On Varieties of Closed Categories and Dependency of Diagrams of Canonical Maps.*

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Abstract

We present a series of diagrams D_n in Symmetric Monoidal Closed Categories such that there is infinitely many different varieties of SMCC (in the sense of universal algebra) defined by diagrams of this series as equations. Similar result will hold for weaker closed categories. We discuss the notion of dependency of diagrams in connection with this result.

1 Introduction.

Canonical maps in closed categories may be seen as instances of morphisms of the free closed category generated by an infinite set of atoms.

There exist many types of closed categories, for example Cartesian Closed Categories (CCC), Symmetric Monoidal Closed Categories (SMCC) etc. Closed categories were first introduced and studied in the 1960ies and 1970ies (see [5]). G. Lambek was the first to notice and explicitly use a close connection with proof theory (see [7, 8, 9]).

Grigori Mints in the 1970ies (see [11, 12]) has shown that this connection extends to much deeper aspects of the structure of closed categories than was initially expected, for example, that the equiality of morphisms in free closed categories can be faithfully represented using normalization in certain systems of natural deduction and lambda calculus.

His works have opened the way to the use of even more advanced proof-theoretic methods. For example, he suggested to one of the authors (his graduate student at the time) the idea of adapting a method of decreasing of the depth of formulas in proof theory to the study of commutativity of diagrams in closed categories. This approach helped to obtain many coherence theorems of category theory (see for example [13, 14]).

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Typical examples of closed categories are the category of vector spaces over a field, the category of modules over a ring, the category of semi-modules over a semi-ring, the category of pointed sets, etc. If the ring is commutative with unit then the category is a SMCC. SMCCs will be our main interest in this paper.

A typical example of a CCC is the category of sets. In closed categories of certain types, for example CCCs, the "maximality" theorem holds (cf. [1]). If to the standard identities defining equality of morphisms in the free CCC is added a new identity[1] between morphisms that have the same domain and the same codomain (i.e., form a diagram) then all the diagrams become commutative.

The situation is completely different in the case of SMCCs and closed categories with weaker theories. The "maximality theorem" does not hold for SMCCs. The negative result is even stronger. Unlike the case of CCC, in SMCC and closed categories with weaker structure, the graphs of naturality conditions (Kelly-Mac Lane graphs [5]) play very important role. There are diagrams $f, g\colon A \to B$ in the free SMCC where f, g have the same graph such that new identity $f \sim g$ added as a new axiom does not imply the commutativity of all diagrams with the same graph. Still, the commutativity of a diagram implies the commutativity of some other diagrams; we may say that the commutativity of these diagrams *depends* on the commutativity of the given one.

The main new result presented in this paper is the description of an infinite series of diagrams $D_1, ..., D_k, ...$ (with the same graph of naturality conditions) in the free SMCC such that for each k there exists a model K_k where $D_1, ..., D_k$ are non-commutative but for some $n, k < n$, $D_n, ...$ are commutative.

In universal algebra "variety" is a class of algebras defined by axioms that have the form of identities. Categories may be seen as partial many-sorted algebras. In this terminology, our new result is that there exist infinitely many different varieties of SMCC between the free category and the category of graphs[2].

This fact justifies the study of dependency of diagrams. Proof-theoretical methods have shown their strength in the study of commutativity and non-commutativity of diagrams in free closed categories [6, 11, 13, 14]. They provide, in particular, efficient deciding algorithms [15]. In all probability they will be also very useful in the study of dependency of diagrams in equationally defined subclasses of the class of closed categories and in concrete non-free models.

2 Algebra and Logic in Closed Categories: some Basic Facts

The connections between structural proof theory and categorical algebra are well known. In this section we follow roughly the schema introduced already in the works of Lambek [8, 9] and Mints [11, 12].

A SMCC is defined by the following data:

[1]It is assumed that all identities are closed w.r.t. substitution.

[2]For example, if we add D_n as unique new axiom (declare that D_n is commutative) the resulting variety will be different from the variety defined by D_k as the new axiom. The result is valid also for all the weaker structures of closed category considered in this paper.

- A category K;
- An object $I \in Ob(K)$
- The bifunctors $\otimes{:}K \times K \to K$ (tensor) and $\multimap{:}K^{op} \times K \to K$ (internal *hom*-functor);
- the following families of maps (basic natural transformations): $1_A{:}A \to A$, $a_{ABC}{:}(A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $a^{-1}_{ABC}{:}A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ $b_A{:}A \otimes I \to A$, $b^{-1}_A{:}A \to A \otimes I$, $c_{AB}{:}A \otimes B \to B \otimes A$ $\epsilon_{AB}{:}A \otimes (A \multimap B) \to B$, $d_{AB}{:}A \to B \multimap A \otimes B \quad (A, B, C \in Ob(K))$.

