Local Computation with Gaussian Potentials

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Abstract. Gaussian or multivariate normal distributions are very popular and important probability models. *Gaussian potentials* [5] are multivariate normal density functions. Such a distribution is often given as a product of *conditional Gaussian densities*, which are more general than Gaussian potentials. These are related to *Gaussian hints* [11] and *Gaussian belief functions* [9, 10]. Gaussian potentials, which form a *cancellative* semigroup, can be embedded into a valuation algebra of pairs. In this *separative extension*, marginalization may be defined only partially. Gaussian potentials form a cancellative semigroup, so families of conditional Gaussian distributions are in the separative extension of Gaussian potentials. Since combination of Gaussian potentials is matrix additition of concentration matrices, division is substraction of concentration matrices, which leads to a new representation of a pair of two concentration matrices by a symmetric matrix, and to a valuation of *symmetric Gaussian potentials*. With this extension valuation algebra, local computation can be performed on join trees. Construction sequences cover factorizations of Gaussian potentials into conditional Gaussian potentials.

1 Introduction

The abstract framework of valuation algebras [5, 12] covers different information representation systems such as relational databases and probability densities. Valuations represent information on some domain, which is defined by a set of variables. Knowledge can be combined and it can be focussed (marginalized) to a domain of interest. Starting from a cancellative semigroup, an algebra of pairs can be defined, where marginalization is defined only partially [5, 12, 2]. How to introduce division into a valuation algebra has already been formulated in an abstract way in [7]. An equivalence relation can be introduced into this algebra, which identifies elements that represent the same information, in the same way as different quotients of integers represent the same rational. This particular equivalence relation can be shown to be congruent with combination and marginalization. Based on this equivalence relation, marginalization can be extended, leading to a separative extension valuation algebra with partial marginalization. This is the topic of Section 2.

Gaussian potentials form a cancellative semigroup, so Gaussian potentials can be embedded into the separative extension of Gaussian potentials, which includes families of conditional Gaussian distributions. An alternative representation of such pairs as symmetric Gaussian potentials can be defined, where the two concentration matrices are replaced by their difference, which is, in general, only symmetric but not positive definite. This representation has the advantage that equivalent pairs of Gaussian potentials have a unique representation. This is the topic of Section 3.

Often, a probability distribution, whose marginals are all well defined, is given as a product of conditional distributions, whose marginals may only be partially defined. Construction sequences [13, 5] capture factorizations of densities into conditionals in such a way that the Lauritzen-Spiegelhalter [8, 5, 12] and HUGIN [4, 5, 12] algorithms can be applied, which exploit division introduced in the separative extension. This is the topic of Section 4.

2 Cancellative Semigroups

Introducing division into a semigroup has two purposes: First, this is an enrichment in elements; in the case of Gaussian potentials, conditional Gaussian potentials are added. This is important since a Gaussian potential is often given as a product of conditional Gaussian potentials. The second purpose is that more efficient local computation schemes such as the Lauritzen-Spiegelhalter and HUGIN architectures can be used with Gaussian potentials. In this section, first, cancellativity is introduced as a sufficient condition for a semigroup to be embeddable into a group. Then, a lattice of labels or domains is imposed on the elements of the semigroup, and a marginalization operation is defined.

2.1 Embedding into a Group

Rational numbers can be expressed as pairs of non-zero integer numbers, or, more precisely, the semigroup $(\mathbb{Z}, \otimes_{\mathbb{Z}})$ of non-zero integer numbers is embedded into the group $(\mathbb{Q}, \otimes_{\mathbb{Q}})$ of non-zero rational numbers $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}$. The pair or *quotient* $(z \otimes_{\mathbb{Z}} z, z) = \frac{z \otimes_{\mathbb{Z}} z}{z} \in \mathbb{Q}$ is a representation of $z \in \mathbb{Z}$ in the larger set $\mathbb{Q}$ of rationals. Since $z \otimes_{\mathbb{Z}} z = z \otimes_{\mathbb{Z}} z'$ for $z, z' \in \mathbb{Z}$ implies $z = z'$, the mapping $z \mapsto (z \otimes z, z)$ is injective. More generally, $(\Phi, \otimes)$ is a *commutative semigroup* if

(A1) Φ is associative and commutative under $\otimes$, i.e. for $\phi, \psi, \chi \in \Phi$,

$$\begin{aligned} \phi \otimes (\psi \otimes \chi) &= (\phi \otimes \psi) \otimes \chi, \qquad \text{and} \\ \phi \otimes \psi &= \psi \otimes \phi. \end{aligned}$$

