Topological Dynamics and Decidability of Infinite Constraint Satisfaction

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Puzzle 1

$$a, b, c, d, \ldots \in \mathbb{A}$$

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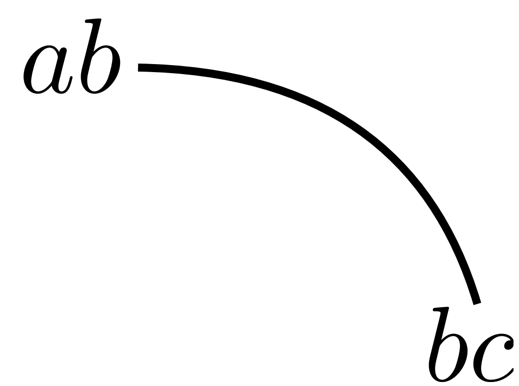

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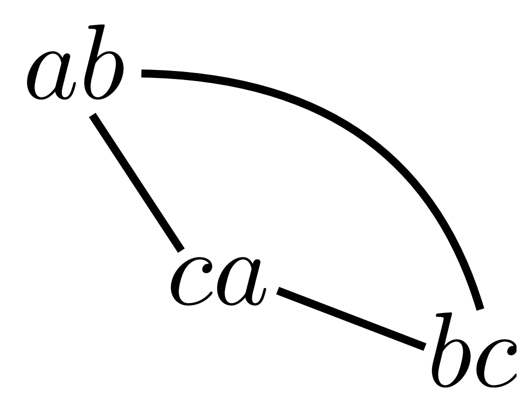

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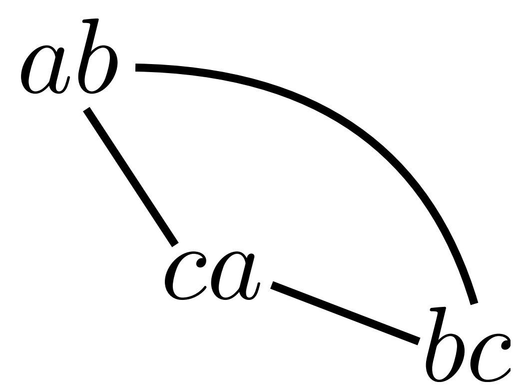
Is it 3-colorable?



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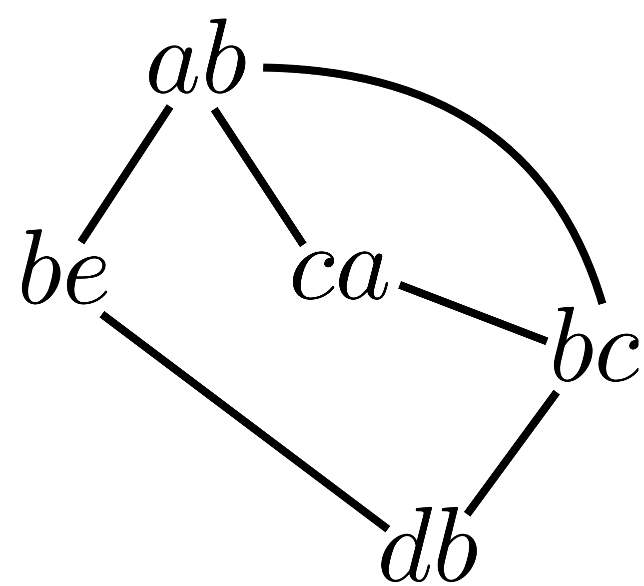

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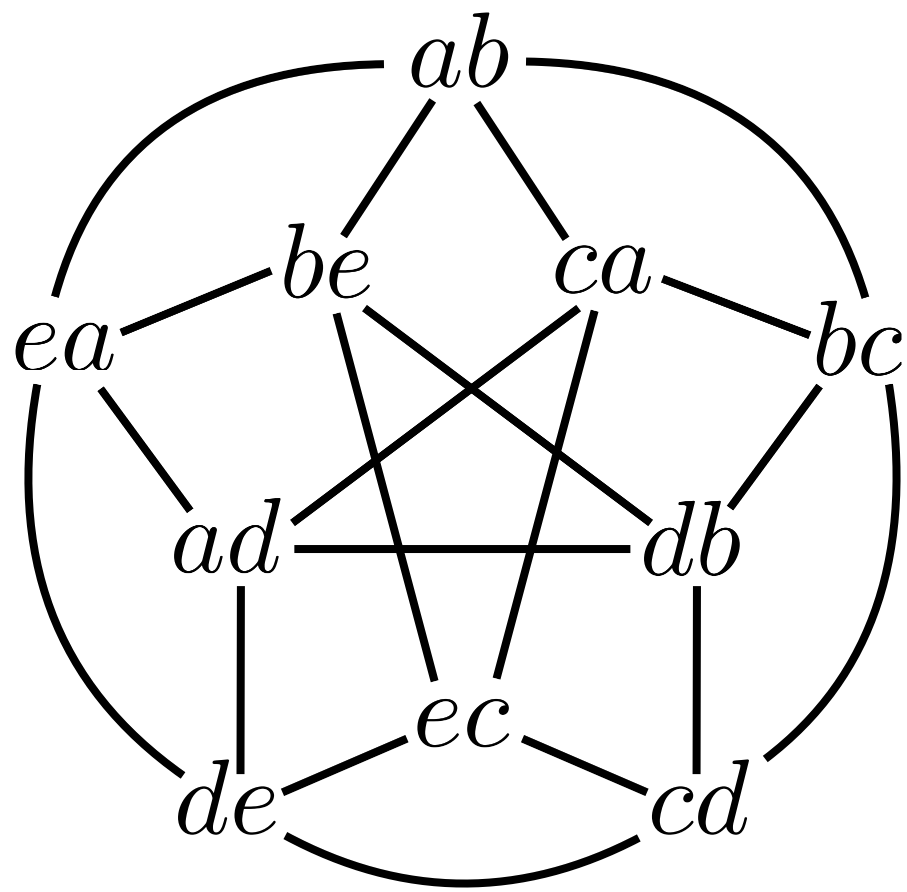