These data satisfy certain equations that we shall not describe in detail (see, e.g., [14]). The main groups of equations are:

- equations between components of basic natural transformations, such as $c_{AB} \circ c_{BA} = 1_{A \otimes B}$, Mac Lane's "pentagon" and "hexagon";
- naturality conditions, functoriality axioms for $\otimes$ and $\multimap$, general category axioms that involve arbitrary morphisms of K, the axioms (involving e, d, $f{:}A \otimes B \to C, g{:}A \to (B \multimap C)$) that make $\otimes$ left adjoint of $\multimap$.

The action of functors and composition on morphisms may be regarded as an application of the following *rules*:

$$\frac{f{:}A \to B \quad g{:}C \to D}{f \otimes g{:}A \otimes C \to B \otimes D}(\otimes) \quad \frac{f{:}A \to B \quad g{:}C \to D}{f \multimap g{:}B \multimap C \to A \multimap D}(\multimap)$$

$$\frac{f{:}A \to B \quad g{:}B \to C}{g \circ f{:}A \to C}(cut).$$

The basic natural transformations correspond in this setting to axiom schemas ($1_A{:}A \to A$, $c_{AB}{:}A \otimes B \to B \otimes A$ etc.).

The free SMCC $\mathbf{F}(\mathbf{A})$ over a set of atoms $\mathbf{A}$ may be built as follows.

- The objects are formulas built from atoms and the constant I using $\otimes$ and $\multimap$ as connectives.
- The morphisms are the expressions $f{:}A \to B$ derivable from axiom schemas corresponding to basic natural transformations by rules $\otimes$, $\multimap$ and *cut* above considered up to the smallest equivalence relation $\equiv$ such that all the equations of the SMCC mentioned above are satisfied.

The relation $\equiv$ is a congruence w.r.t. the application of $\otimes$, $\multimap$ and *cut* (composition). It is substitutive, i.e., $f \equiv g \Rightarrow \sigma f \equiv \sigma g$ for every substitution $\sigma = [A_1, ..., A_k/a_1, ..., a_k]$ because all the morphisms in $\mathbf{F}(\mathbf{A})$ are obtained from components of natural transformations (axiom schemes). If not stated otherwise we shall assume that $\mathbf{A}$ is infinite[3].

Let A be a formula built from atoms and I using $\otimes$ and $\multimap$ as connectives. From the categorical point of view, it represents a functor and every occurrence of an atom or I in A is co - or contravariant. In logic to categorical variance corresponds the notion of sign of an occurrence. It is defined by induction on the process of the construction of A.

Definition 2.1

- *If $A = a$ or $A = I$ the occurrence of a (of I) in A is positive (covariant).*
- *In $A \otimes B$ the signs of occurrences are the same as the signs of corresponding occurrences in A and B. In $A \multimap B$ the signs of occurrences lying in B are the same as in B and the signs of occurrences lying in A are opposite to the signs in A.*
- *The signs of occurrences in the sequent $A \to B$ are the same as in $A \multimap B$.*

Definition 2.2 *A sequent S is called balanced iff every atom has exactly two occurrences with opposite signs in S.*

In the case of SMCC the following proposition holds [4].

[3] A presentation of the free Cartesian Closed Category (CCC) along similar lines may be obtained if we add the following natural transformations (families of maps):

$$0_A{:}A \to I, \quad \delta_A{:}A \to A \otimes A,$$

$$l_{AB}{:}A \otimes B \to A, \quad r_{AB}{:}A \otimes B \to B$$

and the identities that will make I the terminal object and $\otimes$ the cartesian product with projections l, r. These new data may replace the transformations a, b, c because associativity, commutativity and the property of unit I can be obtained from $0, \delta, l, r$ and the new identities.

Maximality of the theory of CCC proved in [1] means that the only possible relation $\sim$ different from $\equiv$ in case of CCC is the relation that makes equivalent all the morphisms with the same source and target.

Let us mention some other types of closed categories:

- Monoidal Closed Categories, where the commutativity isomorphism c_{AB} is absent;
- Symmetric Closed Categories (non-monoidal), where the functor $\otimes$ is absent and c_{AB} is replaced by the isomorphism $\xi_{ABC}{:}A \multimap (B \multimap C) \to B \multimap (A \multimap C)$;
- Closed Categories (without $\otimes$ and any symmetry isomorphism).

For all these types of categories a free category over a set of atoms $\mathbf{A}$ is constructed similarly to $\mathbf{F(A)}$. Each model for the theory of SMCC is at the same time a model for MCC, SCC and CC (but not of CCC). Our main results concerning SMCC will imply similar result for these closed categories.

[4] The pairs of occurrences of the same variable in balanced sequents correspond to the edges of the so called "graph" introduced by Kelly and Mac Lane, and also to "axiom links" in linear logic. The combinatorial proofs of a similar proposition were published as early as in [2, 5]. They use the fact that all the axioms that define the relation $\equiv$ may be written in a balanced form and the rather tricky definition of composition of graphs. The proof using Gentzen-style sequent calculus $\mathbf{L(A)}$ described below is much more straightforward. It uses *cut*-elimination and the properties of "linear" rules of $\mathbf{L(A)}$. There is no similar proposition that would hold in the CCCs case.