A commutative semigroup $(\Phi, \otimes)$ is *cancellative* if, for $\phi, \psi, \psi' \in \Phi$, $\phi \otimes \psi = \phi \otimes \psi'$ implies that $\psi = \psi'$. Define

$$\Phi^* = \{(\phi, \psi) : \quad \phi, \psi \in \Phi\}, \tag{1}$$

the set of pairs of elements of Φ. Then, the mapping $h : \Phi \to \Phi^*$, $\phi \mapsto h(\phi)$,

$$h(\phi) = (\phi \otimes \phi, \phi) \tag{2}$$

is injective. Multiplication $\otimes_{\mathbb{Q}}$ in $\mathbb{Q}$ is just multiplication of the numerators and the denominators. More generally, define a multiplication $\otimes^* : \Phi^* \times \Phi^* \to \Phi^*$ by

$$(\phi_1, \psi_1) \otimes^* (\phi_2, \psi_2) = (\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2). \tag{3}$$

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Then, $(\Phi^*, \otimes^*)$ is a commutative semigroup which extends $(\Phi, \otimes)$, i.e. for $\phi, \psi \in \Phi$

$$h(\phi \otimes \psi) \quad = \quad h(\phi) \otimes^* h(\psi). \tag{4}$$

In the rational numbers, there are different representations of the "same" rational by pairs of integer numbers, for instance $\frac{1}{2}$, $\frac{3}{6}$, and $\frac{4}{8}$ are identified. Here, the connecting property is that $1 \otimes 6 = 2 \otimes 3$, $3 \otimes 8 = 6 \otimes 4$. More generally, the relation $=^*$ in Φ^* defined by

$$(\phi_1, \psi_1) =^* (\phi_2, \psi_2) \quad \Longleftrightarrow \quad \phi_1 \otimes \psi_2 = \psi_1 \otimes \phi_2 \tag{5}$$

is an *equivalence relation*. Even more, this relation is *congruent* with $\otimes^*$, i.e., for $\eta_1, \eta_2, \eta_1', \eta_2' \in \Phi^*$, $\eta_1 =^* \eta_1'$ and $\eta_2 =^* \eta_2'$ imply that

$$\eta_1 \otimes^* \eta_2 \quad =^* \quad \eta_1' \otimes^* \eta_2'. \tag{6}$$

It can be shown [2] that the *separative extension* Φ^* forms a group with inverses $(\phi, \psi)^{-1} = (\psi, \phi)$ and unit element (χ, χ) for $\phi, \psi, \chi \in \Phi$, which are unique up to equivalence modulo $=^*$. In particular, $h(\phi)^{-1} = (\phi, \phi \otimes \phi)$.

2.2 Marginalization

If Φ is a set of information pieces, which may be called *valuations* [5], the elements of Φ may be labeled by a domain they refer to. Therefore, assume D to be a lattice of subsets of a set r of *variables*. Then, each element $\phi \in \Phi$ is labeled by a domain $d(\phi) \in D$. To every variable $X \in r$ is associated a set Ω_X of values, called the *frame* of X, and, similarly, to any set x of variables is associated the Cartesian product $\Omega_x = \times_{X \in x} \Omega_X$ of *configurations* of x. If Φ consists of real functions f, then $d(f) = x$ just stands for the definition domain $\Omega_x = \mathbb{R}^x$, i.e. $f : \mathbb{R}^x \to \mathbb{R}$. Here, $\mathbf{x} \in \mathbb{R}^x$ can be seen as a function $\mathbf{x} : x \to \mathbb{R}$. The product $f \otimes g$ of two functions $f : \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^t \to \mathbb{R}$ is the real-valued function $(f \otimes g) : \mathbb{R}^{s \cup t} \to \mathbb{R}$ defined by

$$(f \otimes g)(\mathbf{x}) \quad = \quad f(\mathbf{x}^{\downarrow s}) \cdot g(\mathbf{x}^{\downarrow t})$$

for all configurations $\mathbf{x} \in \mathbb{R}^{s \cup t}$ and where $\mathbf{x}^{\downarrow s}$ and $\mathbf{x}^{\downarrow t}$ are the restrictions $\mathbf{x}|_s$ and $\mathbf{x}|_t$ of $\mathbf{x} : s \cup t \to \mathbb{R}$ to s and t, respectively. More generally,