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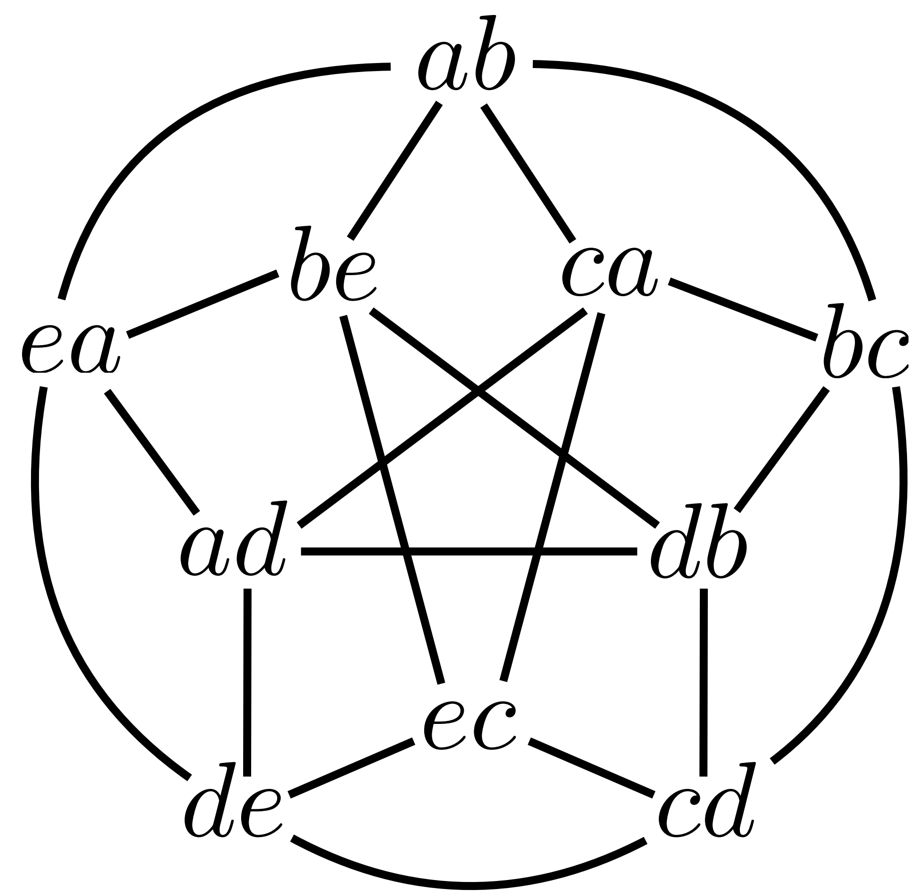
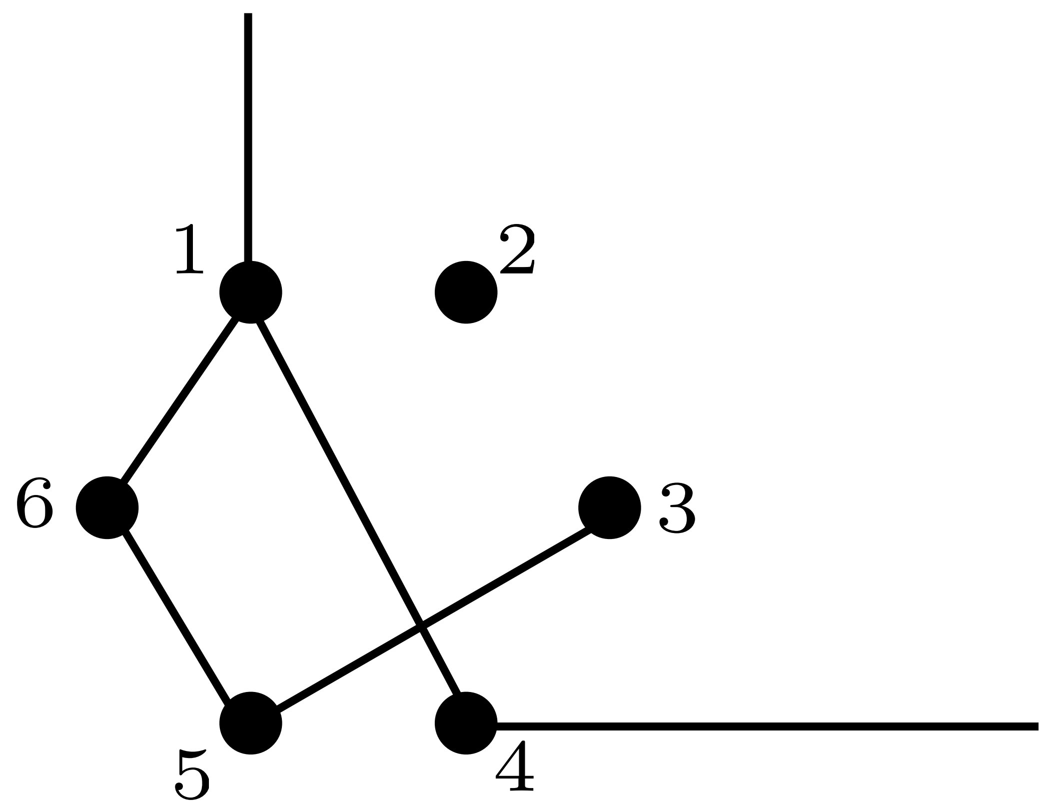


No.

Is 3-colorability decidable?

Doubly periodic graphs



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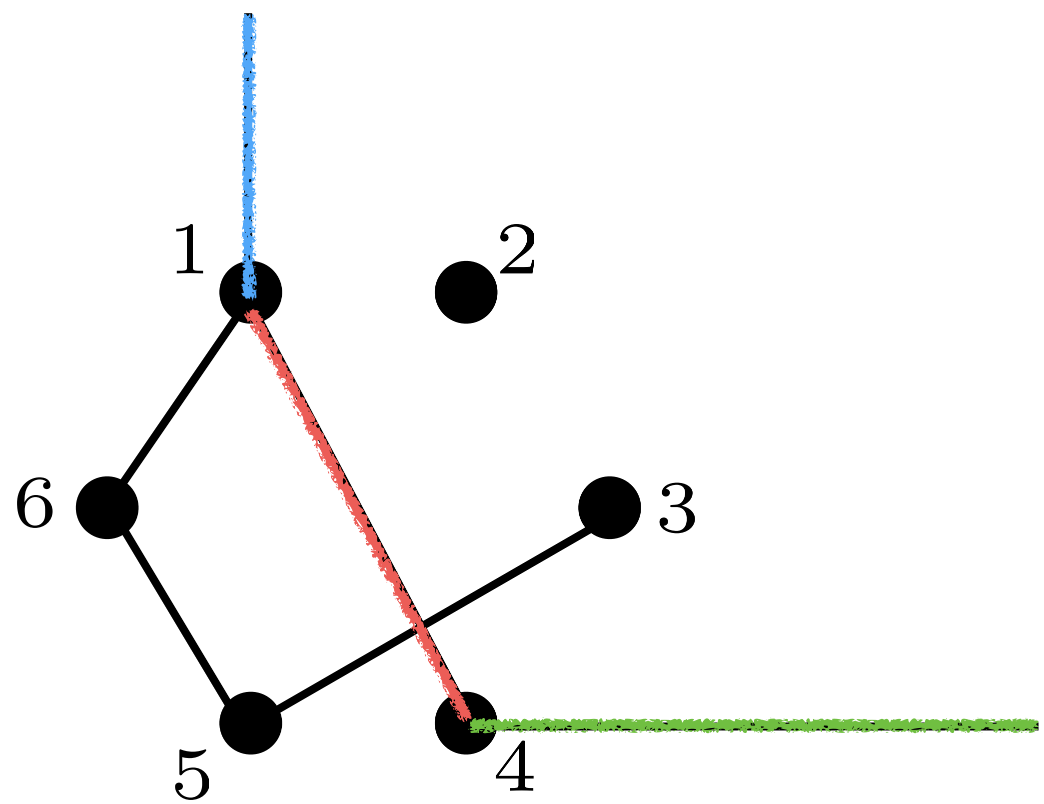