Theorem 2.3 *Let $f, g: A \to B$. There exist $f', g': A' \to B'$ where the sequent $A' \to B'$ is balanced such that f, g and $A \to B$ can be obtained from f', g' and $A' \to B'$ by identification of variables and $f' \equiv g'$ iff $f \equiv g$.*

In this paper we shall consider other equivalence relations on morphisms of $\mathbf{F(A)}$. Below $\sim$ will denote any substitutive equivalence relation that contains $\equiv$, respects the graphs and is a congruence w.r.t. $\otimes$, $\multimap$ and *cut*. Obviously such a $\sim$ will define a structure of a SMCC on $\mathbf{F(A)}$. In connection with our main results we shall consider the relations $\sim_K$ generated by interpretations in certain SMCCs (models) K. More precisely, let us consider any function (valutaion) $v: \mathbf{A} \to Ob(K)$. Since $\mathbf{F(A)}$ is free, every v defines a unique structure-preserving functor (interpretation) $| - |_v: \mathbf{F(A)} \to K$ where $|a|_v = v(a)$. Assume that f, g have the same graph. The relation $\sim_K$ is defined by $f \sim g \Leftrightarrow |f|_v = |g|_v$ for every v in K. We shall say that the diagram $f', g': A' \to B'$ depends on $f, g: A \to B$ if for every SMCC K the equivalence $f \sim_K g$ implies $f' \sim_K g'$.

Theorem 2.4 *Let $\sim$ be the smallest substitutive equivalence relation on the derivations of $\mathbf{F(A)}$ such that $f \sim g$, $\sim$ contains $\equiv$, respects the graphs and is a congruence w.r.t. $\otimes$, $\multimap$ and cut. The diagram $f', g': A' \to B'$ depends on $f, g: A \to B$ iff $f' \sim g'$.*

This theorems shows that syntactic methods may be used for verification of dependency of diagrams, since the standard construction for the smallest equivalence relation $\sim$ uses certain syntactic calculus with pairs of derivations as derivable objects.

The interest of presentation of $\mathbf{F}(\mathbf{A})$ using "algebraic"axioms and rules above is that it opens a way to reformulations using different axiom and rule systems, already well studied in logics [5]. We shall consider in this paper one such reformulation, the sequent calculus for Intuitionistic Multiplicative Linear Logic (IMLL), cf. [3]. The calculus $\mathbf{L}(\mathbf{A})$ is defined as follows:

Axioms

$A \to A \quad (1_A) \qquad \to I \quad \text{(unit)}$

Structural Rules

$$\frac{\Gamma \to A \quad A, \Delta \to B}{\Gamma, \Delta \to B}\text{(cut)} \qquad \frac{\Delta \to I \quad \Sigma \to A}{\Delta, \Sigma \to A}\text{(wkn)} \qquad \frac{\Gamma \to A}{\Gamma' \to A}\text{(perm)}$$

Logical rules

[5] G.E.Mints considered systems similar to $\mathbf{F(A)}$ (he called them "Hilbert-type systems") and the systems of natural deduction for CC, SCC, MCC, SMCC and CCC categories. Future study has shown that the systems of natural deduction are better for the development of deciding algorithms but sequential calculi are more flexible when the transformations of derivations and diagrams are studied.

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \otimes B}\ (\to\otimes) \qquad \frac{A, B, \Gamma \to C}{A \otimes B, \Gamma \to C}\ (\otimes\to)$$

$$\frac{A, \Gamma \to B}{\Gamma \to A \multimap B}\ (\to\multimap) \qquad \frac{\Gamma \to A \quad B, \Delta \to C}{\Gamma, A \multimap B, \Delta \to C}\ (\multimap\to)$$

Here Γ, Δ, Σ are lists of formulas. A list of formulas $\Gamma = A_1, ..., A_n$ may be seen as an abbreviation of $\overline{\Gamma} = (...(A_1 \otimes ...) \otimes A_n) \otimes I$. The transformation **C** of L-derivations into F-derivations and **D** of F-derivations into L-derivations are described in detail in [14, 10]. Let us present here just one case as an example:

$$\mathbf{C}\left(\frac{\Gamma \xrightarrow{\psi} A \quad B, \Delta \xrightarrow{\varphi} C}{\Gamma, A \multimap B, \Delta \to C}\right) =$$

$$= (\mathbf{C}(\varphi) \circ ((e_{AB} \circ (\mathbf{C}(\psi) \otimes 1_{A \multimap B})) \otimes 1_{\overline{\Delta}})) \circ \zeta : \overline{\Gamma, A \multimap B, \Delta} \to C,$$

with $\zeta : \overline{\Gamma, A \multimap B, \Delta} \to (\overline{\Gamma} \otimes (A \multimap B)) \otimes \overline{\Delta}$ a central isomorphism (unique up to $\equiv$).

Via **C** we define the equivalence $\equiv$ on L-derivations as induced by $\equiv$ on F-derivations. The relation $\equiv$ on **L(A)**-derivations is a congruence w.r.t. the application of rules. It is substitutive as well [6]. If we consider another equivalence relation $\sim$ on morphisms of **F(A)** it is also transferred to **L(A)** via **C**. (Similarly, for every valuation $v : \mathbf{A} \to K$ the interpretation $|-|_v$ on derivations is defined via **C**: $|d|_v = |\mathbf{C}(d)|_v$.)