(A2) (*Labeling:*) for $\phi, \psi \in \Phi$,

$$d(\phi \otimes \psi) = d(\phi) \cup d(\psi). \tag{7}$$

For a non-negative continuous $f : \mathbb{R}^s \to \mathbb{R}$ such that

$$\int_{\mathbf{x} \in \mathbb{R}^s} f(\mathbf{x}) d\mathbf{x} \quad < \quad \infty,$$

the marginals $f^{\downarrow t} : \mathbb{R}^t \to \mathbb{R}$ are well defined for every subset $t \subseteq s$ by

$$f^{\downarrow t}(\mathbf{y}) \quad = \quad \int_{\mathbf{z} \in \mathbb{R}^{s-t}} f(\mathbf{y}, \mathbf{z}) d\mathbf{z}, \qquad \mathbf{y} \in \mathbb{R}^t.$$

Such a function f may be called *density* [5]. Densities are the model upon which the axioms of valuation algebras are built. Taking up natural properties of integration of densities, marginalization is an operation $\downarrow$: $\Phi \otimes D, (\phi, x) \mapsto \phi^{\downarrow x}$ that satisfies the following properties (A3)-(A5):

(A3) (*Marginalization:*) For $\phi \in \Phi$, $x \in D$, $x \subseteq d(\phi)$,

$$d(\phi^{\downarrow x}) = x. \tag{8}$$

(A4) (*Transitivity:*) For $\phi \in \Phi$ and $x \subseteq y \subseteq d(\phi)$,

$$(\phi^{\downarrow y})^{\downarrow x} = \phi^{\downarrow x}. \tag{9}$$

(A5) (*Combination:*) For ϕ, $\psi \in \Phi$ with $d(\phi) = x$, $d(\psi) = y$ and $z \in D$ such that $x \subseteq z \subseteq x \cup y$,

$$(\phi \otimes \psi)^{\downarrow z} = \phi \otimes \psi^{\downarrow z \cap y}. \tag{10}$$

A direct consequence of the transitivity and combinations axioms is that

$$(\phi \otimes \psi)^{\downarrow z} \quad = \quad \phi^{\downarrow d(\phi) \cap z} \otimes \psi^{\downarrow d(\psi) \cap z} \tag{11}$$

for $d(\phi) \cap d(\psi) \subseteq z \subseteq d(\phi) \cup d(\psi)$ [5], that is, as a long as no common variables are involved, marginalization of a product can be calculated on the factors independently. For technical reasons, the following properties are added:

(A6) (*Domain:*) $\phi \in \Phi$ with $d(\phi) = x$ implies

$$\phi^{\downarrow x} = \phi. \tag{12}$$

(A7) (*Identity Element:*) There is an element $e \in \Phi$, $d(e) = \emptyset$, such that for any $\phi \in \Phi$

$$\phi \otimes e = e \otimes \phi = \phi, \tag{13}$$

and

$$e \otimes e = e. \tag{14}$$

In the case of continuous densities, the identity element corresponds to the constant 1. A structure $(\Phi, D, d, \otimes, \downarrow)$ satisfying (A1) through (A7) is called a *valuation algebra with full marginalization* [5, 12]. If the semigroup $(\Phi, \otimes)$ is cancellative, it is a special case of a *separative valuation algebra* [5, 12, 2].

It has already been seen that $(\Phi^*, \otimes^*)$ is a group. Then, the purpose is to define marginalization $\downarrow^*$ in Φ^*, at least partially and such that it exends marginalization in Φ. First note that the equivalence relation $=^*$ in Φ^* is not *domain-congruent*, i.e. $\eta =^* \eta'$ does not imply $d^*(\eta) = d^*(\eta')$, for instance for all $\phi, \psi \in \Phi$ it holds that $(\phi, \phi) =^* (\psi, \psi)$. Consider two positive densities f, g, $x = d(f)$ and $y = d(g)$, i.e. $f(\mathbf{x}) > 0$ and $g(\mathbf{y}) > 0$ for all $\mathbf{x} \in \mathbb{R}^x$ and $\mathbf{y} \in \mathbb{R}^y$. It can be shown that the semigroup of positive densities is cancellative [2]. Then, for all z, $y \subseteq z \subseteq x \cup y$,

$$\int_{\mathbf{x} \in \mathbb{R}^{x-z}} \frac{f(\mathbf{x}, \mathbf{z})}{g(\mathbf{z}^{\downarrow y})} d\mathbf{x} \quad = \quad \frac{\int_{\mathbf{x} \in \mathbb{R}^{x-z}} f(\mathbf{x}, \mathbf{z})}{g(\mathbf{z}^{\downarrow y})} d\mathbf{x}$$

for all $\mathbf{z} \in \mathbb{R}^z$. Motivated by this example, define labeling $d^* : \Phi^* \to D$, $\eta \mapsto d^*(\eta)$, by

$$d^*(\eta) \quad = \quad d(\phi) \cup d(\psi), \qquad \eta = (\phi, \psi). \tag{15}$$