v_{ij1} — v_{ij4}

v_{ij1} — $v_{i(j+1)5}$

v_{ij4} — $v_{(i+1)j5}$

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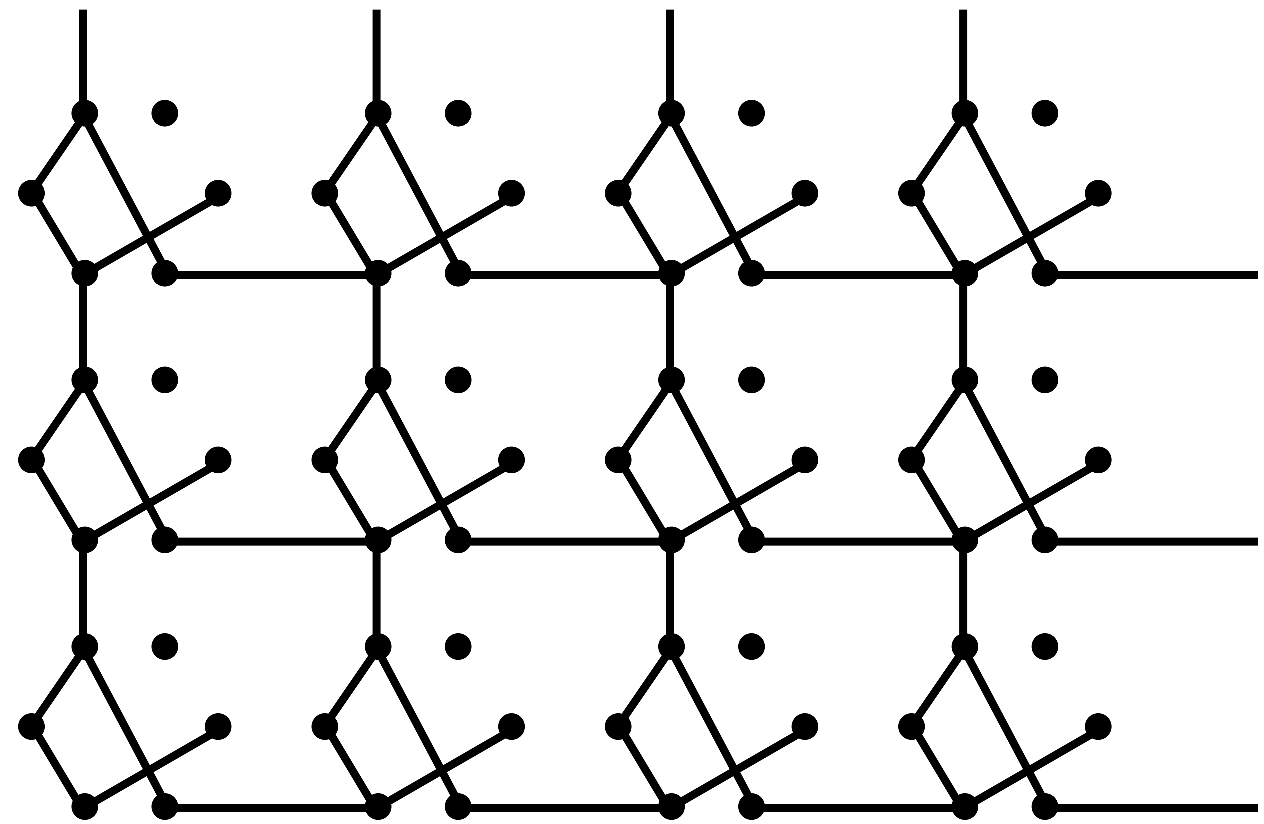

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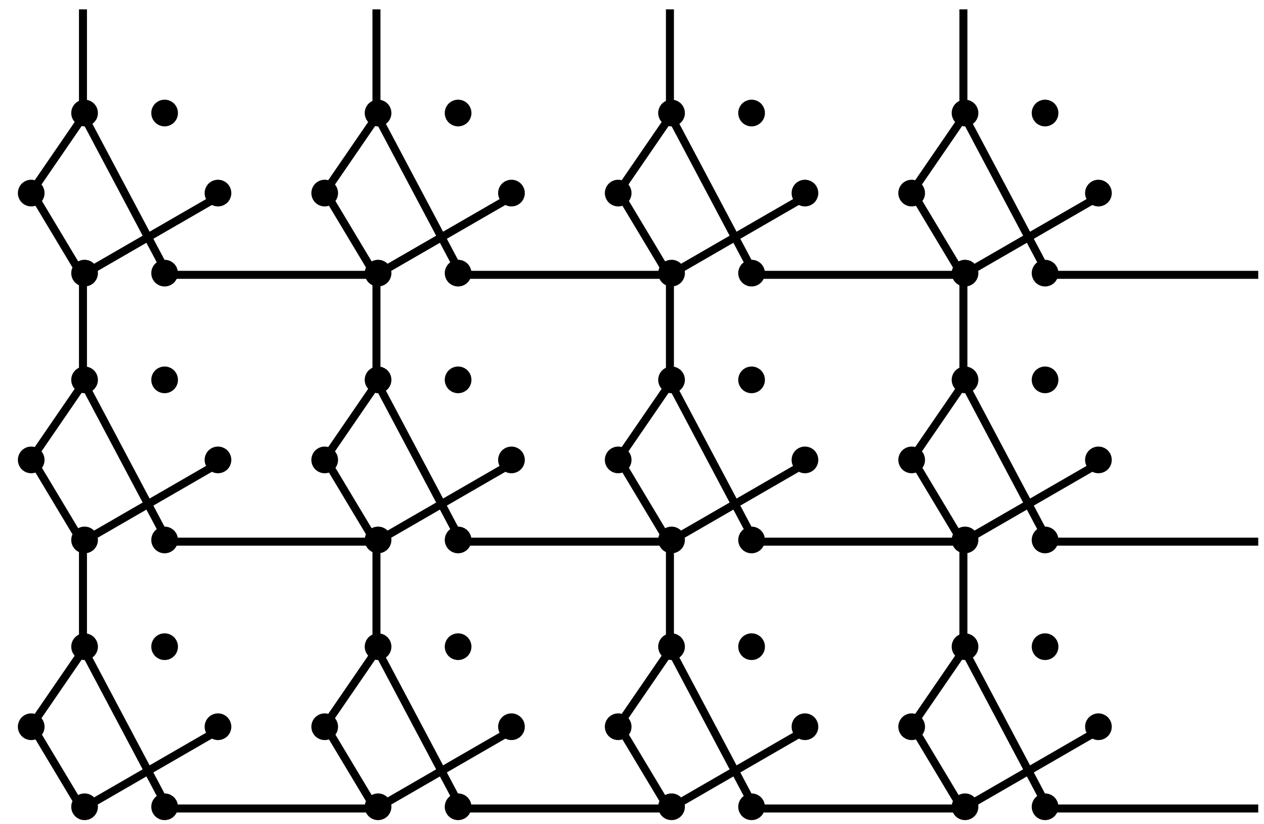

Thm: [Freedman'98]

3-colorability of doubly periodic graphs is undecidable.