The notion of sign and balanced sequent is generalized naturally to **L(A)**. If $\Gamma = A_1, ..., A_n$ then the signs of occurrences of atoms in $\Gamma \to A$ are the same as in $A_1 \multimap (A_2 \multimap ...(A_n \multimap A)...)$. $\Gamma \to A$ is balanced if every atom occurs there exactly twice with opposite signs.

One of the most common transformations of derivations is cut-elimination.

Theorem 2.5 *For every derivation $d : \Gamma \to A$ in* **L(A)** *there exists a cut-free derivation $d' \Gamma \to A$ such that $d' \equiv d$. (This also implies $d' \sim d$ for any $\sim$.) If $\Gamma \to A$ is balanced then all the sequents in its cut-free derivation are balanced* [7].

[6] A sequent calculus for CCC may be described in a similar way. It is enough to modify it as follows:
(a) To replace the axiom $\to I$ by $\Delta \to I$ (Δ being an arbitrary list of formulas);
(b) To add the structural rule of contraction

$$\frac{A, A, \Gamma \to B}{A, \Gamma \to B}.$$

The resulting calculus represents exactly the Intuitionistic Propositional Logic with I as constant "true", $\otimes$ as conjunction and $\multimap$ as implication. The transformations **C** and **D** are modified accordingly, and the equivalence relation on derivations of **L(A)** is induced by equivalence in **F(A)** via **C**. (The sequent calculi for MC, SC and Non-Symmetric Closed Vategories and the transformations **C** and **D** are built in an analogous way.)

[7] In fact, not only *cut* but also all the trivial applications of *wkn* (with $\to I$ as left premise)

Cut-elimination here is a standard algorithm used in proof-theory. For comparison, a cut-elimination procedure described in the algebraic notation in [6] is much more heavy and difficult to use. A similar theorem is true for all the logical calculi corresponding to the closed categories considered above.

Another useful transformation is the reduction of formula's depth described in detail in [14].

Definition 2.6 *The sequent* $\Gamma \to A$ *is called 2-sequent if* A *contains no more than one connective and each member of* Γ *no more than two connectives.*

Some formulas may be replaced by isomorphic ones (reducing further the number of possibilities).

Definition 2.7 $\Gamma \to A$ *is called a pure 2-sequent if* A *has one of the forms* $x, a \otimes b, a \multimap x$ *and each member of* Γ *has one of the forms* $x, a \multimap x, a \multimap (b \otimes c), (a \otimes b) \multimap x, (a \multimap x) \multimap y$. *Here* x, y *stand for* I *or atoms,* a, b *are atoms.*

Every derivation can be transformed into a derivation of some 2-sequent using two operations (followed by isomorphisms to obtain a *pure* 2-sequent):

$$\Gamma \stackrel{d}{\to} B \mapsto \frac{\Gamma \stackrel{d}{\to} B \quad p \stackrel{id}{\to} p}{\Gamma, B \multimap p \to p}(p\, fresh)$$

and *cut* with left premises of the form $p \multimap C, A[p] \to A[C]$ or $C \multimap p, A[p] \to A[C]$ (p fresh) [8]. There also exists the inverse transformation using substitutions $[C/p]$ and *cuts* with $\to C \multimap C$. (Due to the previous theorem *cut* can always be eliminated.)

Theorem 2.8 *(Reduction to 2-sequents.) Let* d_1, d_2 *be two derivations of the same (balanced) sequent* S. *Then there exist two derivations* d_1', d_2' *of the same (balanced) pure 2-sequent* S' *such that for any relation* $\sim$ *(including* $\equiv$ *itself)* d_1, d_2 *are* $\sim$*-equivalent iff* d_1', d_2' *are* $\sim$*-equivalent.*

Another useful property is faithfullness. Using this property we may reduce the problem of equivalence of derivations that have identical inferences in the end to the equivalence of derivations of their premises.

Definition 2.9 *(Cf. [10]) An equivalence* $\sim$ *is faithful w.r.t. the rule* R *if, for any two derivations* φ, φ' *of the same sequent ending by the inference of* R *having*

can be eliminated, and several successive applications of *perm* can be replaced by one application. By abuse of terminology we shall assume below that in all *cut*-free derivations these simplifications are done as well. For all the logical calculi mentioned above (except for CCC) and any given sequent there exists only a finite number of *cut*-free derivations in this sense.

[8] Here a single occurrence of C is replaced by p. The form depends on the variance (sign) of this occurrence of C in A. One takes a standard derivation of these sequents, which always exists in "symmetric" calculi, i.e., logical systems for CCC, SMCC, SCC categories. The theorem that follows holds in each of these systems.

as premises some derivations of the same sequents, $\varphi \sim \varphi'$ iff the derivations of the premises are $\sim$-equivalent.