Now, a *domain operator* $\mathcal{M} : \Phi^* \to D$ and (partial) marginalization $\downarrow^*$: $\Phi^* \times D, (\eta, z) \mapsto \eta^{\downarrow^* z}$ for $z \in \mathcal{M}(\eta)$ are going to be defined. If $\eta = h(\phi) \otimes^* h(\psi)^{-1}$ for some $\phi, \psi \in \Phi$, $x = d(\phi)$, and $y = d(\psi)$, then define

$$\mathcal{M}(\eta) \quad = \quad \{z : \quad y \subseteq z \subseteq x \cup y\}, \tag{16}$$

and

$$\eta^{\downarrow^* z} \quad = \quad h(\phi^{\downarrow x \cap z}) \otimes^* h(\psi)^{-1}, \qquad z \in \mathcal{M}(\eta). \tag{17}$$

Else, $\eta = (\phi, \psi)$ for some $\phi, \psi \in \Phi$, $x = d(\phi)$, and $y = d(\psi)$, define

$$\mathcal{M}(\eta) \quad = \quad \{z : \quad y \subseteq z \subseteq x \cup y\}, \tag{18}$$

and

$$\eta^{\downarrow^* z} \quad = \quad (\phi^{\downarrow x \cap z}, \psi), \qquad z \in \mathcal{M}(\eta). \tag{19}$$

Note that $\downarrow^*$ extends marginalization in Φ, i.e. $z \in \mathcal{M}(h(\phi))$ for all $z \subseteq d(\phi)$, since $h(\phi) = h(\phi) \otimes h(e)^{-1}$ for all $\phi \in \Phi$, and for all $\phi \in \Phi$, $z \subseteq d(\phi)$,

$$h(\phi^{\downarrow z}) \quad = \quad h(\phi)^{\downarrow^* z}. \tag{20}$$

This definition of marginalization does not take into account that there may be other representants $\eta' =^* \eta$ with different $\mathcal{M}(\eta') \neq \mathcal{M}(\eta)$. In order to extend this definition of marginalization, it is now shown that $=^*$ is *congruent with marginalization*, i.e. $\eta =^* \eta'$, and $z \in \mathcal{M}(\eta), \mathcal{M}(\eta')$ implies $\eta^{\downarrow^* z} =^* \eta'^{\downarrow^* z}$. Let $\eta = (\phi, \psi)$ and $\eta' = (\phi', \psi')$ with $x = (\phi)$, $y = d(\psi)$, $x' = d(\phi')$, $y' = d(\psi')$ and $z \in D$ with y, $y' \subseteq z \subseteq x \cup y, x' \cup y'$. Then, $\phi \otimes \psi' = \psi \otimes \phi'$ implies $x \cup y' = y \cup x'$, so, by the combination axiom it holds that

$$\begin{aligned} \phi^{\downarrow x \cap z} \otimes \psi' \quad &= \quad (\phi \otimes \psi')^{\downarrow (x \cup y') \cap z} \\ &= \quad (\psi \otimes \phi')^{\downarrow (y \cup x') \cap z} \\ &= \quad \psi \otimes \phi'^{\downarrow x' \cap z}. \end{aligned}$$

This shows that $(\phi^{\downarrow x \cap z}, \psi) =^* (\phi'^{\downarrow x' \cap z}, \psi')$, and also

$$h(\phi^{\downarrow x \cap z}) \otimes^* h(\psi)^{-1} \quad =^* \quad h(\phi'^{\downarrow x' \cap z}) \otimes^* h(\psi')^{-1},$$

so $=^*$ is indeed congruent with marginalization. Since $=^*$ is congruent with marginalization, it is sound to define

$$\mathcal{M}^*(\eta) \quad = \quad \{z \in \mathcal{M}(\eta') : \quad \eta' =^* \eta, d^*(\eta') = d^*(\eta)\}, \tag{21}$$

and for $z \in \mathcal{M}^*(\eta)$

$$\eta^{\downarrow^* z} \quad = \quad \eta'^{\downarrow^* z} \tag{22}$$

where $\eta' =^* \eta$, $d^*(\eta') = d^*(\eta)$, $z \in \mathcal{M}(\eta')$. Furthermore, the following properties hold [2]:

(A3′) (*Marginalization:*) For $\eta \in \Phi^*$, $x \in \mathcal{M}^*(\eta)$,

$$d^*(\eta^{\downarrow^* x}) = x. \tag{23}$$

(A4′) (*Transitivity:*) If $\eta \in \Phi^*$ and $x \subseteq y \subseteq d^*(\eta)$, then $x \in \mathcal{M}^*(\eta)$ implies $x \in \mathcal{M}^*(\eta^{\downarrow^* y})$ and $y \in \mathcal{M}^*(\eta)$ and

$$(\eta^{\downarrow^* y})^{\downarrow^* x} = \eta^{\downarrow^* x}.$$

(A5′) (*Combination:*) If $\eta, \kappa \in \Phi^*$ with $d^*(\eta) = x$, $d^*(\kappa) = y$ and $z \in D$ such that $x \subseteq z \subseteq x \cup y$, then $z \cap y \in \mathcal{M}^*(\kappa)$ implies $z \in \mathcal{M}^*(\eta \otimes^* \kappa)$ and

$$(\eta \otimes^* \kappa)^{\downarrow^* z} = \eta \otimes^* \kappa^{\downarrow^* z \cap y}.$$

(A6′) (*Domain:*) $\eta \in \Phi^*$ with $d^*(\eta) = x$ implies that $x \in \mathcal{M}^*(\eta)$ and

$$\eta^{\downarrow^* x} = \eta. \tag{24}$$

A structure $(\Phi^*, D, d^*, \otimes^*, \mathcal{M}^*, \downarrow^*)$ that satisfies (A1)-(A2),(A3′)-(A6′), and (A7) is called a *valuation algebra* (with partial marginalization) [5, 12]. Therefore, it is justified to refer to (Φ^*, D) as separative extension.

3 Embedding Gaussian Potentials

3.1 Gaussian Potentials

Consider a set r of variables. A Gaussian distribution over a finite subset s of these variables is determined by its *mean value vector* $\mu : s \to \mathbb{R}$ and the *concentration matrix* $K : s \times s \to \mathbb{R}$, the inverse of the variance-covariance matrix, which is assumed to be *positive definite*. Then, for $t \subseteq s$, $\mu^{\downarrow t}$ and $K^{\downarrow t}$ are the restrictions to t and $t \times t$, respectively. If, on the contrary $t \supseteq s$, then $\mu^{\uparrow t}$ and $K^{\uparrow t}$ denote the vector or matrix obtained from μ or K by setting $\mu(X) = 0$ and $K(X, Y) = 0$ for $X, Y \in t - s$.

Such a pair (μ, K), where both μ and K are relative to a finite subset $s \subseteq r$, is called a *Gaussian potential*. A Gaussian potential (μ, K), $s = d(\mu, K)$ represents a *Gaussian density*

$$\phi_{\mu,K}(\mathbf{x}) \quad = \quad \left|2\pi K^{-1}\right|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)' K (\mathbf{x}-\mu)}, \qquad \mathbf{x} \in \mathbb{R}^s.$$

The set s is the label of the potential, $d(\mu, K) = s$. Further, define the operation of combination between two Gaussian potentials (μ_1, K_1) and (μ_2, K_2) with domains s and t respectively as follows:

$$(\mu_1, K_1) \otimes (\mu_2, K_2) = (\mu, K), \tag{25}$$

where

$$K \quad = \quad K_1^{\uparrow s \cup t} + K_2^{\uparrow s \cup t}, \tag{26}$$

$$\mu \quad = \quad K^{-1} \left(K_1^{\uparrow s \cup t} \cdot \mu_1^{\uparrow s \cup t} + K_2^{\uparrow s \cup t} \cdot \mu_2^{\uparrow s \cup t} \right). \tag{27}$$

