Logics for Non-Cooperative Games with Expectations

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- We introduce the logics $\mathsf{E}(\mathfrak{G})$ for representing probabilistic expectation over classes $\mathfrak{G}$ of Łukasiewicz games.

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- Based on these logics we define a new class of games where players aim is to randomise their strategic choices in order to affect the other players' expectations over an outcome as well as their own.
- We characterise the complexity for both the logics $\mathsf{E}(\mathfrak{G})$ and the newly introduced games.

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- The actions available to players correspond to valuations that can be made to variables under their control.
- The use of Łukasiewicz logics makes it possible to more naturally represent much richer payoff functions for players.

Łukasiewicz Logics I

R. Cignoli, I. M. L. D'Ottaviano, D. Mundici. *Algebraic Foundations of Many-alued Reasoning*

- Language:

$$\phi \oplus \psi \quad | \quad \neg\phi \quad | \quad \overline{0}$$

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- (Infinite-Valued Łukasiewicz Logic $Ł_\infty$) Valuations $v : Var \to [0,1]$:

$$v(\phi \oplus \psi) = \min(v(\phi) + v(\psi), 1) \quad | \quad v(\neg\phi) = 1 - v(\psi) \quad | \quad e(\overline{0}) = 0$$

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- Ł$_\infty$ is complete with respect to valuations into $[0, 1]$.
- In Ł$_\infty$ functions definable by formulas exactly coincide with the class of continuous piecewise linear polynomial functions on $[0, 1]^n$.

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- Each $Ł_k$ is complete with respect to valuations into L_k.
- Łukasiewicz games are defined using the logics $Ł_k^c$, which are obtained from $Ł_k$ by adding constants $\overline{c}$ for each $c \in L_k$.
- The class of functions definable in $Ł_k^c$ coincides with the set of all functions $f : (L_k)^n \to L_k$, for all n.

Łukasiewicz Games: Definition I

E. M., M. Wooldridge. Łukasiewicz Games (2013).

A Łukasiewicz game $\mathcal{G}$ on $Ł_k^c$ is a tuple

$$\mathcal{G} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\phi_i\} \rangle$$

where:

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2. $\mathsf{V} = \{p_1, p_2, \ldots\}$ is a finite set of propositional variables;
3. $\mathsf{V}_i \subseteq \mathsf{V}$ is the set of propositional variables under control of player P_i, so that the sets V_i form a partition of V.

Łukasiewicz Games: Definition II

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4. S_i is the strategy set for player i that includes all valuations $s_i : \mathsf{V}_i \to L_k$ of the propositional variables in V_i, i.e.

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5. $\phi_i(p_1, \ldots, p_t)$ is an $Ł_k^c$-formula, built from variables in V, whose associated function

$$f_{\phi_i} : (L_k)^t \to L_k$$

corresponds to the *payoff function* of P_i, and whose value is determined by the valuations in $\{\mathsf{S}_1, \ldots, \mathsf{S}_n\}$.

EXAMPLES

- Traveler's Dilemma.
- Auctions.
- Coordination Games.
- Matching Pennies.
- Weak-Link Games.

Łukasiewicz Games: Classes I

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- The *type* of $\mathcal{G}$ is the triple $\langle n, m, \delta \rangle$, where
  1. n is the number of players,
  2. m is the number of variables in V,
  3. δ is the variable distribution function for $\mathcal{G}$.

Łukasiewicz Games: Classes II

- Let $\mathcal{G}$ and $\mathcal{G}'$ be two games of type $\langle n, m, \delta \rangle$ and $\langle n, m, \delta' \rangle$.

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- $\mathcal{G}$ and $\mathcal{G}'$ belong to the same class $\mathfrak{G}$ if there exists a permutation $\jmath$ of the indices $\{1, \ldots, n\}$ such that, for all P_i,

$$\delta\left(P_{\jmath(i)}\right) = \delta'(P_i).$$

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- What matters is the distribution of the variables rather than which variables are assigned to the players.

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$$exp_{\varphi_i}(\pi_i, \pi_{-i}) = \sum_{\vec{s}=(s_1,\ldots,s_n)\in\mathsf{S}} \left(\left(\prod_{j=1}^{n} \pi_j(s_j) \right) \cdot f_{\varphi_i}(\vec{s}) \right)$$

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- Modal formulas are build from the atomic ones with the connectives of $Ł\Pi\frac{1}{2}$.
- Nested modalities are not allowed.

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F. Esteva, L. Godo, F. Montagna. The ŁΠ and ŁΠ1/2 logics: two complete fuzzy logics joining Łukasiewicz and product logic.

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$$v(\phi \odot \psi) = v(\phi) \cdot v(\psi) \quad | \quad v(\phi \to \psi) = \begin{cases} 1 & v(\phi) \leq v(\psi) \\ \frac{v(\psi)}{v(\phi)} & \end{cases}$$

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- ŁΠ$\frac{1}{2}$ is complete with respect to valuations into $[0,1]$.
- ŁΠ$\frac{1}{2}$ is the logic of piecewise rational functions on $[0,1]^n$.

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2. $e : (\mathrm{NModF} \times \mathsf{S}) \to L_k$ is a valuation of non-modal formulas, such that, for each $\varphi \in \mathrm{NModF}$

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3. $\pi_i : \mathsf{S}_i \to [0, 1]$ is a probability distribution, for each P_i.