In the case of SMCC faithfulness is easy to prove for the rules $\rightarrow \multimap$, $\rightarrow \otimes$, $\otimes \rightarrow$, wkn. Far from obvious, but true (see [10], theorem 4.15), it holds also for $\multimap \rightarrow$. It may not hold for other systems.

3 Commutative and Non-Commutative Diagrams in the Free SMCC.

Below we shall call diagrams not only pairs $f, g{:}A \rightarrow B$ in $\mathbf{F(A)}$ but also pairs of $\mathbf{L(A)}$-derivations of the same sequent.

A sequent S is called proper iff it does not contain occurrences of subformulas of the form $A \multimap B$ where B is constant (contains only I) and A is not constant.

Theorem 3.1 *(The Kelly-Mac Lane coherence theorem reformulated for* $\mathbf{L(A)}$*, cf. [5].) Let $f, g{:}\Gamma \rightarrow A$ and the sequent $\Gamma \rightarrow A$ be proper. If f and g have the same graph[9] then $f \equiv g$.*

Example 3.2 *If the sequent is not proper, f may be non-equivalent to g. The following diagram (called "triple-dual" diagram) is not commutative*

$$(1)\qquad \begin{array}{ccc} ((a \multimap I) \multimap I) \multimap I & \xrightarrow{\;1\;} & ((a \multimap I) \multimap I) \multimap I \\ & {\scriptstyle k_a \multimap 1} \searrow \qquad \nearrow {\scriptstyle k_{a \multimap I}} & \\ & a \multimap I & \end{array}$$

where a is a variable and $k_a = (1 \multimap e_{aI}) \circ d_{a(a \multimap I)}{:}a \rightarrow (a \multimap I) \multimap I$ is the standard "embedding of a into its second dual".

Non-commutativity of this diagram may be checked formally in $\mathbf{F}(\mathbf{A})$ (the equivalence relation $\equiv$ is decidable). It is non-commutative also in certain models such as the SMCC of vector spaces or the SMCC of modules over a commutative ring with unit. One may note that it is always commutative in the full subcategory of vector spaces of finite dimension. On the contrary, it is not always commutative for finitely generated modules.

It will be useful to consider another diagram which is commutative (with respect to any relation $\sim$) iff the triple-dual diagram is commutative:

$$(2)\;\; f, g{:}\; (a \otimes b \multimap I), ((b \multimap I) \multimap I), ((a \multimap I) \multimap I) \rightrightarrows I.$$

[9] In particular, if the sequent is balanced.

The fact that (2) is commutative iff (1) is commutative is more easily checked in $\mathbf{L}(\mathbf{A})$. It is also a good illustration of application of proof-theoretical methods.

Let $f_0 = 1_{((a\multimap I)\multimap I)\multimap I}$, and let g_0 denote the derivation corresponding to $k_{a\multimap I}\circ(k_a \multimap 1_I)$. We perform a *cut*-elimination. It is easily checked that *cut*-free derivations will be equivalent to the derivations ending by $\to \multimap$. By faithfulness (in this case corresponding to adjunction) we pass from the pair

$$f_0, g_0: ((a \multimap I) \multimap I) \multimap I \to ((a \multimap I) \multimap I) \multimap I$$

to the pair

$$f_0^-, g_0^-: (((a \multimap I) \multimap I) \multimap I), ((a \multimap I) \multimap I) \to I.$$

Afterwards we perform the reduction to a pure 2-sequent, first applying (simultaneously to f_0^-, g_0^-) *cut* with

$$h{:}\ b \multimap (a \multimap I), ((b \multimap I) \multimap I) \to ((a \multimap I) \multimap I) \multimap I,$$

(it is an easy exercice to find the derivation h) and then (again via *cut*) the isomorphism $i{:}a \otimes b \multimap I \to b \multimap (a \multimap I)$. All *cuts* can be eliminated afterwards. The result is the pair of derivations $f, g{:}\, a \otimes b \multimap I, (b \multimap I) \multimap I, (a \multimap I) \multimap I \to I$ (one has $(b \multimap I) \multimap I$, and another $(a \multimap I) \multimap I$ as the main formula of last application of $\multimap \to$). One may return to f_0, g_0 via substitution $[b \multimap I/a]$, *cuts* with isomorphisms and $\to (b \multimap I) \multimap (b \multimap I)$, and application of the rule $\to \multimap$. Since all the steps preserve $\sim$, $f_0 \sim g_0 \Leftrightarrow f \sim g$.

The pair of derivations f, g is also an example of so called critical pair. There is a full description of non-equivalent pairs of derivations in $\mathbf{L}(\mathbf{A})$ based on the notion of critical pairs. The idea of critical pair was suggested by Voreadou [16], but the proof of her main theorem used an erroneous lemma. Here we give the formulation of the corrected theorem that was proved in [14]. Without loss of generality (due to theorems 2.8, 2.3), and to simplify the formulations we shall consider only the case of balanced pure 2-sequents.