The Gaussian potential (μ, K) represents the Gaussian density

$$\phi_{\mu,K}(\mathbf{x}) \quad = \quad \phi_{\mu_1,K_1}(\mathbf{x}^{\downarrow s}) \cdot \phi_{\mu_2,K_2}(\mathbf{x}^{\downarrow t}) \tag{28}$$

for $\mathbf{x} \in \mathbb{R}^{s \cup t}$. Further, for a Gaussian potential (μ, K) on domain s, marginalization to a set $t \subseteq s$ is defined by

$$(\mu, K)^{\downarrow t} = (\mu^{\downarrow t}, \left((K^{-1})^{\downarrow t} \right)^{-1}), \tag{29}$$

which represents the marginal Gaussian density

$$\int\limits_{\mathbf{y} \in \mathbb{R}^{s-t}} \phi_{\mu,K}(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^t. \tag{30}$$

An alternative formula for marginalization is

$$\left((K^{-1})^{\downarrow t} \right)^{-1} \quad = \quad K^{\downarrow t} - K^{\downarrow t, s-t} (K^{\downarrow s-t})^{-1} K^{\downarrow s-t, t}, \tag{31}$$

see [1, 3]. According to (31), inversion of a matrix is not needed on the whole domain $d(\mu, K)$ but only on $d(\mu, K) - t$. Gaussian potentials can be shown to be a valuation algebra [5]. The identity element is $(\mu_\emptyset, K_\emptyset)$ for constant functions $\mu_\emptyset(\diamond) = K_\emptyset(\diamond, \diamond) = 0$, where $\diamond$ denotes the unique configuration of $\Omega_\emptyset$.

3.2 Pairs of Gaussian Potentials

Since the diagonal elements of a symmetric positive definite matrix are positive, the intersection of the domains of the sum $K = {K_1}^{\uparrow s \cup t} + {K_2}^{\uparrow s \cup t}$ of two concentration matrices can be uniquely determined, i.e. K_2 can be obtained unambiguously from K_1 and K, so the semigroup of Gaussian potentials is cancellative. Therefore, there is a separative extension of Gaussian potentials. Furthermore, the embedding of Gaussian potentials into pairs of Gaussian potentials can be simplified since

$$h(\mu, K) \quad = \quad [(\mu, 2K), (\mu, K)] \quad =^* \quad [(\mu, K), (\mu_\emptyset, K_\emptyset)]\,.$$

Then, a pair $[(\mu_1, K_1), (\mu_2, K_2)] \in \Phi^*$ can be marginalized to t (in the sense of $\mathcal{M}$) when $d(\mu_2, K_2) \subseteq t \subseteq d(\mu_1, K_1) \cup d(\mu_2, K_2)$.

An important case of a pair of Gaussian potentials is when it is of the form $(\phi, \phi^{\downarrow t})$ for some $\phi \in \Phi$ and $t \subseteq d(\phi) = s$. Such a pair represents a family $\{\phi_{\mu,K}(\mathbf{x}|\mathbf{y})\}_{\mathbf{y}\in\mathbb{R}^t}$ of *conditional Gaussian densities*

$$\phi_{\mu,K}(\mathbf{x}|\mathbf{y}) \quad = \quad \frac{\phi_{\mu,K}(\mathbf{x}, \mathbf{y})}{\int_{\mathbf{x}\in\mathbb{R}^{s-t}} \phi_{\mu,K}(\mathbf{x}, \mathbf{y}) d\mathbf{x}}, \qquad \mathbf{x} \in \mathbb{R}^{s-t}. \tag{32}$$

3.3 Symmetric Gaussian Potentials

Combination of Gaussian potentials essentially consists in addition of concentration matrices. So, since the denominator of a pair of Gaussian potentials corresponds to division, the inverse operation of combination, it is intuitive to ask whether this corresponds to some sort of substraction. The difference of two positive definite matrices is only symmetric in general. This is an *algebraic* motivation to consider such “concentration matrices”, which are symmetric but not necessarily positive definite. However, there is also an *analytic* motivation. Consider a conditional Gaussian density (32). Such a density is of the form

$$\begin{aligned} \phi_{\mu,K}(\mathbf{x}|\mathbf{y}) \quad &= \quad \frac{\left|2\pi K^{-1}\right|^{-\frac{1}{2}}}{\left|2\pi {K_t}^{-1}\right|^{-\frac{1}{2}}} \frac{e^{-\frac{1}{2}(\mathbf{x}-\mu)' K (\mathbf{x}-\mu)}}{e^{-\frac{1}{2}(\mathbf{x}^{\downarrow t}-\mu^{\downarrow t})' K_t (\mathbf{x}^{\downarrow t}-\mu^{\downarrow t})}} \\ &= \quad \frac{\left|2\pi K^{-1}\right|^{-\frac{1}{2}}}{\left|2\pi {K_t}^{-1}\right|^{-\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mu)' (K - K_t)(\mathbf{x}-\mu)} \end{aligned}$$

for $K_t = \left((K^{-1})^{\downarrow t}\right)^{-1}$. More precisely, is there a valuation algebra Φ^Δ whose “concentration matrices” are symmetric matrices and which is isomorphic to Φ^* (modulo $=^*$)? The answer is affirmative [1].

