Positional Scoring Rules for the Allocation of Indivisible Goods

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Fair division of indivisible goods

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- an allocation $\overrightarrow{\pi} : \mathcal{A} \to 2^{\mathcal{O}}$
- such that $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$ (preemption),
- $\bigcup_{i \in \mathcal{A}} \pi_i = \mathcal{O}$ (no free-disposal),
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Plenty of real-world applications: course allocation, operation of Earth observing satellites, …

Centralized allocation

A classical way to solve the problem:

- Ask each agent i to give a score (weight, utility...) $w_i(o)$ to each object o
- Consider all the agents have **additive** preferences

$$\rightarrow u_i(\pi) = \sum_{o \in \pi} w_i(o)$$

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  - the lexicographic minimum over $(u_1(\pi), \dots, u_n(\pi))$ – refinement of the egalitarian solution

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Our starting point: *What can we do with ordinal preferences?*

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k-Approval	1	1	0	0	0	0

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5 objects, 3 agents...

- $1 : o_1 \succ o_2 \succ o_3 \succ o_4 \succ o_5$
- $2 : o_4 \succ o_2 \succ o_5 \succ o_1 \succ o_3$
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Let's consider allocation $\pi = \langle \{o_1\}, \{o_4, o_2\}, \{o_3, o_5\} \rangle$.

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- **Borda:** $u_1(\pi) = 5$; $u_2(\pi) = 5 + 4 = 9$; $u_3(\pi) = 4 + 3 = 7$.
- **Lexicographic:** $u_1(\pi) = 16$; $u_2(\pi) = 24$; $u_3(\pi) = 12$.
- **QI:** $u_1(\pi) = 1 + 4\varepsilon$; $u_2(\pi) = 2 + 7\varepsilon$; $u_3(\pi) = 2 + 5\varepsilon$.
- **2-approval:** $u_1(\pi) = 1$; $u_2(\pi) = 2$; $u_3(\pi) = 1$.

Positional scoring allocation rules

Back to our resource allocation problem...

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$\rightsquigarrow$ 12 **positional scoring allocation rules**
(transposition to resource allocation of positional scoring rules in voting)

The problems studied

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For each pair (scoring vector, social criterion), what is the complexity of...

1. **Optimal Allocation Value (OAV):** is it possible to find an allocation of utility $\geq K$?
2. **Optimal Allocation (OA):** does π belong to the set of optimal allocations?
3. **Find Optimal Allocation (FOA):** find an optimal allocation.

Main results

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For $\min_i u_i(\pi)$ (egalitarianism):

- **Bad news:** **hard** (NP-complete, coNP-complete, NP-hard for OAV, OA, FOA resp.) for Borda, lexicographic and QI scoring functions.

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Most results for min carry over to leximin.

Results: summary

	OA	OAV	FOA
$F_{s,+}$	in P	in P	pol. time
$F_{s,\min}$	coNP-comp*	NP-comp*	NP-hard*
k-app or $m \in O(1)$	in P	in P	pol. time
lex or ε-qi	coNP-comp	NP-comp	NP-hard
borda	coNP-comp	NP-comp	
lex or borda or ε-qi, if $n \in O(1)$	in P	in P	pol. time
$F_{s,\text{leximin}}$	coNP-comp*	NP-comp*	NP-hard*
lex or ε-qi	in coNP	NP-comp	NP-hard
borda	in coNP	NP-comp	
lex or borda or ε-qi, if $n \in O(1)$	in P	in P	pol. time

About approximation

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Question: *Is it possible to efficiently compute* ***good*** *(but potentially suboptimal) allocations?*

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Our approach: Instead of giving general approximation results[1], we:

- focus on a **simple** allocation protocol;
- and try to analyze how good the allocations it gives are.

[1] Actually there is one in the paper, for (lexico, min).

An elicitation-free protocol...

Ask the agents to pick in turn their most preferred object among the remaining ones, according to some **predefined sequence** σ.

Example

3 agents 1, 2, 3 / 6 objects / sequence 123321 $\rightarrow$ 1 chooses first (and takes her preferred object), then 2, then 3, then 3 again...

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Here we focus on **regular sequences** σ of the kind $(1 \ldots n)^*$, but our results are similar for alternating sequences like $(1 \ldots nn \ldots 1)^*$.

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More formally:

- **Multiplicative** Price of Elicitation-Freeness: worst case ratio $u_c^{\text{opt}}/u_c(\sigma)$, for a sequence σ.
- **Additive** Price of Elicitation-Freeness: worst case difference $u_c^{\text{opt}} - u_c(\sigma)$, for a sequence σ.

Some theoretical bounds for Borda

For **classical utilitarianism** ($\sum_i u_i(\pi)$):

$$1 + \frac{n-1}{m} + \Theta(\frac{1}{m^2}) \leq MPEF \leq 2 - \frac{1}{n} + \Theta(\frac{1}{m^2}), \text{ when } m \to +\infty.$$

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For **egalitarianism** ($\min_i u_i(\pi)$):

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See the paper for more!

Some experimental results for Borda

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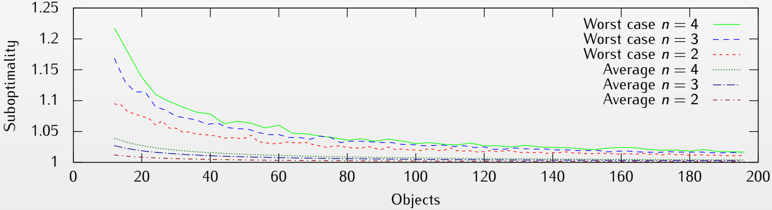

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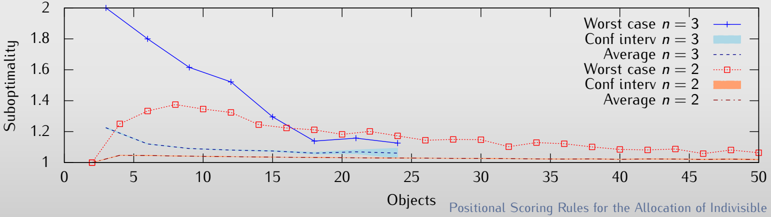

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Possible future work: manipulation, link with envy-freeness...