Definition 3.3 *A pair of derivations of the same balanced pure 2-sequent S is critical if*

$$(1)\ \ d_1 \equiv \frac{\Gamma, A' \multimap I \overset{d_1'}{\to} A \quad I \overset{1_I}{\to} I}{\Gamma, A' \multimap I, A \multimap I \to I} \multimap \to, \quad d_2 \equiv \frac{\Gamma, A \multimap I \overset{d_2'}{\to} A' \quad I \overset{1_I}{\to} I}{\Gamma, A' \multimap I, A \multimap I \to I} \multimap \to, perm;$$

(2) a cut-free derivation of S can end only by some application of $\multimap \to$;

(3) the derivations d_1', d_2' are not $\equiv$-equivalent to derivations ending by $\multimap \to$.

The pair is minimal if Γ does not contain single atoms as its members.

Let α be some substitution of I for variables. In [14] the "substitutions with purification" were defined. Let $d{:}\Gamma \to A$ be a derivation of some 2-sequent. Then $\alpha * d$ is the derivation obtained from d by α and *cuts* with isomorphisms that will make its final sequent pure. The derivation $\alpha * d$ is defined up to $\equiv$, but its final sequent is defined without ambiguity.

Theorem 3.4 *(Cf. [14].) Let d_1, d_2 be derivations of a balanced sequent $\Gamma \to A$ and d'_1, d'_2 the corresponding derivations of a balanced pure 2-sequent. Then $d_1 \equiv d_2$ iff there exists a substitution α of I for variables such that $\alpha * d'_1, \alpha * d'_2$ is a minimal critical pair*[10].

4 The "Triple-Dual" Conjecture.

Conjecture 4.1 *Commutativity of the triple-dual diagram implies commutativity of all the diagrams of canonical maps $f, g{:}A \to B$ with balanced $A \to B$. More precisely: let $\sim$ be the smallest equivalence relation that satisfies all axioms of SMCC, is substitutive and the triple-dual diagram is commutative w.r.t. $\sim$. Then for all $f, g{:}A \to B$ with balanced $A \to B$ in $\mathbf{F(A)}$ we have $f \sim g$.*

An argument in favor of this conjecture is that the following theorem holds.

Theorem 4.2 *(Soloviev, 1990 [13].) If $\sim$ is the smallest equivalence relation that satisfies all the axioms of SMCC, is substitutive, the triple-dual diagram is commutative w.r.t. $\sim$, and for all f, g and atom a*

(∗) $[a \multimap I/a]f \sim [a \multimap I/a]f \Rightarrow f \sim g$,

then $f \sim g$ for all $f, g{:}A \to B$ in $\mathbf{F(A)}$ with the same graph[11].

As recently checked Antoine El Khoury, the commutativity of the triple-dual diagram implies (without the assumption (*)) the commutativity of all diagrams $f, g{:}A \to B$ with balanced $A \to B$ containing no more than 3 variables.

5 Main Results

Commutativity of the diagrams considered below **does not imply** the commutativity of the triple-dual diagram, so the equivalence relation generated by these identities are between $\equiv$ (minimal relation) and the relation generated by commutativity of the triple-dual.

First non-trivial "intermediate" equation was obtained due to a suggestion of M. Spivakovsky, developed later by L. Mehats and S. Soloviev [10].

The diagram (3) studied in [10] was obtained from

$$(2)\ \ f, g\colon (a \otimes b \multimap I), ((b \multimap I) \multimap I), ((a \multimap I) \multimap I) \overrightarrow{\rightarrow} I$$

[10]The conditions on the left premises that require verification of equivalence are applied to the (finite number of) derivations with smaller final sequent. This theorem may be used recursively to obtain deciding algorithms for $\equiv$. In [15] an algorithm of low polynomial complexity was described.

[11]Equivalently: with balanced $A \to B$.

by *cut* with (unique) $h{:}(((a \multimap I) \otimes (b \multimap I)) \multimap I) \multimap I \to (a \otimes b \multimap I)$.

Let k be a field, $k[x,y]$ the related polynomial ring in two variables, $I = k[x,y]/(x^2, xy, y)$ and $M(k,I)$ the SMCC generated by I and k (as an I-module). It was shown in [10] that in $M(k,I)$ (3) is commutative while (2) and (1) are not (lemma 5.8). So, if we add to the axioms of SMCC the equation corresponding to (3), the diagrams (2) and (1) will remain non-commutative.

In this paper we describe certain sequence $D_2, ..., D_k, ..., D_m, ...$ of diagrams and certain models K_k such that in K_k the diagrams $D_2, ..., DG_k$ are not commutative and there exists $m > k$ such that $DG_m, ...$ are commutative (we don't know whether $DG_k, ..., DG_{m-1}$ are commutative).

Below we shall write A^* instead of $A \multimap I$. Let A^n denote the n-th "tensor power" of an object A, $A^n = (A \otimes ...) \otimes A$, and f^n the n-th "tensor power" of a morphism f, $f^n = (f \otimes ...) \otimes f$. For example, $e^n_{aI}{:}(a^* \otimes a)^n \to I^n$.

Let b^n_I be defined by $b^1_I = b_I{:}I \otimes I \to I,$, ..., $b^n_I = b \circ (b^{n-1}_I \otimes 1_I){:}I^{n+1} \to I$.