Define Φ^Δ to be the set of pairs (μ, K) for some finite set $s \subseteq r$ of variables where $\mu : \mathbb{R}^s \to \mathbb{R}$ and $K : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ is symmetric, i.e. $K(X, Y) = K(Y, X)$ for all $X, Y \in s$, and define $d^\Delta(\mu, K) = s$. Such a (μ, K) is called *symmetric Gaussian potential*. However, a technical problem arises when defining combination of such pairs: Since K has not to be positive definite any more, it is not necessarily invertible. Based on the mapping $h^\Delta : \Phi \to \Phi^\Delta$,

$$h^\Delta(\mu, K) \quad = \quad (K\mu, K), \qquad (\mu, K) \in \Phi, \tag{33}$$

combination $\otimes^\Delta$ in Φ^Δ may be defined by

$$\begin{aligned} (\mu_1, K_1) \otimes^\Delta (\mu_2, K_2) \quad = \quad & ({\mu_1}^{\uparrow x\cup y} + {\mu_2}^{\uparrow x\cup y}, \\ & {K_1}^{\uparrow x\cup y} + {K_2}^{\uparrow x\cup y}) \end{aligned} \tag{34}$$

for $(\mu_1, K_1), (\mu_2, K_2) \in \Phi^\Delta$, $x = d^\Delta(\mu_1, K_1)$ and $y = d^\Delta(\mu_2, K_2)$. The domain and marginalization operators $\mathcal{M}^\Delta$ and $\downarrow^\Delta$ in Φ^Δ can be (partially) defined for $(\mu, K) \in \Phi^\Delta$, $x = d^\Delta(\mu, K)$ by

$$\mathcal{M}^\Delta(\mu, K) \quad = \quad \{t : \quad K^{\downarrow x-t} \text{ positive definite}\}, \tag{35}$$

and, in light of (31), for $t \in \mathcal{M}^\Delta(\mu, K)$ and $s = x - t$

$$\begin{aligned} (\mu, K)^{\downarrow^\Delta t} \quad = \quad & (\mu^{\downarrow t} - K^{\downarrow t,s}(K^{\downarrow s})^{-1}\mu^{\downarrow s}, \\ & K^{\downarrow t} - K^{\downarrow t,s}(K^{\downarrow s})^{-1}K^{\downarrow s,t}) \end{aligned} \tag{36}$$

where $K^{\downarrow x_1,x_2}$ denotes the restriction of $K : \mathbb{R}^x \times \mathbb{R}^x \to \mathbb{R}$ to $K^{\downarrow x_1,x_2} : \mathbb{R}^{x_1} \times \mathbb{R}^{x_2} \to \mathbb{R}$ for $x_1, x_2 \subseteq x = d^\Delta(\mu, K)$. It can be shown [2] that the valuation algebra of pairs of Gaussian potentials and the valuation algebra of symmetric Gaussian potentials are isomorphic in the sense that there is a class isomorphism $i : \Phi^* \to \Phi^\Delta$ such that $\eta =^* \eta'$ and $d^*(\eta) = d^*(\eta')$ imply that

$$i(\eta) \quad = \quad i(\eta'), \tag{37}$$

and for all $\eta, \kappa \in \Phi^*$,

$$\begin{aligned} i(\eta \otimes^* \kappa) \quad &= \quad i(\eta) \otimes^\Delta i(\kappa), & (38) \\ \mathcal{M}^*(\eta) \quad &= \quad \mathcal{M}^\Delta(i(\eta)), & (39) \end{aligned}$$

and, for all $t \in \mathcal{M}^*(\eta)$,

$$i(\eta^{\downarrow^* t}) \quad = \quad i(\eta)^{\downarrow^\Delta t}. \tag{40}$$

Furthermore, note that $\mathcal{M}^*$ is defined only with respect to a whole equivalence class, whereas $\mathcal{M}^\Delta$ is defined in terms of a unique symmetric Gaussian potential, and combination does not require any matrix inversion. It is not possible to directly define a (labeled) quotient valuation algebra $\Phi^* / =^*$ since $=^*$ is not domain-congruent.