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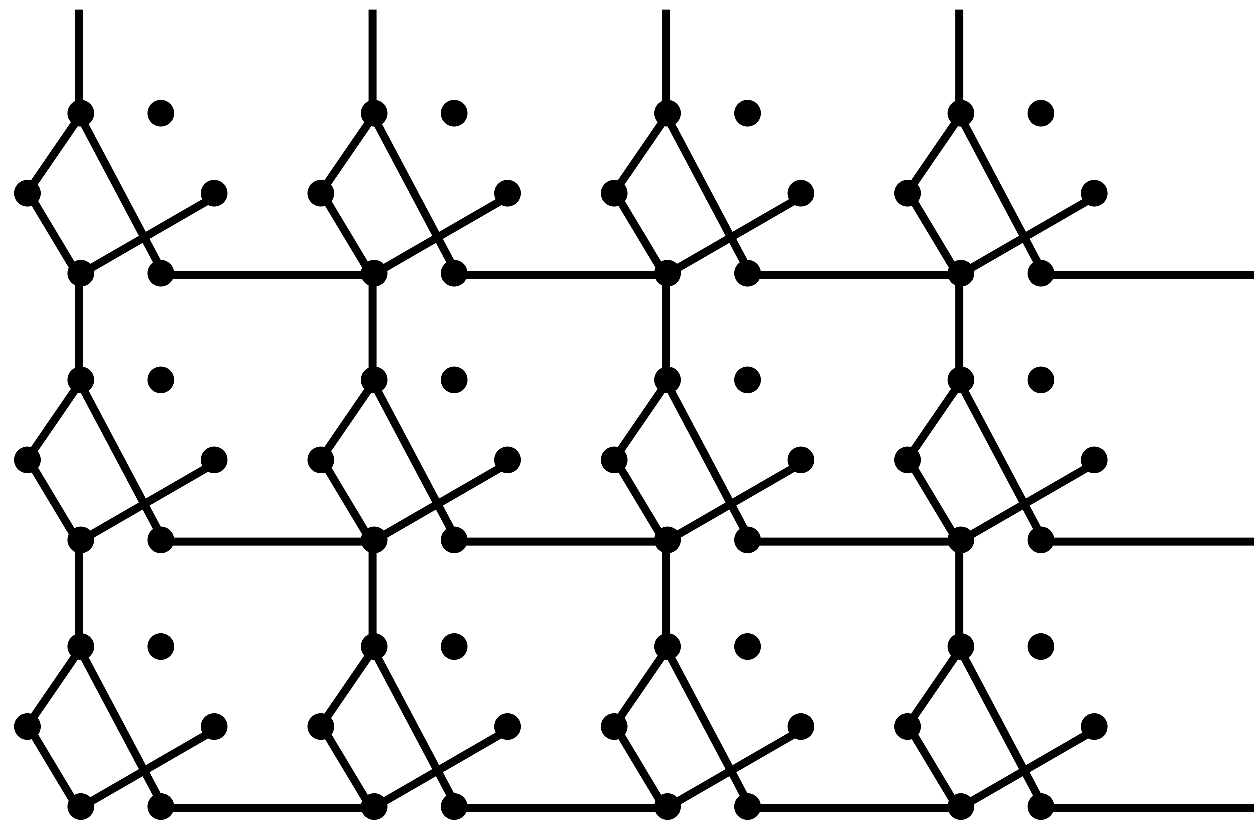

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What if we only use $=$?

Puzzle 11

A system of equations over $\mathbb{Z}_2$:

- variables: ab $a \neq b$
- equations: $ab + bc + ca = 0$

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ca + ae + ec &= 0\\
ac + cd + da &= 0\\
cb + be + ec &= 0\\
db + be + ed &= 0\\
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\end{aligned}$$

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Topological dynamics

For decidability, we will use

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Every continuous action of $\mathrm{Aut}(\mathbb{Q},<)$ on a non-empty compact space has a fixpoint.

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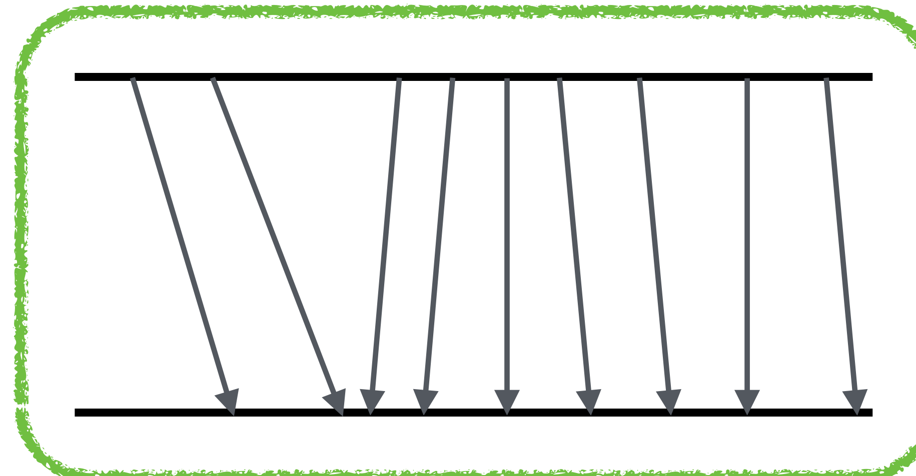

