M.Briseid showed that the $(f^n(x_0))$ **is redundant** to assume: rate of convergence using only $b \geq d(x, f(x))$ (Fixed Point Theory 2007, Int. J. Math. Stat. 2010).

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- E.M.Briseid showed that for bounded metric spaces the existence of a x_0-uniform rate of convergence **implies** that f is asymptotically contractive (JMAA 2007). Also: new uniformity results generalizing Reich et al (2007).

Example 2: Generalized p-contractions

Definition: [Rhoades 1977] (X, d) metric space and $p \in \mathbb{N}$.
$f : X \to X$ is called **generalized p-contractive** if

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}\big(\mathbf{x} \neq \mathbf{y} \to \mathbf{d}(\mathbf{f}^{\mathbf{p}}(\mathbf{x}), \mathbf{f}^{\mathbf{p}}(\mathbf{y})) < \operatorname{diam}\{\mathbf{x}, \mathbf{y}, \mathbf{f}^{\mathbf{p}}(\mathbf{x}), \mathbf{f}^{\mathbf{p}}(\mathbf{y})\}\big).$$

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Theorem: [Kincses/Totik 1990]
(K, d) **compact** metric space and $f : K \to K$ continuous and generalized p-contractive for some $p \in \mathbb{N}$. Then f has a unique fixed point ξ and for every $x \in K$ $\lim\limits_{\mathbf{n} \to \infty} \mathbf{f}^{\mathbf{n}}(\mathbf{x}) = \xi$.

Definition: [Briseid, J. Nonlinear Convex Anal. 2008]
(X, d) metric space, $p \in \mathbb{N}$. $f : X \to X$ is called **uniformly generalized** p-**contractive** with modulus $\eta : \mathbb{Q}_+^* \to \mathbb{Q}_+^*$ if for all $x, y \in, \varepsilon \in \mathbb{Q}_+^*$

$$\mathbf{d(x,y)} > \varepsilon \to \mathbf{d(f^p(x), f^p(y))} + \eta(\varepsilon) < \text{ diam } \{\mathbf{x, y, f^p(x), f^p(y)}\}.$$

New existence without compactness

Theorem (Briseid, J. Nonlinear Convex Anal. 2008)

(X, d) **complete** metric space and $p \in \mathbb{N}$. $f : X \to X$ be a **uniformly** continuous and **uniformly** generalized p-contractive with moduli ω, η. Let $(f^n(x_0))$ be bounded by $b \in \mathbb{Q}_+^*$. Then f has a unique fixed point ξ and $(f^n(x_0))$ converges to ξ with rate of convergence $\Phi : \mathbb{Q}_+^* \to \mathbb{N}$,

$$\Phi(\varepsilon) := \begin{cases} p\lceil (b - \varepsilon)/\rho(\varepsilon) \rceil & \text{if } b > \varepsilon, \\ 0, & \text{otherwise} \end{cases}$$

with

$$\rho(\varepsilon) := \min\left\{\eta(\varepsilon), \frac{\varepsilon}{2}, \eta\left(\frac{1}{2}\omega^p\left(\frac{\varepsilon}{2}\right)\right)\right\}.$$

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Existence of fixed point by **uniform uniqueness** via **uniform continuity** and **uniform generalized p-contractivity!**

Fluctuation Bounds

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(x)}\| \leq \|\mathbf{x}\|$ for all $x \in X$.

$$\mathbf{A_n(x)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(x)}, \text{ where } \mathbf{S_n(x)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(x)} \quad (\mathbf{n \geq 0})$$

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For every $x \in X$, the sequence $(A_n(x))_n$ converges.

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

By logical metatheorems (K. TAMS 2005, Gerhardy/K. TAMS 2008, K. Springer Monograph 2008):

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists \mathbf{n} \leq \mathbf{\Phi}(\varepsilon, \mathbf{g}, \mathbf{b}, \eta)\, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{n}; \mathbf{n} + \mathbf{g}(\mathbf{n})] \left(\|\mathbf{A_i}(\mathbf{x}) - \mathbf{A_j}(\mathbf{x})\| < \varepsilon\right),$$

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$$\exists n \leq \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n; n + g(n)] \left(\|A_i(x) - A_j(x)\| < \varepsilon\right),$$

where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^{(K)}(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N}, \; h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$

Computable rate of convergence iff the **norm of limit is computable**!

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

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As a corollary to our analysis of Birkhoff's proof, Avigad and Rute showed

Theorem (Avigad, Rute (2012))

$(A_n(x))$ admits at most

$$\mathbf{2\log(M)} \cdot \frac{\mathbf{b}}{\varepsilon} + \frac{\mathbf{b}}{\gamma} \cdot \mathbf{2\log(2M)} \cdot \frac{\mathbf{b}}{\varepsilon} + \frac{\mathbf{b}}{\gamma}$$

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many fluctuations.

Partly possible because Birkhoff's proof only uses boundedly many (in the data) instances of the law-of-excluded-middle for $\exists$-statements!