4 Conditionals and Construction Sequences

In practical applications, Gaussian potentials are often given as a product of conditional Gaussian densities, especially with models defined on the basis of probability networks such as Bayesian networks. A *conditional* for h given t is a valuation η in a separative extension Φ^*, $\eta =^* (\phi, \phi^{\downarrow t})$ for some $\phi \in \Phi$ and $t \subseteq d(\phi)$, $h = d(\phi) - t$. h is called a *head* and t a *tail* of η. Note that head and tail need not be unique [2], i.e. it may also be a conditional for some different head h' and tail t'. A conditional for $h = d^*(\eta)$ given $t = \emptyset$ is called a *density*. If $\eta \in \Phi^*$ is a conditional for h given t, then $t \in \mathcal{M}^*(\eta)$. However, for every subset $t' \subseteq t$, $t' \notin \mathcal{M}^*(\eta)$. Furthermore, densities in Φ^* are those elements which correspond to an element of Φ, i.e. $\eta \in \Phi^*$ being a density implies that there is a $\phi \in \Phi$ such that $\eta =^* h(\phi)$. It follows that densities are fully marginalizable. Note also that the product of two conditionals is a conditional again, and every marginal of a conditional is a conditional again, that is conditionals form a valuation subalgebra.

A *conditional Gaussian potential* is then a conditional η in the separative extension of Gaussian potentials or the corresponding symmetric Gaussian potential $i(\eta)$. It can be proved [1] that $(\mu, K) \in \Phi^\Delta$ is a conditional for h given t if and only if K is symmetric non-negative definite and if $K^{\downarrow h}$ is positive definite and has the same rank as K. In the case of symmetric non-negative definite matrices, any principal submatrix of the same rank as K is positive definite [3], so here, head and tail are not unique.

A sequence $\eta_1, \eta_2, \ldots, \eta_n$ of conditionals with heads h_i and tails t_i, $i = 1, \ldots, n$ is called *construction sequence* [13, 5] if

- $t_1 = \emptyset$,
- $t_i \subseteq (d^*(\eta_1) \cap \ldots \cap d^*(\eta_{i-1}))$, and
- $h_i \cap (d^*(\eta_1) \cap \ldots \cap d^*(\eta_{i-1})) = \emptyset$.

A construction sequence factors a density into conditionals, i.e.

$$\eta \quad = \quad \eta_1 \otimes^* \eta_2 \otimes^* \cdots \otimes^* \eta_n$$

is a density [2]. Moreover, for every $k = 1, \ldots, n$, it holds that

$$\eta_{1,\ldots,k} \quad = \quad \eta_1 \otimes^* \eta_2 \otimes^* \cdots \otimes^* \eta_k$$

is a density [2].

If the factors $\eta_1, \eta_2, \ldots, \eta_n$ on a join tree form a construction sequence, then all marginals of $\eta = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_n$ are well defined. However, since marginals of conditionals are only partially defined, it has to be verified that the algorithms for local computation can be executed. Indeed, if the factors form a construction sequence, then local computation is possible [13, 5, 12].

5 Conclusion

It has been shown that valuation algebras are suitable for an algebraic understanding of conditional Gaussian distributions. Starting from the cancellative semigroup of Gaussian potentials, an algebra of pairs of Gaussian potentials has been defined, where marginalization is defined only partially. An equivalence relation has been introduced into this valuation algebra and shown to be congruent with combination and marginalization, allowing to extend marginalization, such that a pair can be marginalized whenever an equivalent pair can be marginalized [2]. An alternative representation of such pairs as symmetric Gaussian potentials has been shown. Here, the classes of equivalent elements of the same domain are mapped to a unique symmetric Gaussian potential [1], so the domain operator can be defined in terms of a single potential and not of a whole class of potentials. Construction sequences have been introduced, which factor a Gaussian potential into conditional Gaussian potentials. Furthermore, it has been shown elsewhere [13, 5, 12] that local computation can be applied to such a construction sequence.

Future work should include an implementation of symmetric Gaussian potentials, as well as showing the relationship between conditional Gaussian potentials and Gaussian hints [11, 6], as well as between symmetric Gaussian potentials and extended matrices of Gaussian belief functions [9, 10].

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