To obtain the diagrams $D_2, ..., D_k, ...$ we notice that there exists

$$h_k{:}(((a \multimap I)^k \multimap I) \multimap I) \to (a^k \multimap I).$$

In $\mathbf{F(A)}$ $h_k = \pi_{((a^*)^k)^{**} a^k I}(1_{((a^*)^k)^{**}} \otimes \pi_{a^k (a^*)^k I}(b^{k-1}_I \circ (e^k_{aI} \circ \xi)))$ Here ξ is an appropriate central isomorphism[12].

The diagram

$$(D^0_2)\ f^0_2, g^0_2{:}(a^2)^*, a^{**}, a^{**} \rightrightarrows I$$

is obtained from the diagram (2) by substitution of a for b (in other words, by identification of variables a and b). The diagram D_2

$$(DG_2)\ f_2, g_2{:}((a^*)^2)^{**}, a^{**}, a^{**} \to I$$

is obtained from D^0_2 by *cut* with (the derivation corrsponding to) h_2.

We define[13] the diagram D^0_m, $m \geq 2$, as the result of substitution of a^{m-1} for b into diagram (2). The morphisms obtained from f, g by this substitution are denoted f^0_m, g^0_m.

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[12]In $\mathbf{L(A)}$ to h_k corresponds the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{a \to a \quad I \to I}{a, a \multimap I \to I} \quad \dfrac{a \to a \quad I \to I}{a, a \multimap I \to I}}{a, a \multimap I, a, a \multimap I \to I}}{\cdots}}{a, a \multimap I, ..., a, a \multimap I \to I}$$

$$\dfrac{a, a \multimap I, ..., a, a \multimap I \to I}{a, ..., a, (a \multimap I), ..., (a \multimap I) \to I}$$

$$\cdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a^k, (a \multimap I)^k \to I}}{(a \multimap I)^k, a^k \to I}}{a^k \to ((a \multimap I)^k \multimap I)} \quad \dfrac{}{I \to I}}{a^k, ((a \multimap I)^k \multimap I) \multimap I \to I}}{((a \multimap I)^k \multimap I) \multimap I \to (a^k \multimap I)}$$

[13]The index m will correspond to the number of factors in tensor products.

The diagram D_m is obtained from D_m^0 by *cut* with h_m (f_m, g_m are resulting derivations):

$$(D_m)\ f_m, g_m{:}((a^*)^m)^{**}, a^{**}, (a^{m-1})^{**} \rightrightarrows I.$$

In order to obtain the models K_k we shall consider certain SMCCs of commutative semimodules over commutative semirings.

For all basic definitions concerning semirings and semimodules see [4]. Below we shall denote the "addition" of the semiring I by $+$ and "multiplication" by $*$. In case of a semimodule M we shall denote by $+_M$ its additive operation and $*_M$ the action of I on M; the index M will often be omitted.

Proposition 5.1 *I-semimodiles over a commutative semiring I and their homomorphisms form a SMCC with tensor product $\otimes$ and internal hom-functor $\multimap$ defined in usual way.*

We consider the categories of semimodules over the semiring $I_n = \{0, ..., n\}$ with *max* as addition and with "bounded multiplication" $*$ as multiplication:

$p * q = p \cdot q$ if $p \cdot q < n$ and $p * q = n$ otherwise.

Obviously I_n is a commutative semiring. When n is irrelevant or clear from the context it will be omitted.

Notice that in this category $b_M{:}M \otimes I \to M$ is defined by $b_M(x \otimes p) = p * x$, in partucular if $M = I$ then $b_I(p_1 \otimes p_2) = p_1 * p_2$. For an element $p_1 \otimes ... \otimes p_n \in I \otimes ... \otimes I$, $b_I^{n-1}(p_1 \otimes ... \otimes p_n) = p_1 * ... * p_n$.

We shall consider the semimodules M over I that have some additional properties.

(Top) There is a "top" element $T_M \in M, T_M \neq 0_M$ such that for all $x \in M$, $x + T_M = T_M + x = T_M$, if $x \in M, x \neq 0_M$ then $n * x = T_M$ and if $0 \neq k \in I$ then $k * T_M = T_M$.

Obviously, I itself does satisfy these consditions if we take $T_I = n$. For I_s considered as a semimodule over I s must be not greater than n.

Lemma 5.2 *Let M_1, M_2 be two semimodules over I with top elements T_1 and T_2 respectively. Let $f{:}M_1 \to M_2$ be a homomorphism of semimodules, different from constant 0. Then $f(T_1) = T_2$ and for $x \in M_1$, $x \neq 0$, $f(x) \neq 0$. As a consequence, two morphisms $f, g{:}M_1 \to M_2$ always coincide at least on 0 and T_1.*

Definition 5.3 *Let us call an $I-$semimodule r-reducible for some $r \in I, 1 < r < n$ if for every $x \in M$ $r * x = T_M$.*

Example 5.4 *Let $M = I_2 = \{0, 1, 2\}$ considered as a semi-module over the semi-ring $I_4 = \{0, 1, 2, 3, 4\}$ (with the ordinary multiplication "bounded by 2" as* *the action). It satisfies* **(Top)** *with* $T_M = 2$ *and is 2-reducible. Of course* M *of this example is also 3- and 4-reducible.*

We shall consider the semimodules M such that

(r-red) M is $r-$reducible for some $r \in I, 1 < r < n$.