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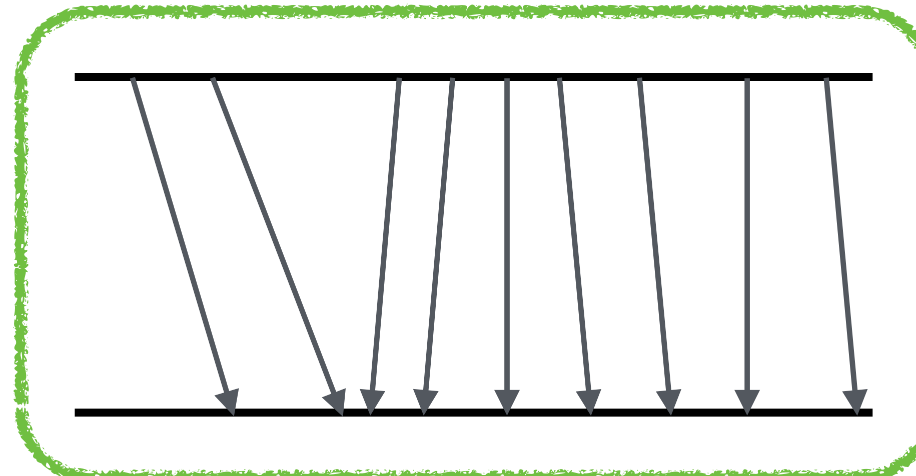

Thm: [Pestov'98]

Every continuous action of $\boxed{\mathrm{Aut}(\mathbb{Q},<)}$ on a non-empty compact space has a fixpoint.

(proof: by Ramsey Theorem)

CSPs

An instance:

- a set V of variables
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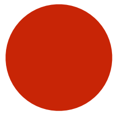
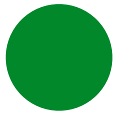
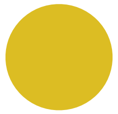
Decidable, NP-complete

CSP examples

Graph 3-colorability:

variables - nodes

values - , , 

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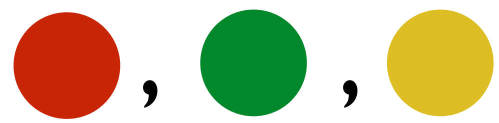
values - 

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Solving equations mod 2:

variables - variables

values - 0 , 1

constraints - $(x, y, z; R)$, f.e. eqn. $x + y + z = 0$

$$R \subseteq \{0,1\}^3$$

$$R = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$$

CSPs and homomorphisms

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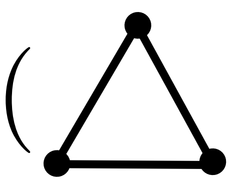
Instance solutions = homomorphisms $\mathcal{A} \to \mathcal{B}$

CSPs for fixed templates

$\mathcal{B}$ - a relational structure

$\mathrm{CSP}(\mathcal{B})$ - instances with template $\mathcal{B}$

Example:

3-colorability = $\mathrm{CSP}($  $)$

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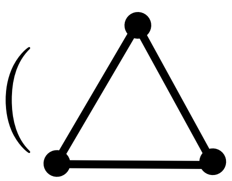
3-colorability = $\mathrm{CSP}($  $)$

Conjecture: [Feder-Vardi'98]

For any $\mathcal{B}$, the problem $\mathrm{CSP}(\mathcal{B})$
is eitner NP-complete or in PTIME

Infinite CSPs

- infinite templates ($\mathcal{B}$)

  -- example: instance = set of triples $(x, y, z) \in V^3$

  solution = $f : V \to \mathbb{Q}$ such that

  $x < y < z$ or $z < y < x$

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Equivariant functions: $f : X \to Y$ such that

$$f(\pi \cdot x) = \pi \cdot f(x)$$

Orbit-finite sets

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Define orbit-finite sets by set expressions:

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- set builders $\{e : v_1, \ldots, v_k \in \mathbb{A} : \phi\}$

  expression

  variables

  f.o. formula with =

- unions, tuples

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Examples:

$$\{(a, b) : a, b \in \mathbb{A} : a \neq b\}$$

$$\{\{(a, b, c), (b, c, a), (c, a, b)\} : a, b, c \in \mathbb{A} : \top\}$$

CSPs with atoms

An instance:

- a set V of variables
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- a set of constraints: $(\underbrace{x_1, \ldots, x_k}_{\substack{\cap \\ V}}; \underset{\substack{\cap \\ T^k}}{R})$

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equivariant?

Examples

Graph from Puzzle I:

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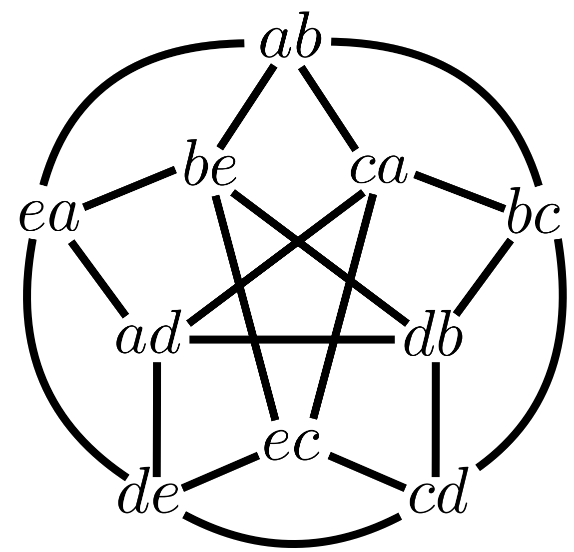

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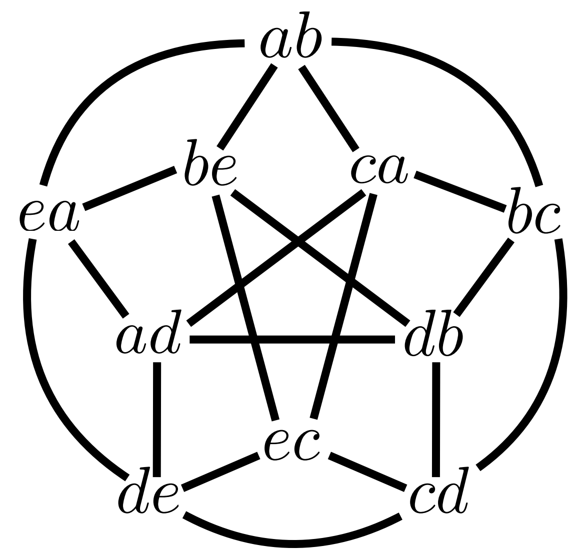

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Examples

Equation system from Puzzle II:

$$\begin{aligned} ab + bc + ca &= 0 \\ ab + ba &= 1 \end{aligned}$$

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$$
\begin{gathered}
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\cup \\
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$$