Bounds on Metastability

Strong nonlin. ergodic theorem (Wittmann)

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Halpern iterations: $f : C \to C$ nonexpansive, $x_0 \in C$, $\alpha_n \in [0,1]$

$$\mathbf{x_{n+1} := \alpha_{n+1}\, x_0 + (1 - \alpha_{n+1})\, f(x_n).}$$

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Using weak compactness, Wittmann proved in 1992:

Theorem (R. Wittmann 1992): $C \subseteq X$ closed and convex, $x_0 \in C$ and $Fix(f) \neq \emptyset$. Under suitable conditions on (α_n) (e.g. $\alpha_n := \frac{1}{n+1}$) (x_n) converges strongly towards the fixed point of f that is closest to x_0.

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Remark: Wittmann's theorem is a **nonlinear generalization of the Mean ergodic theorem**: for $\alpha_n := 1/(n+1)$, $C := X$ and **linear** f, the Halpern iteration coincides with the Cesàro means.

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- In Wittmann's proof the use of **weak compactness** gets in the end **eliminated** via a quantitative projection argument but could be fully effectivized (K., Adv. Math. 2011)
- Wittmann's result has been generalized to **uniformly smooth Banach spaces** (Shioji-Takahashi 1997) and **CAT(0)-spaces** (Saejung 2010) using **Banach limits (AC)**.

Rate of Metastability in Saejung's Theorem

Theorem (K./Leuştean, Adv. Math. 2012; improved 2013)

X CAT(0), $C \subseteq X$ convex, $diam(C) \leq M$, (x_n) as above, $\varepsilon \in (0,1)$:

$$\forall \mathbf{g} : \mathbb{N}^{\mathbb{N}} \exists \mathbf{k} \leq \mathbf{\Sigma}(\varepsilon, \mathbf{g}, \mathbf{M}) \, \forall \mathbf{i}, \mathbf{j} \in [\mathbf{k}; \mathbf{k} + \mathbf{g}(\mathbf{k})] \, \big(\rho(\mathbf{x_i}, \mathbf{x_j}) \leq \varepsilon\big),$$

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$$\mathbf{\Sigma}(\varepsilon, \mathbf{g}, \mathbf{M}) := \left\lceil \frac{12\mathbf{M}^2 \chi_{\mathbf{L}}^*(\varepsilon^2/12)}{\varepsilon^2} \right\rceil, \text{ with } \mathbf{L} := \widetilde{\mathbf{h}^*}^{(\lceil \mathbf{M}^2/\varepsilon_0^2 \rceil)}(\mathbf{0}) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil,$$

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$$\chi^*_{\mathbf{k}}(\varepsilon) := \left\lceil \frac{12\mathbf{M}^2(\mathbf{k}+1)}{\varepsilon} + \frac{144\mathbf{M}^4(\mathbf{k}+1)^2}{\varepsilon^2} \right\rceil - 1, \; \varepsilon_0 := \varepsilon^2/24(\mathbf{d}+1)^2,$$

$$\Theta_{\mathbf{k}}(\varepsilon) := \left\lceil \frac{3\mathbf{M}^2 \chi^*_{\mathbf{k}}(\varepsilon/3)}{\varepsilon} \right\rceil, \; \mathbf{\Delta}^*_{\mathbf{k}}(\varepsilon, \mathbf{g}) := \frac{\varepsilon}{3\mathbf{g}_{\varepsilon,\mathbf{k}}\left(\Theta_{\mathbf{k}}(\varepsilon) - \chi^*_{\mathbf{k}}(\varepsilon/3)\right)},$$

$$\mathbf{g}_{\varepsilon,\mathbf{k}}(\mathbf{n}) := \mathbf{n} + \mathbf{g}\left(\mathbf{n} + \chi^*_{\mathbf{k}}\left(\frac{\varepsilon}{3}\right)\right), \; \mathbf{h}(\mathbf{k}) := \max\left\{\left\lceil \frac{\mathbf{M}^2}{\mathbf{\Delta}^*_{\mathbf{k}}(\varepsilon^2/4, \mathbf{g})} \right\rceil, \mathbf{k}\right\} - \mathbf{k},$$

$$\mathbf{h}^*(\mathbf{k}) := \mathbf{h}\left(\mathbf{k} + \left\lceil \frac{1}{\varepsilon_0} \right\rceil\right) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil, \; \widetilde{\mathbf{h}^*}(\mathbf{k}) := \mathbf{k} + \mathbf{h}^*(\mathbf{k}).$$

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- A **quadratic** full rate of convergence for the asymptotic regularity $\rho(x_n, f(x_n)) \to 0$:

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Also unbounded C and modified Halpern iterations: $M \geq 4\|p - x_0\|$ (Schade/K., Fixed Point Theory Appl. 2012)

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- Rate of asymptotic regularity for Bruck's iteration of Lipschitzian pseudocontractions in Hilbert space (**Körnlein/K., 2013 submitted)**.

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