Theorem 5.5 *(1) All semi-modules over* I *satisfying* **(top)** *form an SMCC.*

(2)Let r *be fixed,* $1 < r < n$. *All semi-modules* M *over* I *satisfying* **(top)** *and* **(r-red)** *form an SMCC.*

This theorem will permit us to consider the SMCC generated by $\otimes$ and $\multimap$ from I and some given semimodule, for example $I = \{0,1,2,3,4\}$ and $M = \{0,1,2\}$ and be sure that all the objects of this category will have a top element and be r-reducible[14].

Lemma 5.6 *Let* M *be an* r*-reducible semimodule over* I *and* $f{:}M \to I$ *(we may say also that* $f \in M \multimap I$*). Then for all* $x \in M$ $f(x) \geq n/k$.

Let h_m^- denote $a^m) \otimes (a^*)^m \xrightarrow{\xi} (a^* \otimes a)^m \xrightarrow{e_{aI}^m} I^m \xrightarrow{b_I^{m-1}} I$ Consider the SMCC K of semimodules satisfying **(top)** and **(r-red)** over I. Now we can easily prove the following lemma.

Lemma 5.7 *Let* n, r *be as above,* m *such that* $(n/r)^m \geq n$ *and* v *an interpretation defined by* $v(a) = M \in Ob(K)$. *Then the morphism* $|h_m^-|{:}(M^m) \otimes (M^*)^m \to I$ *takes the value* 0 *if its argument is* 0 *and* $T_I = n$ *otherwise.*

Corollary 5.8 *Under the same conditions, the morphism* $|h_m|$ *takes only two values:* 0 *when its argument is* 0 *and* $T_{(M^m)^*}$ *otherwise (for every* $M \in Ob(K)$*).*

Lemma 5.9 *For all* n, r, m *as in lemma 5.7 and every interpretation* v *in the SMCC* K *of* I*-semimodules satisying* **top** *and* **r-red***, the diagram* $|D_m|_v$ *is commutative. So it is commutative with respect to the relation* $\sim_K$.

Let $2 \leq k$, $n = 3^k + 1, l = n/2$. Let $I = I_n$, $M = \{0, 1, ..., l\}$. Notice that M is $n/2$-reducible. Consider the SMCC K_k of all semimodules satisfying **top** and **l-red** generated by I and M.

Lemma 5.10 *Let the interpretation* v *be defined by* $v(a) = M \in Ob(K_k)$. *The diagrams* $|D_2|_v, ..., |D_k|_v$ *are non-commutative.*

[14]These properties are easier to verify and use than, for example, the property of being a semilattice. Notice that the structure of the objects of the SMCC generated by I and M is not necessarily simple. For example, the semimodule $M \multimap I$ will be generated (non freely) by $[2]{:}1 \mapsto 2, [3]{:}1 \mapsto 3$, $(M \multimap I) \otimes (M \multimap I)$ will have four generators, etc.

To prove this lemma we verify that the image $Im(|h_j|_v)$ contains a certain element p different from 0 and $T_{(M^j)^*}$, and there exist elements $\psi \in M^{**}, \varphi \in (M^{j-1})^{**}$ such that the two arrows of the diagram D_j^0 take different values on the argument (p, ψ, φ) $(2 \leq j \leq k)$.

Theorem 5.11 *There exist infinitely many different varieties of SMCC. Each of these varieties is defined by taking a (single) diagram D_m of the sequence above as a new axiom.*

To prove this theorem we use the fact that for any $k \geq 2$ there exists m (it is enough to take $m \geq log_2(3^k + 1)$) such that D_k cannot belong to the smallest equivalence relation generated by D_m. In other words, the commutativity of D_m does not imply the commutativity of D_k (by lemmas 5.9, 5.10).

6 Conclusion

To verify the commutativity of a diagram in a model can be very difficult. Instead of verifying commutativity of diagrams case by case one may hope that if one diagram is commutative then the commutativity of another will follow.

By theorem 5.11 there exist infinitely many distinct equivalence relations $\sim_K$ on derivations of $\mathbf{F(A)}$ (or $\mathbf{L(A)}$). This fact shows the importance of the study of dependency of diagrams in SMCC and closed categories with weaker structure.

Taking into account the existence of efficient deciding algorithms for commutativity of diagrams in free closed categories, the first step would be to verify whether a diagram is commutative in the free case. If it is not commutative, the study of dependency may follow. In particular, one may find some "key" diagrams whose commutativity will imply the commutativity of others (cf. the axiomatization of equivalence relations by critical pairs considered in [10]).

We believe that this direction of research will provide a new and promising application of proof theory to categorical algebra.

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