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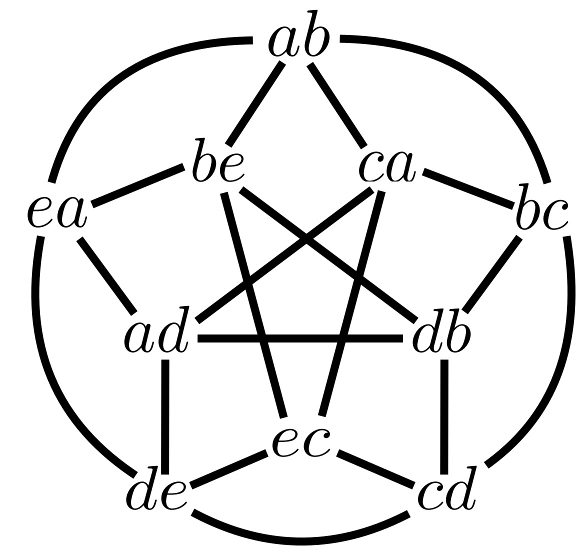
Complexity: exponential slowdown

- 3-colorability: NEXPTIME-complete
- equation solving: EXPTIME

Examples

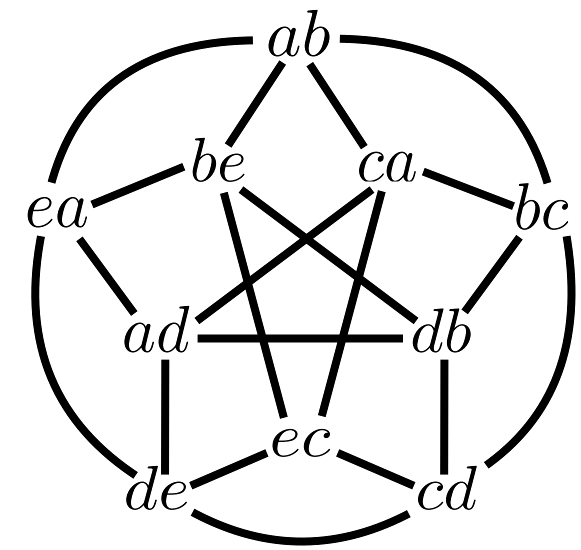
Puzzle I:



Examples

Puzzle I:

- one orbit of nodes



Examples

Puzzle I:

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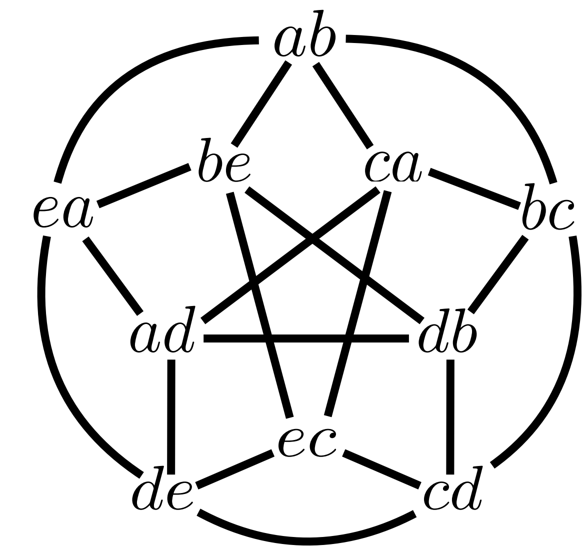

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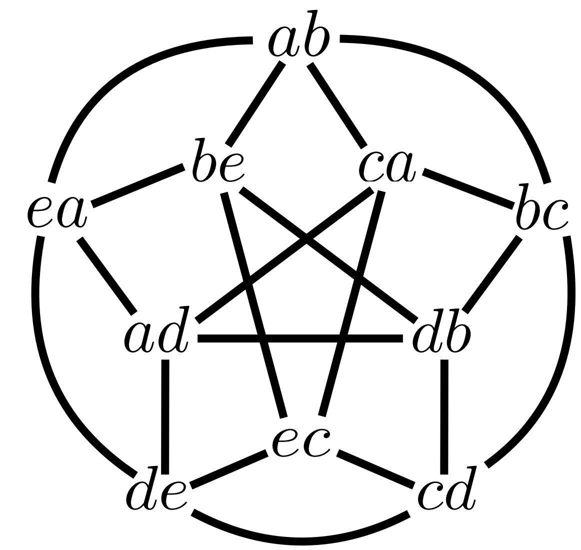

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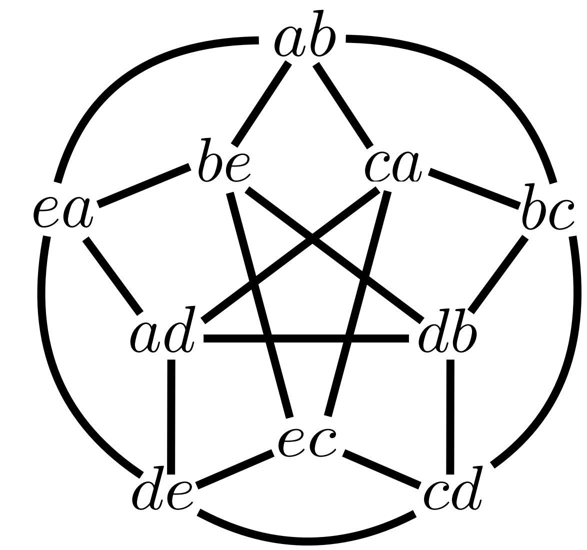

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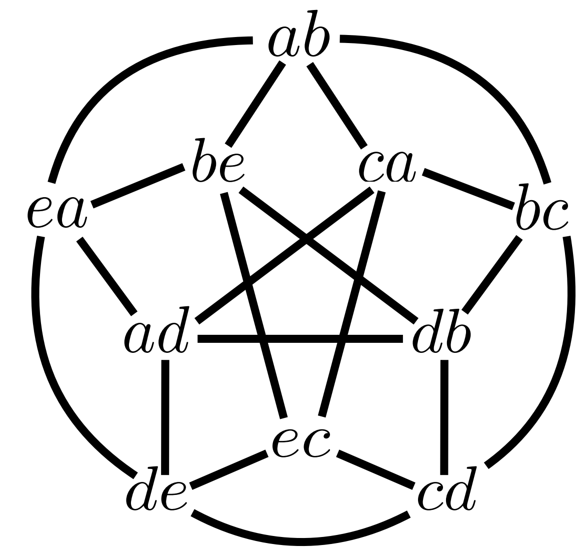

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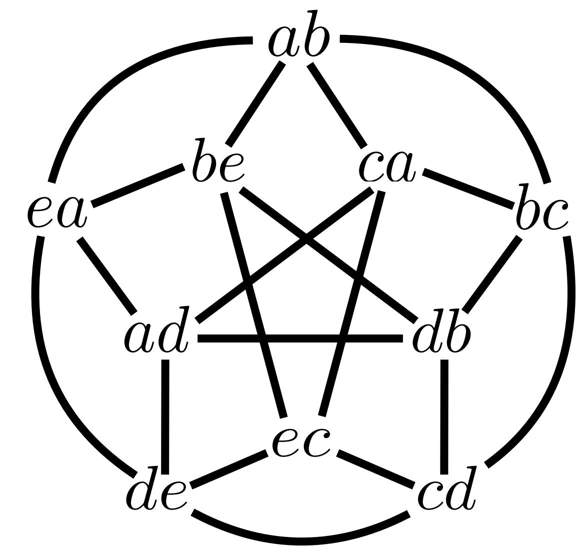