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$$\|\mathsf{E}\varphi\|_{\mathbf{M},\vec{s}} = exp_{\varphi}(\pi_1, \ldots, \pi_n) = \sum_{\vec{s}=(s_1,\ldots,s_n)\in \mathsf{S}} \left(\left(\prod_{j=1}^{n} \pi_j(s_j) \right) \cdot e(\varphi, \vec{s}) \right).$$

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A modal formula Φ is satisfiable if there exists a model $\mathbf{M}$ such that $\|\Phi\|_{\mathbf{M},\vec{s}} = 1$.

Axiomatization

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3. Probabilistic axioms for E, with $\varphi, \psi, \bar{r} \in$ NModF:

   3.1 $\mathsf{E}(\neg\varphi) \leftrightarrow \neg\mathsf{E}\varphi$
   3.2 $\mathsf{E}(\varphi \oplus \psi) \leftrightarrow [(\mathsf{E}\varphi \to \mathsf{E}(\varphi \& \psi)) \to \mathsf{E}\psi]$
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4. Independence axioms for E, where $p_{1_i}, \ldots, p_{m_i}$ is the tuple of variables assigned to P_i, for all tuples $r_{1_1}, \ldots, r_{m_1}, \ldots, r_{1_n}, \ldots, r_{m_n} \in (L_k)^m$:

   4.1 $\mathsf{E}\left(\bigwedge_{i=1}^{n}\left(\bigwedge_{j_i=1_i}^{m_i}\left(\Delta\left(p_{j_i} \leftrightarrow \bar{r}_{j_i}\right)\right)\right)\right) \leftrightarrow \bigodot_{i=1}^{n}\left(\mathsf{E}\left(\bigwedge_{j_i=1_i}^{m_i}\Delta\left(p_{j_i} \leftrightarrow \bar{r}_{j_i}\right)\right)\right)$

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   4.1 $\mathsf{E}\left(\bigwedge_{i=1}^{n}\left(\bigwedge_{j_i=1_i}^{m_i}\left(\Delta\left(p_{j_i} \leftrightarrow \bar{r}_{j_i}\right)\right)\right)\right) \leftrightarrow \bigodot_{i=1}^{n}\left(\mathsf{E}\left(\bigwedge_{j_i=1_i}^{m_i}\Delta\left(p_{j_i} \leftrightarrow \bar{r}_{j_i}\right)\right)\right)$

5. The following inference rules for E, with $\varphi, \psi \in$ NModF:

   5.1 Necessitation: from φ derive $\mathsf{E}\varphi$
   5.2 Monotonicity: from $\varphi \to \psi$ derive $\mathsf{E}\varphi \to \mathsf{E}\psi$

FUNCTIONAL REPRESENTATION

Let $\mathfrak{G}$ be a class of Łukasiewicz games on $Ł_k^c$ and let $mŁ_k^c$ be the m-variable fragment of $Ł_k^c$. The following statements are equivalent:

Functional Representation

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1. There exists a state $\sigma : mŁ_k^c \to [0,1]$ such that for all tuples $r_{1_1}, \ldots, r_{m_1}, \ldots, r_{1_n}, \ldots, r_{m_n} \in (L_k)^m$

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2. There exists a probability distribution $\pi_i : \mathsf{S}_i \to [0,1]$ for each P_i, such that, for all $\varphi \in mŁ_k^c$,

$$\sigma(\varphi) = exp_{\varphi}(\pi_1, \ldots, \pi_n).$$

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1. $\Gamma \vdash_{\mathsf{E}(\mathfrak{G})} \Phi$.
2. For every model $\mathbf{M}$ such that, for each $\Psi \in \Gamma$,
$$\|\Psi\|_{\mathbf{M}} = 1,$$
also
$$\|\Phi\|_{\mathbf{M}} = 1.$$

Outline

GAMES WITH EXPECTATIONS

A game with expectations $\mathcal{E}_{\mathcal{G}}$ on $\mathsf{E}(\mathfrak{G})$ is a tuple

$$\mathcal{E}_{\mathcal{G}} = \langle \mathsf{P}, \mathsf{V}, \{\mathsf{V}_i\}, \{\mathsf{S}_i\}, \{\varphi_i\}, \{\mathsf{M}_i\}, \{\Phi_i\} \rangle,$$

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3. Φ_i is an $\mathsf{E}(\mathfrak{G})$-formula such that every atomic modal formula occurring in Φ_i has the form $\mathsf{E}\psi$, with $\psi \in \{\varphi_1, \ldots, \varphi_n\}$.

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- A game with expectations $\mathcal{E}_{\mathcal{G}}$ on $\mathsf{E}(\mathfrak{G})$ is said to have a *Nash Equilibrium*, whenever there exists a model $\mathbf{M}^*$ that is a best response model for each player P_i.

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- By Nash's Theorem, every $\mathcal{E}_{\mathcal{G}}$ of this form admits an Equilibrium

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- While P_1 aims at matching P_2's expectation, P_2's goal is quite the opposite, since she wants their expectations to be as far as possible.
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- Checking the existence of an equilibrium for a game with expectation is in 2-EXPTIME.

THANKS!