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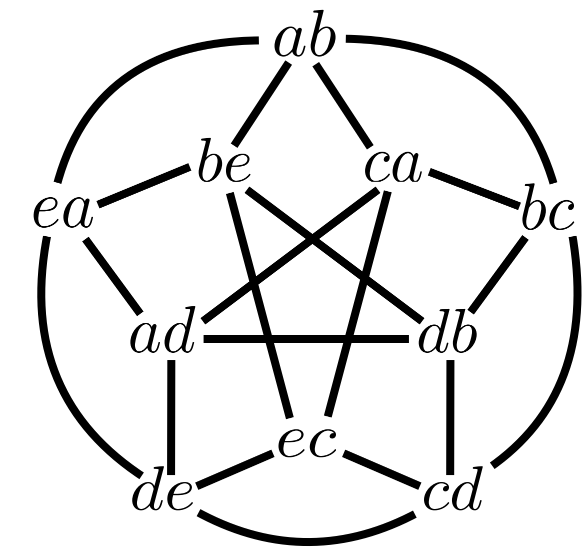

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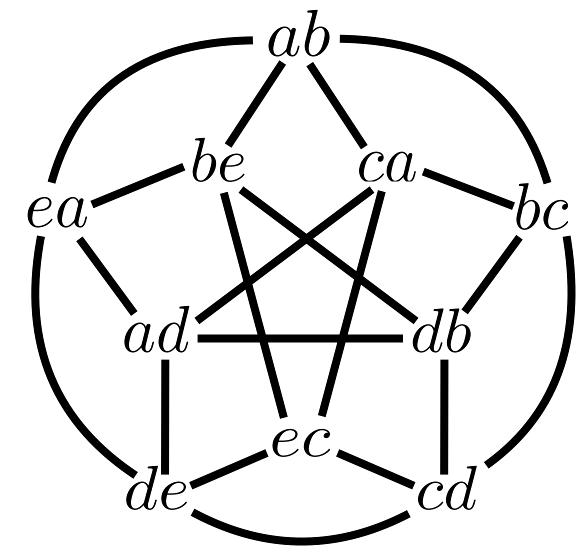

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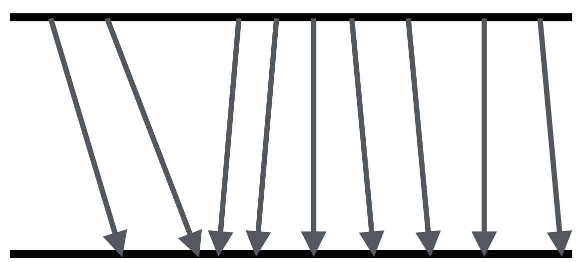

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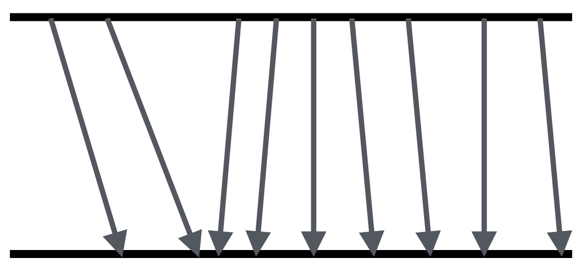

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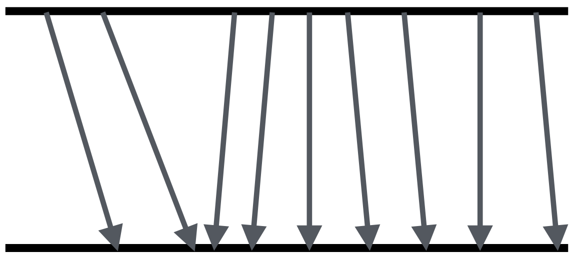

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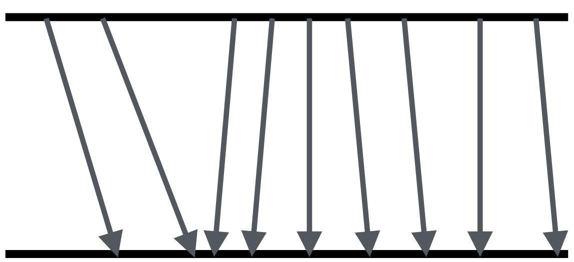

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- definable sets remain definable
- equivariant functions are monotone-equivariant

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And existence of monotone-equivariant solutions is decidable!

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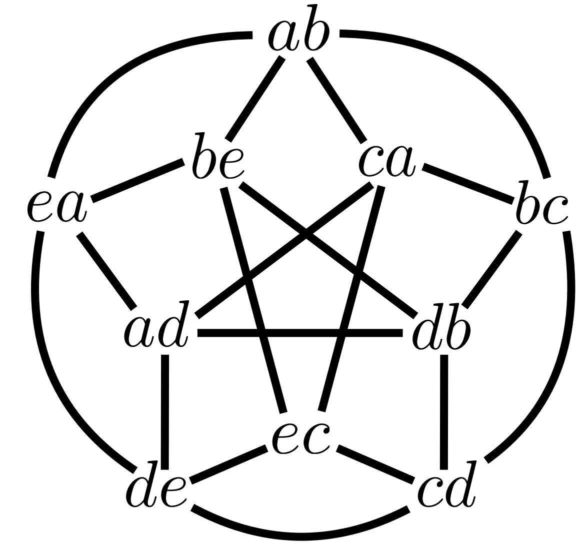

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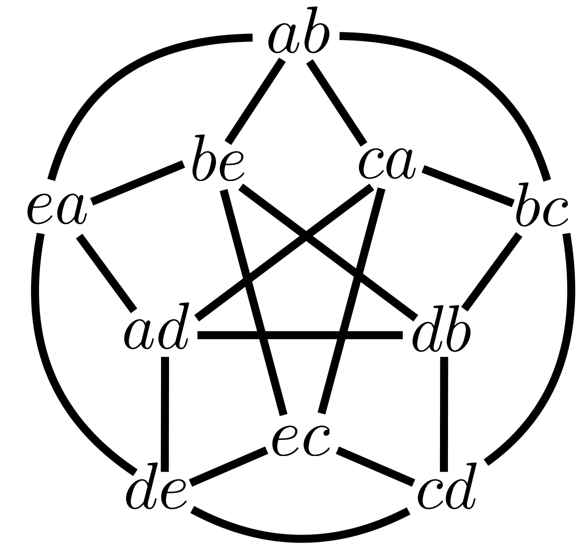

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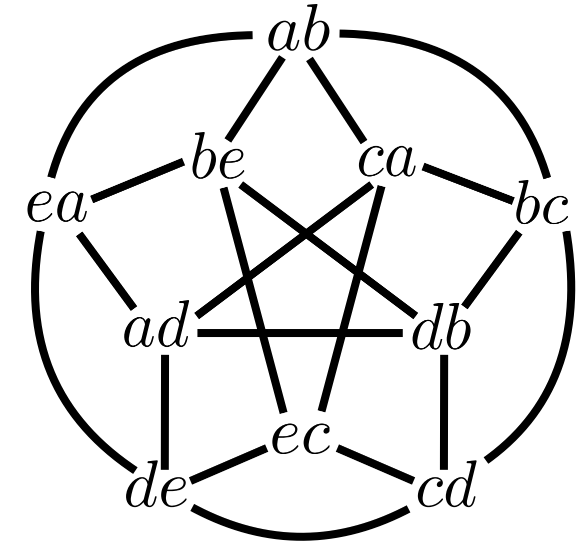

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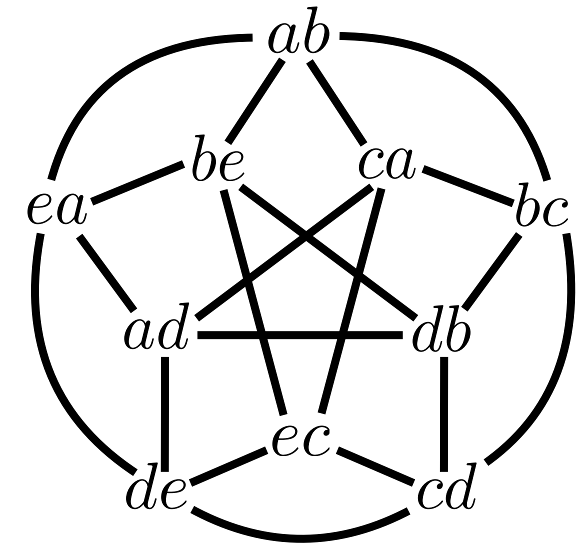

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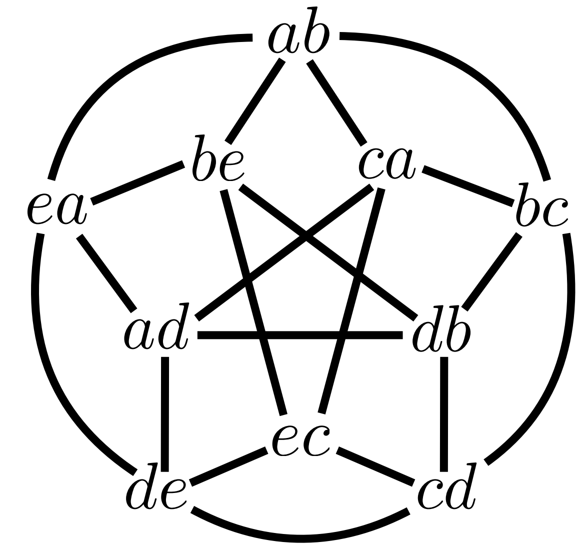

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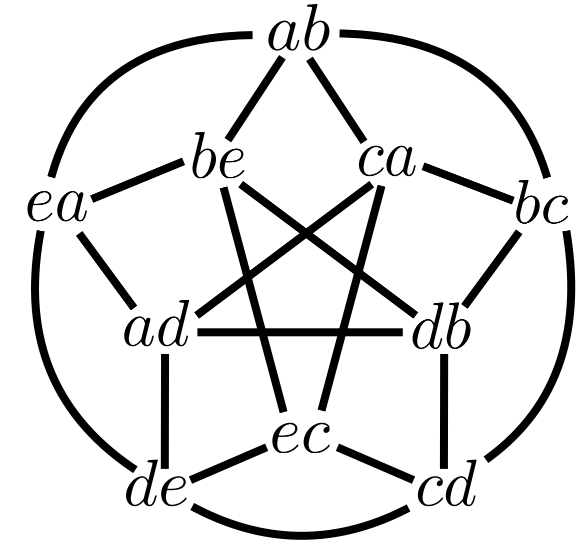

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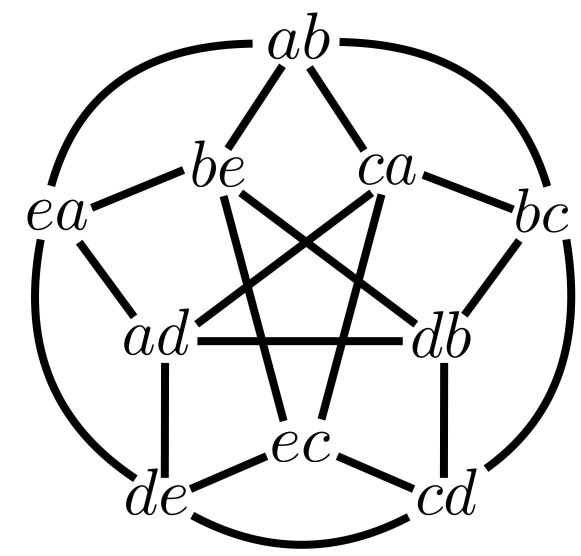

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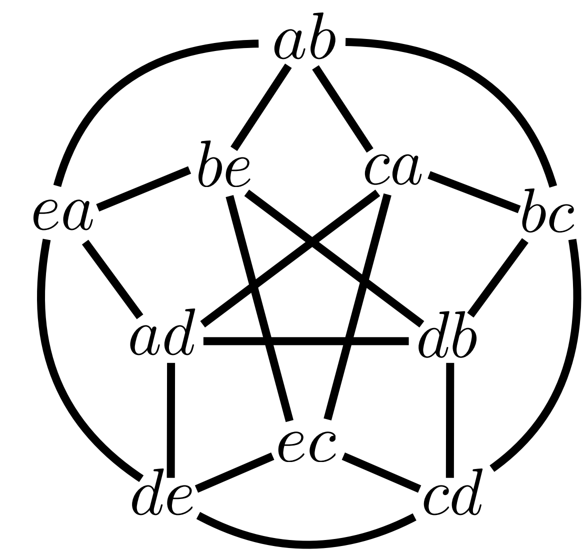

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Examples: for definable instances,

- 3-colorability is NEXPTIME-complete
- Horn-SAT is EXPTIME-complete
- 2-colorability is PSPACE-complete
- etc.

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Orbit-finite CSPs: the following is undecidable:

Given orbit-finite relational structures $\mathcal{A}$ and $\mathcal{B}$, is there a homomorphism from $\mathcal{A}$ to $\mathcal{B}$?

Paper

B. Klin, E. Kopczyński, J. Ochremiak, S. Toruńczyk:
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